SPECIAL PARITY OF PERFECT MATCHINGS IN BIPARTITE GRAPHS

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Received 17 July 1984
Revised 8 February 1988

Let G be a bipartite graph in which every edge belongs to some perfect matching, and let D be a subset of its edge set. It is shown that M ∩ D has the same parity for every perfect matching M if and only if D is a cut, and equivalently if and only if (G, D) is a balanced signed-graph. This gives necessary and sufficient conditions on the sign pattern of an n × n real matrix under which all nonzero terms in its permanent expansion have the same sign.

1. Introduction

The starting point for the study of this paper is the following problem. What conditions on the sign pattern of an n × n real matrix A = (a_{ij}) ensure that all nonzero terms in its permanent expansion have the same sign? An analogous question obtained by replacing the word “permanent” by “determinant” has been studied in [1] and has recently been reconsidered using graph theoretic-approach [5, 7, 8]. When all nonzero terms of A’s permanent (resp. determinant) expansion have the same sign we say that A has a signed permanent (resp. a signed determinant). While the study of signed determinants used digraphs whose adjacency matrix is A, we consider here the bipartite graph B(A) = (X, E, Y) where X = {x_1, ..., x_n}, Y = {y_1, ..., y_n} and E = {[x_i, y_j]: a_{ij} ≠ 0}. Let D = {[x_i, y_j]: a_{ij} < 0}. We note that B(A) together with D is a signed-graph [2], denoted here by (B(A), D). Since every perfect matching of B(A) corresponds to a nonzero term in the permanent expansion of A we see that A has a signed permanent if and only if whenever M and N are perfect matchings of B(A), |D ∩ M| ≡ |D ∩ N| (mod 2). Given two sets P and Q we define the P-parity of Q to be |P ∩ Q| (mod 2). We say that Q is P-even (resp. P-odd) if its P-parity is 0 (resp. 1).

Let G = (X, E, Y) be a bipartite graph and let D ⊆ E. We give here two different necessary and sufficient conditions (Theorems 5 and 8) for all perfect matchings of G to have the same D-parity. The condition of Theorem 5 is stated in terms of the D-parity of certain cycles in G. It is interesting to note that the necessary and sufficient condition for an n × n matrix A = (a_{ij}) to have signed determinant is stated in terms of the D-parity of all cycles of the digraph whose adjacency matrix is A; here again D = {(i, j): a_{ij} < 0}. The condition of Theorem

† Research supported in part by NSF grant # DMS-8702930.

8 leads to an $O(|V| \cdot |E|)$ algorithm to recognize $n \times n$ matrices with signed-permanents, where $|V| = n$ and $|E|$ is the number of nonzero entries in the matrix.

We conclude this section with several basic definitions. In this paper all graphs are undirected and have no loops. A walk $W$ in a graph $G = (V, E)$ is a sequence of vertices $(x_1, x_2, \ldots, x_n)$ such that $[x_i, x_{i+1}] \in E$ for $1 \leq i \leq n - 1$. The multiset of edges of $W$ is $E(W) = \{[x_i, x_{i+1}] : i < n\}$. The length of $W$ is $|E(W)| = n - 1$. We say that $W$ is closed if $x_i = x_n$ and that it is simple if $x_i \neq x_j$ for $i \neq j$, except possibly $x_1 = x_n$. A closed and simple walk is called a cycle. Note that a cycle of length $l > 2$ has no repeated edges while a cycle of length 2 consists of one repeated edge. Let $D \subseteq E$. A walk $W$ is $D$-even ($D$-odd) if $E(W)$ is $D$-even ($D$-odd).

Let $M$ be a matching. A walk $W$ is weakly $M$-alternating if precisely one of each pair of adjacent edges $[x_i, x_{i+1}]$, $[x_{i+1}, x_{i+2}]$ of $W$ belongs to $M$ ($i = 1, \ldots, n - 2$). A weakly $M$-alternating closed walk is $M$-alternating if precisely one of $[x_{n-1}, x_n]$, $[x_1, x_2]$ is in $M$. An augmenting path is a weakly $M$-alternating path whose first and last edges are not in $M$.

2. Alternating cycles

In this section, all graphs have at least one perfect matching, denoted by $M$. It is clear that we can say nothing about edges of a graph which do not belong to any perfect matching. We say that an edge of a graph $G$ is essential if it belongs to some perfect matching of $G$. We say that a graph $G$ is clean if all its edges are essential. We denote the symmetric difference of two sets by $\Delta$.

Lemma 1. An edge $e$ is essential if and only if $e \in M$ or $e \in E(C)$ for some $M$-alternating cycle $C$.

Proof. Suppose that $e$ is an essential edge. Then $e \in M'$ for some perfect matching $M'$. If $e \in M$, we are done. So assume that $e \in M' \setminus M$. Then the connected component of $G' = (V, M \cup M')$ which contains $e$ is an $M$-alternating cycle. Conversely, suppose that $e \in E(C) \setminus M$ for some $M$-alternating cycle $C$. Then $e$ is an edge of the perfect matching $M \Delta E(C)$. $\square$

Lemma 2. If $G$ is a bipartite graph then every weakly $M$-alternating closed walk is $M$-alternating.

Proof. Let $W = (x_1, \ldots, x_n)$ be a weakly $M$-alternating closed walk. Since $|E(W)|$ is even, precisely one of $[x_1, x_2]$ and $[x_{n-1}, x_n]$ belongs to $M$. $\square$

Denote by $E^*(W)$ the set of edges of $W$ which appear in $E(W)$ an odd number of times. Then for every closed walk $W$ there exist cycles $C_1, \ldots, C_t$ such that $E^*(W) = E(C_1) \Delta, \ldots, \Delta E(C_t)$. 

Lemma 3. Let $G$ be a bipartite graph and let $W$ be an $M$-alternating closed walk. Then there exist $M$-alternating cycles $C_1, C_2, \ldots, C_t$ $(t \geq 1)$ such that

$$E^*(W) = E(C_1) \Delta E(C_2) \Delta \cdots \Delta E(C_t).$$

Proof. Let $W = (x_1, \ldots, x_n)$ be an $M$-alternating closed walk in $G$. Then $n \geq 4$. We proceed by induction on $n$. If $n = 4$ then $W$ is an $M$-alternating cycle and the result holds. Assume that $n > 4$ and let $k$ be the smallest integer such that $x_k = x_j$ for some $j < k$. By Lemma 2 the cycle $C = (x_j, \ldots, x_k)$ is $M$-alternating. If $k = n$ we are done, so assume that $1 < k < n$. It follows from Lemma 2 that the closed walk $U = (x_1, \ldots, x_j, x_{k+1}, \ldots, x_n)$ is $M$-alternating. By the inductive hypothesis there exist $M$-alternating cycles $C_1, \ldots, C_{t-1}$ $(t-1 \geq 1)$ such that $E^*(U) = E(C_1) \Delta \cdots \Delta E(C_{t-1})$. Thus $E^*(W) = E(C_1) \Delta \cdots \Delta E(C_t)$, where $C_t = C$.  

We note that a similar argument shows that for every closed walk $W$, there exist cycles $C_1, C_2, \ldots, C_t$ such that $E^*(W) = E(C_1) \Delta \cdots \Delta E(C_t)$. This also follows from the fact that the simple edge subgraph obtained from the edge subgraph $W$ by deleting all edges of even degree and collapsing edges of odd degree to a single edge is Eulerian and hence its edge set is a union of simple cycles. We use this fact in the proof of Theorem 8.

The next result is of interest by itself.

Theorem 4. Let $G$ be a clean bipartite graph and let $M$ be a perfect matching in $G$. Then the $M$-alternating cycles span the cycle space of $G$ over $F_2$; i.e. for every cycle $C$ there exist $M$-alternating cycles $C_1, \ldots, C_t$ such that $E(C) = E(C_1) \Delta \cdots \Delta E(C_t)$.

Proof. Let $C = (x_1, x_2, \ldots, x_m, x_1)$ be a cycle in $G$. We proceed by constructing a weakly $M$-alternating closed walk $W$ such that $E(C) = E^*(W)$. The result would then follow from Lemmas 2 and 3. Intuitively we would like to follow the edges of $C$ in order, and each time we encounter a pair of successive edges both of which are not in $N$, we wish to replace the latter by an $M$-alternating walk based on an $M$-alternating cycle containing it. As it turns out, we may replace sections of $C$ containing more than one edge.

More precisely, we prove the following claim:

(2.1) Let $W = (x_1, x_2, \ldots, x_n)$ be a walk. Then there exists a weakly $M$-alternating walk $W' = (y_1, y_2, \ldots, y_l)$ such that $y_1 = x_1$, $y_2 = x_2$, $y_l = x_n$ and $E^*(W) = E^*(W')$.

We prove (2.1) by induction on $n$. Let $k$ be the smallest integer such that both $[x_{k-1}, x_k]$ and $[x_k, x_{k+1}]$ are not in $M$. (If no such $k$ exist then $W$ is itself an $M$-alternating walk.) Since $G$ is clean there exists an $M$-alternating cycle $(x_k = v_1, x_{k+1} = v_2, v_3, \ldots, v_q, v_1)$. Let $p \geq 2$ be the largest index $i$ such that $v_i = x_{k+i}$ for
i = 1, \ldots, p. Replace the walk \((x_k, x_{k+1}, \ldots, x_{k+p})\) by the weakly \(M\)-alternating walk \(U = (v_1, v_q, v_{q-1}, \ldots, v_2, v_1, v_q, v_{q-1}, \ldots, v_{p+1}, v_p)\). Let \(V_1\) consist of the path \((x_1, x_2, \ldots, x_{k-1})\) followed by \(U\). The walk \(V_1\) is clearly weakly \(M\)-alternating. Furthermore, since \(E^*(U) = E^*(v_p, v_{p-1}, \ldots, v_1) = E^*(x_k, x_{k+1}, \ldots, x_{k+p})\), we have, \(E^*(x_1, \ldots, x_{k+p}) = E^*(V_1)\). Now let \(V_2 = (v_{p+1}, v_p, x_{k+p}, x_{k+p+1}, \ldots, x_n)\). Since \(p \geq 2\), \(V_2\) is shorter than \(W\). Hence, by the inductive hypothesis there exists a weakly \(M\)-alternating walk \(V_1\) joining \(v_{p+1}\) and \(x_n\) whose first edge is \([v_{p+1}, x_{k+p}]\) and \(E^*(V_2) = E^*(V_3)\). Form \(W'\) from \(V_2\) by deleting its first edge. Then the walk \(W'\) consisting of \(V_1\) followed by \(W_2\) is clearly a weakly \(M\)-alternating walk joining \(x_1\) and \(x_n\). Its first edge is \([x_1, x_2]\) and it satisfies \(E^*(W) = E^*(W')\).

The result now follows from applying (2.1) on the cycle \(C\). \(\Box\)

3. D-parity of perfect matchings

**Theorem 5.** Let \(G = (V, E)\) be a graph and \(D \subseteq E\). Suppose \(M\) is a perfect matching of \(G\). Then all perfect matchings of \(G\) have the same \(D\)-parity if and only if every \(M\)-alternating cycle in \(G\) is \(D\)-even.

**Proof.** Assume first that every \(M\)-alternating cycle in \(G\) is \(D\)-even. Suppose that there exists a perfect matching \(N \neq M\) in \(G\) such that \(|N \cap D| \neq |M \cap D| \pmod{2}\). Then \(|D \cap (MA N)|\) is odd. Each connected component of \(G' = (V, MA N)\) is an \(M\)-alternating cycle. It follows that there exists a connected component \(C\) with \(|E(C) \cap D|\) odd. But \(C\) is a \(D\)-odd \(M\)-alternating cycle in \(G\), a contradiction. Conversely, suppose that \(C\) is a \(D\)-odd \(M\)-alternating cycle in \(G\). Then \(N = M \Delta E(C)\) is a perfect matching of \(G\) but \(|N \cap D| \equiv |M \cap D| \pmod{2}\). \(\Box\)

**Corollary 6.** Let \(A\) be an \(n \times n\) real matrix. Then \(A\) has a signed nonzero permanent if and only if \((B(A), D)\) has a perfect matching \(M\) and every \(M\)-alternating cycle in \((B(A), D)\) is \(D\)-even.

The next Corollary follows from Theorems 4 and 5.

**Corollary 7.** Let \(G = (V, E)\) be a clean bipartite graph, and let \(D \subseteq E\). Suppose there exists a perfect matching in \(G\). Then all perfect matchings in \(G\) have the same \(D\)-parity if and only if all cycles in \(G\) are \(D\)-even (i.e. \(G\) is balanced [3]).

**Theorem 8.** Let \(G = (V, E)\) be a clean bipartite graph and let \(D \subseteq E\). Then all perfect matchings of \(G\) have the same \(D\)-parity \(p\) if and only if there exists a subset \(T \subseteq V\) with \(|T| \equiv p \pmod{2}\) such that

\[D = E \cap (T \times (V \setminus T)).\]
**Proof.** Suppose that \( D = E \cap (T \times (V \setminus T)) \) for some \( T \subset V \). Let \( N \) be a perfect matching in \( G \). Then \( |N \cap D| = |N \cap (T \times (V \setminus T))| - |T| - 2 |N \cap (T \times T)| = |T| \pmod{2} \). Hence all perfect matchings in \( G \) have the same \( D \)-parity.

Conversely, suppose that every perfect matching in \( G \) has the same \( D \)-parity. Let \( C \) be a connected component of \( G \). It follows from Theorem 5 that all perfect matchings in \( C \) have the same \( D \)-parity. Hence we may assume that \( G \) is connected. Fix some \( v \in V \). Let \( T \) be the set of all vertices which can be reached from \( v \) by a \( D \)-even walk and let \( T' \) be the set of all vertices which can be reached from \( v \) by a \( D \)-odd walk. Since \( G \) is connected \( V = T \cup T' \). Suppose that \( x \in T \cap T' \) for some \( x \in V \). Then there exists a \( D \)-even walk \( W = (v = u_0, \ldots, u_r = x) \) and a \( D \)-odd walk \( W' = (v = v_0, \ldots, v_i = x) \) joining \( v \) and \( x \). Let \( Q \) be the concatenation of \( W \) and \( W' \). Then \( Q \) is a \( D \)-odd closed walk. Hence as noted after the proof of Lemma 3, there exist cycles \( C_1, \ldots, C_s \) such that

\[
E^*(Q) = E(C_i) \Delta \cdots \Delta E(C_s).
\]

By Theorem 4, each \( C_i \) is the symmetric difference of some \( M \)-alternating cycles. It follows from Theorem 5 that each \( C_i \) is \( D \)-even. Hence by (3.1) \( E^*(Q) \) is \( D \)-even, but \( Q \) is \( D \)-odd, a contradiction. Thus \( T \cap T' = \emptyset \) and hence \( T' = V \setminus T \).

Now let \( e = [x, y] \in E \). Suppose that \( x \in T \). Then there exists a \( D \)-even walk \( W = (z_1, z_2, \ldots, z_n) \) joining \( v = z_1 \) with \( x = z_n \). Consider the walk \( W' = (z_1, \ldots, z_n, y) \). If \( y \in T \) then \( W' \) is \( D \)-even and hence \( e \notin D \). If \( y \in V \setminus T \) then \( W' \) is \( D \)-odd and hence \( e \in D \). Similarly if \( x \in V \setminus T \) we deduce that \( y \in T \) implies \( e \in D \). Thus \( D = E \cap (T \times (V \setminus T)) \). \( \square \)

The following example shows that Theorem 8 does not hold if \( G \) is not bipartite. Let

\[
V = \{u_1, u_2, u_3, w_1, w_2, w_3\}
\]
\[
E = \{[u_i, u_j], [w_i, w_j], [u_i, w_j] : i \neq j, i, j = 1, 2, 3\}
\]
\[
D = \{[u_i, u_j], [w_i, w_j]\}.
\]

Then \( G \) is clean, all its perfect matchings are \( D \)-even but \( D \) is not a cut.

We restate Theorem 8 in matrix terminology. Let \( A = (a_{ij}) \) be an \( n \times n \) real matrix. An entry \( a_{ij} \) is essential if there exists a permutation \( \sigma \in S_n \) such that \( a(\sigma(i)) = j \) and \( a_1 \cdot a_2 \cdots a_{n(n)} \neq 0 \). Following [10] we say a matrix \( A \) is totally supported if all its entries are essential. We note that changing the magnitude of an inessential entry does not affect the value of the permanent. We denote by \([n]\) the set \( \{1, \ldots, n\} \), and for every subset \( X \) of \([n]\), we write \( X^c = [n] \setminus X \).

**Corollary 9.** Let \( A = (a_{ij}) \) be an \( n \times n \) real matrix. Then \( A \) has a nonzero signed permanent with sign \((-1)^p\) if and only if there exist subsets \( X \) and \( Y \) of \([n]\) such that \( |X| + |Y| = p \pmod{2} \), and for every essential entry \( a_{ij} \),

\[
a_{ij} > 0 \quad \text{if} \quad (i, j) \in X \times Y \cup X^c \times Y^c
\]
and
\[ a_{ij} < 0 \quad \text{if} \quad (i, j) \in X \times Y^c \cup X^c \times Y. \]

**Corollary 10.** Let \( A \) be a totally supported matrix. Then \( A \) has a signed permanent with sign \((-1)^p\) if and only if there exist permutation matrices \( P \) and \( Q \) such that
\[
P A Q = k \begin{bmatrix} \geq 0 & \leq 0 \\ \leq 0 & \geq 0 \end{bmatrix}
\]
where \( p = l + k \).

We conclude this section with a brief discussion of the relation between our results and those of [2, 10]. Two matrices \( A \) and \( B \) are **diagonally equivalent** if there exist diagonal matrices \( D_1 \) and \( D_2 \) such that \( D_1 \, A \, D_2 = B \).

It follows from [10, Theorem 3.1] that \( A = (a_{ij}) \) is diagonally equivalent to \( |A| = (|a_{ij}|) \) if and only if all cycles in the bipartite graph associated with \( A \), denoted here \( B(A) \), are \( D \)-even. This result together with [2, Theorem 6.11] implies that if \( A \) is diagonally equivalent to \( |A| \) then \( |\text{per } A| = \text{per } |A| \). Moreover, by 6.13 in [2] we see that when \( A \) is totally supported, or equivalently if \( B(A) \) is clean the converse also holds. Since \( |\text{per } A| = \text{per } |A| \) iff all perfect matchings of \( B(A) \) have the same \( D \)-parity, we see that Corollary 7 essentially follows from Engel and Schneider [2] and Saunders and Schneider [10].

**4. Algorithmic aspect**

We conclude with a discussion of the algorithmic aspects of detecting if all perfect matchings of a given pair \((G = (V, E), D \subseteq E)\) have the same \( D \)-parity. By Theorem 5 it suffices to find a perfect matching and to check if there exists a \( D \)-odd \( M \)-cycle. Finding a perfect matching, if one exists, can be done in \( O(|V|^{3/2}|E|) \) [4, 9]. In order to deal with the second task of detecting the existence of a \( D \)-odd \( M \)-cycle, we note the following. For \( m \) in \( M \) let \( M_m = M - \{m\} \) and let \( G_m = (V, E - \{m\}) \). Then \( G \) contains a \( D \)-odd \( M \)-cycle if and only if for some \( m \) in \( M \), \( G_m \) has an \( M_m \) augmenting path whose \( D \)-parity is opposite to the \( D \)-parity of the edge \( m \). Thus the problem reduces to detecting the existence of an augmenting path between two given vertices \( u \) and \( v \) which has a given \( D \)-parity. This is a hard problem for general graphs. However, when \( G \) is bipartite we can modify known algorithms that search for augmenting paths in bipartite graphs. The idea is to do a breadth first search rooted at \( u \) choosing alternately edges not in \( M \) and edges in \( M \). Thus, if we think of \( u \) as being in level 0, then edges from an even level to an odd level are not in \( M \), while edges from an odd level to an even level are in \( M \). As we search we label vertices with + or
Special parity of perfect matching graphs

... according to the $D$-parity of the search path from the root. A vertex may be labeled by both + and -. This can happen in one of the following two ways. The simpler case is when both labels are acquired via forward-edges (tree-edges). In this case we continue our forward search. The more complex case is when one of the labels is obtained via a back-edge $[x, y]$. Since $G$ is bipartite $x$ is in an even level and $y$ is in an odd level. If $[x, y]$ is a back-edge in the induced breadth first search tree, that is, if $y$ is an ancestor of $x$, then since $[x, y]$ can not be part of an augmenting path from $u$, we ignore it. However, if $[x, y]$ is a cross-edge, that is if $y$ and $x$ are unrelated, we may need to research the subtree rooted at $y$. The researching is necessary only when the traversal of $[x, y]$ results in a second label for $y$. Since a vertex may have at most two labels, every edge is traversed at most twice. Distinguishing between back-edges and cross-edges can be easily implemented. One option is to keep a list of ancestors at each node. Another way is to perform the search in stages. In the first stage we consider all tree-edges then we perform a depth first search and finally we consider all nontree-edges using the depth first search labels to distinguish between back-edges and cross-edges. It follows that the time complexity of the algorithm to find an augmenting path from $u$ to $v$ with a given $D$-parity is $O(|E|)$ and the time complexity of the algorithm to find if there exists a $D$-odd $M$-cycle is $O(|V| \cdot |E|)$.

When the bipartite graph is known to be clean, Theorem 8 provides a linear time algorithm as outlined below. Let $T$ and $T'$ be as in the proof of Theorem 8. It follows from the proof of Theorem 8 that it suffices to check if $T \cap T' = \emptyset$. This is done by labeling vertices with $T$ or $T'$ according to the $D$-parity of the search path. Then $T \cap T' \neq \emptyset$ if and only if there exists a vertex labeled by both $T$ and $T'$.

Acknowledgment

The authors wish to thank the referee for valuable remarks.

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