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The Loewner driving function of trajectory arcs of quadratic differentials

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ABSTRACT

We obtain a first order differential equation for the driving function of the chordal Loewner differential equation in the case where the domain is slit by a curve which is a trajectory arc of a certain type of quadratic differential. In particular this includes the case when the curve is a path on the square, triangle or hexagonal lattice in the upper half-plane or, indeed, in any domain with boundary on the lattice. We also demonstrate how we use this to calculate the driving function numerically. Equivalent results for other variants of the Loewner differential equation are also obtained.

0. Introduction

Suppose that $\mathbb{H} = \{z \in \mathbb{C}\colon \operatorname{Im}(z) > 0\}$ is the upper half-plane and $\gamma : [0, T) \mapsto \overline{\mathbb{H}}$ is a simple Jordan curve with $\gamma(0) \in \mathbb{R}$ and $\gamma(0, T) = \{\gamma(t)\colon t \in (0, T)\} \subset \mathbb{H}$. Then for each $t \in (0, T)$,

$$H_t = \mathbb{H} \setminus \gamma(0, t]$$

is a simply-connected domain and hence by the Riemann mapping theorem, we can find a conformal map f_t of $\mathbb{H}$ onto H_t. Moreover, we can require that f_t has series expansion

$$f_t(z) = z - \frac{C(t)}{z} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty.$$

Normalized in this way f_t is unique and is said to be *hydrodynamically normalized*. The function $C(t)$ is positive, continuous and strictly increasing; it is called the *half-plane capacity* of $\gamma(0, t]$. Thus we can reparameterize γ such that $C(t) = 2t$ for all t; we will call this *parameterization by half-plane capacity*. With this normalization and parameterization, the function f_t satisfies the differential equation (where f_t' denotes differentiation with respect to z and $\dot{f}_t$ denotes differentiation with respect to t):

$$\dot{f}_t(z) = -\frac{2 f_t'(z)}{z - \xi(t)}, \tag{1}$$

where $\xi(t) = f_t^{-1}(\gamma(t))$ is a continuous real-valued function. This is the *chordal Loewner differential equation*. $\xi(t)$ is called the *driving function* of the slit γ. The converse is also true: given a measurable function ξ, the differential equation (1) with initial condition $f_0(z) \equiv z$ has solution f_t. Each f_t is a conformal map of $\mathbb{H}$ into itself (although $f_t(\mathbb{H})$ is not necessarily a slit domain). Chapters 3 and 4 of [8] give full details of this construction.

Since Schramm's discovery of stochastic Loewner evolution in 1999 (see [19]), there has been huge interest in the chordal Loewner differential equation and its variants. However, the relationship between the slit in $\mathbb{H}$ and its driving function is

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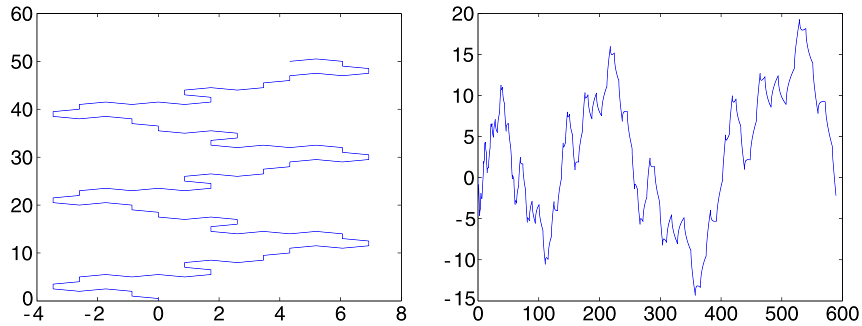

Fig. 1. A path on the hexagonal lattice on the upper half-plane (left) and a plot of its driving function on the y-axis against time on the x-axis (right).

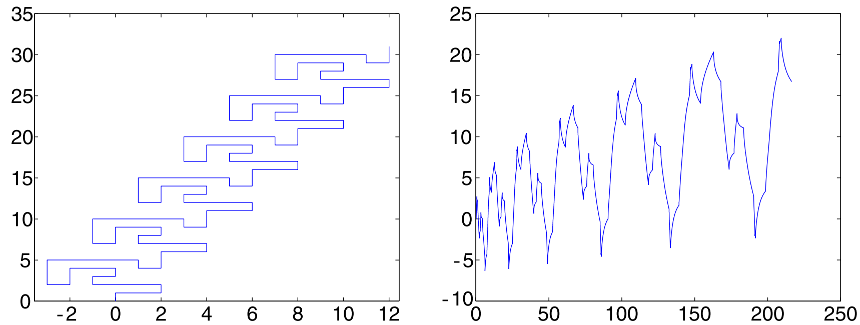

Fig. 2. A path on the square lattice on the upper half-plane (left) and a plot of its driving function on the y-axis against time on the x-axis (right).

not well understood. There are a few papers that relate the behaviour of the slit with the behaviour of the driving function e.g. [11,9]; also, the paper [3] calculates the slit arising from a few driving functions. In this paper, we will obtain a first order differential equation for ξ which we can then solve numerically to compute the driving function ξ in the case where the curve γ is a trajectory arc of a certain type of quadratic differential. We will show that this includes, for example, the case when γ is a path on the square/triangle/hexagonal lattice in the upper half-plane or indeed, in any domain whose boundary lies on such a lattice. So for example, Fig. 1 plots the driving function of a path on the hexagonal lattice in the upper half-plane and Fig. 2 plots the driving function of a path on the square lattice in the upper half-plane.

We also note that we can obtain equivalent results for other variants of the Loewner differential equation for example the radial version or with multiple slits. We will discuss these cases in this paper as well.

The proof of our formulae uses a generalization of the Schwarz–Christoffel formula for computing conformal mappings of $\mathbb{H}$ onto domains bounded by trajectory arcs of rotations of a given quadratic differential.

We also mention that, currently, the common method used to find the driving function of a given slit is to use the Zipper algorithm discovered independently by D.E. Marshall and R. Kühnau to approximate the function f_t which can then be used to determine the driving function. The Zipper algorithm can be viewed as a discrete version of the Loewner differential equation and hence is well suited to studying growth processes. It also has the advantage of being very fast. See [12] and [4].

We end the introduction with the following simple example which illustrates the basic ideas used in this paper. Suppose that $\gamma : [0, T) \mapsto \overline{\mathbb{H}}$ is a Jordan curve such that $\gamma(0) = 0$ and $\gamma(0, T)$ is a straight line arc in $\mathbb{H}$ starting from 0. As above, for each $t \in (0, T)$, we can find a conformal map f_t of $\mathbb{H}$ onto $H_t = \mathbb{H} \setminus \gamma(0, t]$ such that

$$f_t = z - \frac{2t}{z} + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty. \tag{2}$$

Since $\gamma(0, T)$ is a straight line, we can apply the classical Schwarz–Christoffel formula (see [13, p. 189]), to get

$$f_t'(z) = R\big(z - \xi(t)\big)\big(z - C^-(t)\big)^{\mu^-/2}\big(z - C^+(t)\big)^{\mu^+/2}, \tag{3}$$

where $\xi(t) = f_t^{-1}(\gamma(t))$ and $R \in \mathbb{C} \setminus \{0\}$; $C^+(t) > C^-(t)$ are the two preimages of 0 under f_t; and μ^+, μ^- can be given explicitly in terms of the angle $\gamma(0, T)$ makes with $\mathbb{R}$, with

$$\mu^+ + \mu^- = -2.$$

Then by expanding (3) near ∞, we get

$$f_t'(z) = R + \frac{R(\frac{\mu^- C^-(t)}{2} + \frac{\mu^+ C^+(t)}{2} + \xi(t))}{z} + \cdots.$$

By differentiating (2) and comparing this with the above, we deduce that $R = 1$ and

$$2\xi(t) = -\mu^- C^-(t) - \mu^+ C^+(t).$$

In addition, we can show that

$$\dot{C}^{\pm}(t) = \frac{2}{C^{\pm}(t) - \xi(t)},$$

and hence $\xi(t)$ satisfies

$$\dot{\xi}(t) = -\frac{\mu^-}{C^-(t) - \xi(t)} - \frac{\mu^+}{C^+(t) - \xi(t)}.$$

This differential equation can then be solved to find $\xi(t)$. Note that, in this case, the function f_t can be given explicitly (see [12]). Also, this function is the building block of the Zipper algorithm mentioned above.

1. Main results

To state our main results, we have to provide some background in the theory of quadratic differentials. Note that not all the terms used here are standard in the literature. See Chapter 8 of [15] and [20] for more details. A *quadratic differential* on a domain $D \subset \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is the formal expression

$$Q(z)\,dz^2,$$

where $Q(z)$ is a meromorphic function on D. For $\omega \in D$ with $\omega \neq \infty$, $Q(z)$ has Laurent series expansion about ω,

$$Q(z) = \sum_{k=n}^{\infty} a_k (z - \omega)^k$$

for some $n > -\infty$ with $a_n \neq 0$. We define the *degree of ω with respect to $Q(z)\,dz^2$*, $\deg_Q(\omega)$, to be equal to n.

If $\infty \in D$, then near ∞, Q has Laurent series expansion given by

$$Q(z) = \sum_{k=m}^{\infty} b_k z^{-k}$$

for some $m > -\infty$ with $b_m \neq 0$. We define the *degree of ∞ with respect to $Q(z)\,dz^2$*, $\deg_Q(\infty)$, to be equal to $m - 4$. The “4” in this definition ensures that the degree is conformally invariant in a way which we will make precise later. Then $\omega \in D$ is

- a *zero* of $Q(z)\,dz^2$ if $\deg_Q(\omega) > 0$;
- a *pole* of $Q(z)\,dz^2$ if $\deg_Q(\omega) < 0$;
- an *ordinary point* of $Q(z)\,dz^2$ if $\deg_Q(\omega) = 0$.

A *trajectory arc* of $Q(z)\,dz^2$ is a curve $\gamma : (a, b) \mapsto D$ that does not meet any zeroes and poles of $Q(z)\,dz^2$ and satisfies

$$Q\big(\gamma(t)\big)\dot{\gamma}(t)^2 > 0 \quad \text{for all } t \in (a, b).$$

For $\theta \in [0, \pi)$, a *θ-trajectory arc* of $Q(z)\,dz^2$ is a curve $\gamma : (a, b) \mapsto D$ that satisfies

$$\arg\big[Q\big(\gamma(t)\big)\dot{\gamma}(t)^2\big] = 2\theta \quad \text{for all } t \in (a, b).$$

Then γ is a θ-trajectory arc of $Q(z)\,dz^2$ if and only if it is a trajectory arc of $e^{-2i\theta} Q(z)\,dz^2$. Hence, a 0-trajectory arc is simply a trajectory arc and we call a $\pi/2$-trajectory arc an *orthogonal trajectory arc*. It is clear that these definitions are invariant under reparameterization of γ so we will often call the point set of γ a trajectory arc or θ-trajectory arc as appropriate. We call a maximal trajectory arc a *trajectory* and similarly, a maximal θ-trajectory arc is called a *θ-trajectory*. For example, if we consider the quadratic differential $1\,dz^2$ in $\mathbb{C}$, then the θ-trajectories are the straight lines with gradient $\exp(2\theta)$.

We now consider a special type of quadratic differential.

Definition. Let D be a domain with piecewise analytic boundary. An *algebraic quadratic differential* is a quadratic differential, $Q(z)\,dz^2$, on D satisfying the following two properties:

(1) $Q(z)$ extends continuously to a function mapping the prime ends of D into $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$;
(2) we can write

$$\partial D = \bigcup_{j=1}^{n} \overline{\Gamma}_j$$

such that each Γ_j is an open analytic arc with $\Gamma_k \cap \Gamma_j = \emptyset$ for $k \neq j$ and moreover, $\arg[Q(z)\,dz^2]$ is constant on each Γ_j, i.e. each Γ_j is a θ_j-trajectory arc for some $\theta_j \in [0, \pi)$.

Similar quadratic differentials are studied by Kühnau in [7] where he applies them to the study of certain Grötzsch-style extremal problems. We will prove the following theorem on algebraic quadratic differentials on $\mathbb{H}$.

Theorem 1.1. *Suppose that $Q(z)\,dz^2$ is an algebraic quadratic differential on $\mathbb{H}$. Then we have*

$$Q(z) = R \prod_{j=1}^{n} (z - \zeta_j)^{\lambda_j}$$

for some constant $R \in \mathbb{C} \setminus \{0\}$, $\zeta_j \in \mathbb{C}$, $\lambda_j \in \mathbb{R}$ for $j = 1, \dots, n$.

This theorem can be viewed as a generalization of the Schwarz–Christoffel formula to domains bounded by θ_k-trajectory arcs of a given quadratic differential. This result is very similar to Satz 1 in [7] and indeed, the idea that quadratic differentials can be used to generalize the Schwarz–Christoffel formula is not new.

Also, note that if

$$Q(z) = R \prod_{j=1}^{n} (z - \zeta_j)^{\lambda_j}$$

as in Theorem 1.1; then $Q(z)\,dz^2$ is an algebraic quadratic differential on $\mathbb{H}$ if and only if for each $j = 1, \dots, n$, either

- $\zeta_j \in \mathbb{R}$;
- or $\zeta_j \notin \mathbb{R}$ but $\lambda_j \in \mathbb{Z}$ and there exists k such that $\zeta_j = \overline{\zeta_k}$ with $\lambda_k = \lambda_j$.

Now consider an algebraic quadratic differential on a domain D. Note that property (2) of the definition of algebraic quadratic differentials implies that ∂D must be locally connected. Thus each prime end of D corresponds to a single point in ∂D (see [16, p. 27]). If, conversely, a point on ∂D corresponds to a single prime end, then we make no distinction between the two. For example, if $D = \mathbb{H} \setminus i(0, 1]$, then the point $1 \in \partial D$ corresponds to a single prime end but the point $0 \in \partial D$ corresponds to two distinct prime ends.

Let z be a prime end of D. Then we have two cases: either $z \in \Gamma_j$ for some $j = 1, \dots, n$; or there exist exactly two of the (Γ_j), that terminate at the prime end z. In the latter case, we will denote z by z_k and assume that Γ_k, a θ_k-trajectory arc, and Γ_{k-1}, a θ_{k-1}-trajectory arc, terminate at z_k where $\theta_{k-1}, \theta_k \in [0, \pi)$. Then we can define *the degree of z_k in D with respect to $Q(z)\,dz^2$*, $\deg_{D,Q}(z_k)$, as follows:

$$\deg_{D,Q}(z_k) = \begin{cases} 2[|\theta_k - \theta_{k-1}|/\pi + J_k - 1] & \text{if } \theta_k \neq \theta_{k-1}, \\ 2J_k & \text{if } \theta_k = \theta_{k-1}, \end{cases}$$

where J_k is the number of trajectories of $Q(z)\,dz^2$ inside D that end at the prime end z_k. If J_k is infinite, then the degree is not defined. For prime ends z such that $z \in \Gamma_j$ for some $j = 1, \dots, n$, we define

$$\deg_{D,Q}(z) = 0.$$

We will see that this indeed generalizes the concept of degree to points on the boundary. In particular, we will show that for $x \in \partial\mathbb{H}$, if $\deg_{\mathbb{H},Q}(x) \in \mathbb{Z}$, then Q can be extended to a meromorphic function in a neighbourhood of x with

$$\deg_{\mathbb{H},Q}(x) = \deg_Q(x).$$

For example, if $D = \{re^{i\theta}\colon\ \theta \in (0, \frac{5\pi}{4})\}$, then the quadratic differential dz^2 is an algebraic quadratic differential in D with $\Gamma_1 = (0, \infty)$ and $\Gamma_2 = e^{i\frac{5\pi}{4}}(0, \infty)$. Note that Γ_1 is a 0-trajectory arc and Γ_2 is a $\pi/4$-trajectory arc. Let $z_1 = 0$. Since $(-\infty, 0)$ is a trajectory arc of dz^2 that is contained in D and terminates at z_1 on the boundary of D, we have $J_1 = 1$. Hence, by definition,

$$\deg_{D,1}(0) = \frac{1}{2}.$$

We then have the following theorem on the Loewner driving function of a ϕ-trajectory arc of an algebraic quadratic differential $Q(z)\,dz^2$.

Theorem 1.2. *Suppose that $Q(z)\,dz^2$ is an algebraic quadratic differential on $\mathbb{H}$ such that there is a point $\xi_0 \in \mathbb{R}$ with $\deg_{\mathbb{H},Q}(\xi_0) = N \in \{0,1,\ldots\}$. Then*

$$Q(w) = R(w-\xi_0)^N \prod_{j=1}^{n} (w-a_j)^{\alpha_j},$$

where $a_j \in \mathbb{C}$, $\alpha_j \in \mathbb{R}$ for $j = 1,\ldots,n$ and R is a non-zero constant. Let $\gamma : [0,T) \mapsto \overline{\mathbb{H}}$ be a simple curve such that $\gamma(0) = \xi_0$, $\gamma(0,T) \subset \mathbb{H}$ and $\gamma(0,T)$ is a ϕ-trajectory arc of $Q(z)\,dz^2$ ($\phi \in [0,\pi)$) that is parameterized by half-plane capacity. Suppose that the functions f_t map $\mathbb{H}$ conformally onto $\mathbb{H} \setminus \gamma(0,t]$ and are hydrodynamically normalized. Then for $t \in (0,T)$

$$2\xi(t) = -\mu^- C^-(t) - \mu^+ C^+(t) - \left(\sum_{j=1}^{n} \alpha_j A_j(t)\right) + \Sigma_0, \tag{4}$$

and

$$\dot{\xi}(t) = -\frac{\mu^-}{C^-(t)-\xi(t)} - \frac{\mu^+}{C^+(t)-\xi(t)} - \sum_{j=1}^{n} \frac{\alpha_j}{A_j(t)-\xi(t)} \tag{5}$$

with initial condition $\xi(0) = \xi_0$. Here, the functions $A_j(t)$ are defined by

$$A_j(t) = f_t^{-1}(a_j) \quad \text{for } j = 1,\ldots,n;$$

$C^+(t) > C^-(t)$ are the two preimages of ξ_0 under f_t;

$$\mu^{\pm} = \deg_{\mathbb{H}\setminus\gamma(0,t],Q}\big(f_t\big(C^{\pm}(t)\big)\big);$$

and

$$\Sigma_0 = N\xi_0 + \sum_{k=1}^{n} \alpha_k a_k.$$

The example in the introduction is a special case of this theorem with $Q \equiv 1$. Note that the idea of combining quadratic differentials with the Loewner differential equation is not new; it was first considered by Schiffer in [18]. Also see [17].

Theorem 1.2 can then be used to find the driving function in the case when the slit γ consists of consecutive θ_k-trajectory arcs of given quadratic differentials. We will explain how to do this in detail later. One difficulty with using Theorem 1.2 is that the parameterization is inherently given in terms of half-plane capacity. This makes it difficult to calculate the driving function ξ if we do not know anything about the half-plane capacity of the trajectory arc (which, in general, is the case). The next theorem will allow us to compare the parameterization with the length of the slit.

Proposition 1.3. *Let $Q(z)\,dz^2$ be a quadratic differential on $\mathbb{H}$ and let $\gamma : [0,T) \mapsto \overline{\mathbb{H}}$ be a simple curve such that $\gamma(0) \in \mathbb{R}$, $\gamma(0,T) \subset \mathbb{H}$ and $\gamma(0,T)$ is a ϕ-trajectory arc of $Q(z)\,dz^2$ ($\phi \in [0,\pi)$) that is parameterized by half-plane capacity. Suppose that the functions f_t map $\mathbb{H}$ conformally onto $\mathbb{H} \setminus \gamma(0,t]$ and are hydrodynamically normalized. Let*

$$\Phi_t(z) = \frac{Q(f_t(z))\, f_t'(z)^2}{(z-\xi(t))^2}.$$

Then for $t \in (0,T)$, Φ_t has removable singularity at $\xi(t)$ and γ satisfies

$$\dot{\gamma}(t) = -2\sqrt{\frac{\Phi_t(\xi(t))}{Q(\gamma(t))}}. \tag{6}$$

The rest of this paper is organized as follows: In Section 2, we will state some basic results from the theory of quadratic differentials and use them to prove Theorem 1.1. Then we will use Theorem 1.1 to prove Theorem 1.2 and Proposition 1.3 in Section 3. In Section 4 we will discuss how to obtain the driving function numerically using Theorem 1.2 and Proposition 1.3. Finally in Section 5, we will discuss extensions of Theorem 1.2 to the case when we have multiple slits as well as to the radial Loewner differential equation.

2. Algebraic quadratic differentials and generalized Schwarz–Christoffel mapping

The aim of this section is to prove Theorem 1.1. We will first look at some of the basic results in the theory of quadratic differentials that we will need.

Suppose that f is a conformal map from a domain D_2 onto a domain D_1 and suppose that $Q_1(w)\,dw^2$ is a quadratic differential on D_1. If we define

$$Q_2(z) \equiv Q_1\big(f(z)\big) f'(z)^2, \tag{7}$$

then $Q_2(z)\,dz^2$ is a quadratic differential on D_2. We will call this the *transformation law*. It is clear that θ-trajectory arcs are preserved by the transformation law, i.e.

$$\gamma \text{ is a } \theta\text{-trajectory arc of } Q_2(z)\,dz^2 \quad \Leftrightarrow \quad f \circ \gamma \text{ is a } \theta\text{-trajectory arc of } Q_1(w)\,dw^2,$$

and also, for $z \in D_2$

$$\deg_{Q_2}(z) = \deg_{Q_1}\big(f(z)\big).$$

Hence trajectories and $\deg_Q$ are conformally invariant in the above sense.

In fact, the local behaviour of trajectories near a point ω is completely determined by $\deg_Q(\omega)$ (see [20, Section 7]). In particular, we note that if $\deg_Q(\omega) = n$ with $n \geqslant -1$, then there are exactly $n+2$ trajectory arcs of $Q(z)\,dz^2$ that end at ω and form equal angles with each other. The same holds for θ-trajectory arcs (since θ-trajectory arcs are trajectory arcs of $e^{2i\theta} Q(z)\,dz^2$).

This is the reason that we define $\deg_{D,Q}(x)$, the degree of a point on the boundary, in terms of the trajectories ending at the prime end x. If $D = \mathbb{H}$ and Q extends to a meromorphic function on a neighbourhood of some $x \in \mathbb{R} \cup \{\infty\}$ with $\deg_{\mathbb{H},Q}(x)$ finite, then by studying the trajectory structure at x, we can see that

$$\deg_{\mathbb{H},Q}(x) = \deg_Q(x).$$

Note that conformal invariance of trajectories and prime ends implies that $\deg_{D,Q}$ is also conformally invariant.

We will also need the following fact: If $Q(z)\,dz^2$ is a quadratic differential on $\mathbb{H}$ such that $Q(z)$ extends analytically to some interval (a,b) in $\mathbb{R}$ and $Q(z)$ is real on (a,b) (in other words (a,b) is a trajectory or orthogonal trajectory arc of $Q(z)\,dz^2$), then we can extend $Q(z)\,dz^2$ across (a,b) to a quadratic differential on the lower half-plane by reflection. See [20, Section 4]. In particular, if $Q(z)$ extends analytically to the intervals (a,x) and (x,b) (with $a < x < b$) and is real on them, then $Q(z)\,dz^2$ extends to a quadratic differential on $N \setminus \{x\}$ where N is some neighbourhood of x. We use this fact to prove the following lemma.

Lemma 2.1. *Suppose that $Q(z)\,dz^2$ is an algebraic quadratic differential on $\mathbb{H}$. Then for any $z \in \partial\mathbb{H}$, $\deg_{\mathbb{H},Q}(z) \in \mathbb{Z}$ implies that $Q(z)\,dz^2$ extends to a quadratic differential on $N \setminus \{z\}$ where N is some neighbourhood of z.*

Proof. Firstly, if $z \in \Gamma_j$ for some $j = 1, \ldots, n$, then by definition $\deg_{\mathbb{H},Q}(z) = 0$ and $Q(z)\,dz^2$ can be extended to a neighbourhood of z by reflection. Otherwise we write $z = z_k$ and suppose that a θ_{k-1}-trajectory arc, Γ_{k-1}, and a θ_k-trajectory arc, Γ_k, end at z_k. Then, by definition, $\deg_{\mathbb{H},Q}(z_k) \in \mathbb{Z}$ implies that $\theta_k - \theta_{k-1}$ is a multiple of $\pi/2$. Thus Γ_{k-1} and Γ_k are trajectory arcs or orthogonal trajectory arcs of $e^{-2i\theta_{k-1}} Q(z)\,dz^2$. Therefore, $e^{-2i\theta_{k-1}} Q(z)$ is real on $I \setminus \{z_k\}$ for some interval I in $\mathbb{R}$ containing z_k. Hence by reflection, $e^{-2i\theta_{k-1}} Q(z)\,dz^2$ extends to $N_k \setminus \{z_k\}$ where N_k is a neighbourhood of z_k. Hence, $Q(z)\,dz^2$ also extends to a neighbourhood of $N_k \setminus \{z_k\}$. □

We can now prove Theorem 1.1 but first, we explain briefly why we can view Theorem 1.1 as a generalized form of the Schwarz–Christoffel formula. Recall that the Schwarz–Christoffel formula allows one to compute the conformal map of the upper half-plane onto a domain bounded by a polygon. See [13, p. 189] for more details. If we have a conformal map f of $\mathbb{H}$ to some domain D such that the sides of D consist of θ-trajectory arcs of the quadratic differential $Q(w)\,dw^2$. Then $Q(w)\,dw^2$ is an algebraic quadratic differential on D and by the conformal invariance of trajectories, $Q(f(z)) f'(z)\,dz^2$ is an algebraic quadratic differential on $\mathbb{H}$. Theorem 1.1 then implies that

$$Q\big(f(z)\big) f'(z)^2 = R \prod_{j=1}^{n} (z - \zeta_j)^{\lambda_j}.$$

This is precisely the Schwarz–Christoffel formula when $Q(z) \equiv 1$.

Proof of Theorem 1.1. Since $Q(z)\,dz^2$ is an algebraic quadratic differential, we can find

$$z_1 < \cdots < z_m,$$

with

$$\Gamma_k = \begin{cases} (z_{k-1}, z_k) & \text{for } k = 1, \ldots, m, \\ (z_m, \infty) & \text{for } k = m+1, \\ (-\infty, z_0) & \text{for } k = 0, \end{cases}$$

such that each Γ_k is a θ_k-trajectory arc of $Q(z)\,dz^2$ for some $\theta_k \in [0,\pi)$. Let

$$\mathcal{T} = \{\Gamma_0, \Gamma_1, \ldots, \Gamma_{m+1}\}.$$

Then take any $\Gamma \in \mathcal{T}$. Since Γ is a θ-trajectory for some θ, Γ is a trajectory arc of $e^{-2i\theta}Q(z)\,dz^2$. Hence by reflection, we can reflect the quadratic differential $e^{-2i\theta}Q(z)\,dz^2$ across Γ to get a quadratic differential on $\mathbb{H}^- = \{z\colon \operatorname{Im}(z) < 0\}$ which we call $\widetilde{Q}(z)\,dz^2$. Similarly, by rotating $\widetilde{Q}(z)\,dz^2$, we can reflect it across another $\Upsilon \in \mathcal{T}$ to get another quadratic differential $Q^*(z)\,dz^2$ on $\mathbb{H}$. Since Q^* is obtained from Q by rotating twice, we have

$$Q^*(z) = e^{i\sigma}Q(z)$$

for some $\sigma \in [0, 2\pi)$. This shows that

$$\Psi(z) = \frac{Q'(z)}{Q(z)} = \frac{(Q^*)'(z)}{Q^*(z)}$$

can be extended to a meromorphic function in $\mathbb{C} \setminus \{z_1, \ldots, z_m\}$.

Note that the above proof also shows that for each $k = 1, \ldots, m$, there exists a $\beta_k \in \mathbb{R}$ such that $Q(z)^{\beta_k}$ extends analytically to $N_k \setminus \{z_k\}$ where N_k is a neighbourhood of z_k. Then part (1) of the definition of algebraic quadratic differentials implies that the singularity at z_k of $Q(z)^{\beta_k}$ cannot be essential (otherwise we would get a contradiction with the great Picard's theorem). Hence all the critical points (i.e. the zeroes and poles) of $\Psi(z)$ are simple poles.

Thus we can write

$$\Psi(z) = h(z) + \sum_{j=1}^{n} \frac{\lambda_j}{z - \zeta_j},$$

where $\zeta_j \in \mathbb{C}$, and $\lambda_j \in \mathbb{C}$ (for $j = 1, \ldots, n$), and $h(z)$ is an entire function in $\mathbb{C}$. In fact, we must also have $h(z)$ is analytic at $z = \infty$ (otherwise we would get a contradiction with the great Picard's theorem as above). Hence, by Liouville's theorem, $h(z) \equiv C$ and, moreover, for the same reason, we must have $C = 0$. Thus

$$\Psi(z) = \sum_{j=1}^{n} \frac{\lambda_j}{z - \zeta_j}.$$

Hence on the upper half-plane $\mathbb{H}$, we can write

$$Q(z) = R\prod_{j=1}^{n}(z - \zeta_j)^{\lambda_j}.$$

Finally, part (2) of the definition of algebraic quadratic differentials implies that we must have $\lambda_j \in \mathbb{R}$ for $j = 1, \ldots, n$. $\square$

If $\zeta_j \in \mathbb{C} \setminus \mathbb{R}$, then by definition, we must have

$$\lambda_j = \deg_Q(\zeta_j).$$

Moreover, if $\zeta_j \in \mathbb{R}$ and $\deg_{\mathbb{H},Q}(\zeta_j) < \infty$ we also have

$$\lambda_j = \deg_{\mathbb{H},Q}(\zeta_j).$$

We will not prove this fact here but in the following corollary we will consider a special case. The general proof follows readily from it. We will prove the following corollary which is an application of Theorem 1.1 to domains slit by a ϕ-trajectory arc.

Corollary 2.2. *Suppose that $Q(z)\,dz^2$ is an algebraic quadratic differential on $\mathbb{H}$ such that there is a point $\xi_0 \in \mathbb{R}$ with $\deg_{\mathbb{H},Q}(\xi_0) = N \in \{0, 1, \ldots\}$. Then we can write*

$$Q(w) = R(w - \xi_0)^N \prod_{j=1}^{n}(w - a_j)^{\alpha_j}, \tag{8}$$

where $a_j \in \mathbb{C}$, $\alpha_j \in \mathbb{R}$ for $j = 1, \ldots, n$, and also R is some non-zero constant. Let $\gamma : [0, T] \mapsto \overline{\mathbb{H}}$ be a simple curve such that $\gamma(0) = \xi_0$ and $\gamma(0, T)$ is a ϕ-trajectory arc of $Q(w)\,dw^2$ in $\mathbb{H}$ ($\phi \in [0, \pi)$) and $\zeta = \gamma(T) \in \mathbb{H}$ is an ordinary point of $Q(w)\,dw^2$ (i.e. $\deg_Q(\zeta) = 0$). Suppose that f maps $\mathbb{H}$ conformally onto $\mathbb{H} \setminus \gamma(0, T]$. Then f satisfies

$$Q\big(f(z)\big) f'(z)^2 = R'(z - \xi)^2 (z - c^-)^{\mu^-} (z - c^+)^{\mu^+} \prod_{j=1}^{n}(z - A_j)^{\alpha_j}, \tag{9}$$

where R' is some constant; c^-, c^+ are the two preimages of ξ_0 under f satisfying $c^- < c^+$; A_j is the preimage of a_j under f; and ξ is the preimage of ζ; and

$$\mu^\pm = \deg_{\mathbb{H}\setminus\gamma(0,T],Q}\big(f\big(c^\pm\big)\big).$$

Proof. Firstly, Theorem 1.1 and Lemma 2.1 imply that $Q(w)$ can be written as (8). Then by Carathéodory's theorem, f extends continuously to $\partial\mathbb{H}$ and by the Schwarz reflection principle, f extends to a conformal map on a neighbourhood of $\mathbb{R}\setminus\{\xi, c^-, c^+\}$. Since θ-trajectory arcs are conformally invariant, this implies that $\widehat{Q}(z)\,dz^2 = Q(f(z))f'(z)^2\,dz^2$ is an algebraic quadratic differential on $\mathbb{H}$. Thus, using Theorem 1.1 and the fact that f extends to a conformal map on a neighbourhood of $\mathbb{R}\setminus\{\xi, c^-, c^+\}$, we can write

$$\widehat{Q}(z) = R'(z-\xi)^M(z-c^-)^{\mu^-}(z-c^+)^{\mu^+}\prod_{j=1}^n (z-A_j)^{\alpha_j}.$$

Then from the local behaviour of trajectory arcs around a point as discussed above, we find that there are exactly two ϕ-trajectory arcs of $Q(z)\,dz^2$ ending at $\zeta = \gamma(T)$ of which $\gamma(0,T)$ is one of them. So by the conformal invariance of trajectories, there is exactly one ϕ-trajectory arcs of $\widehat{Q}(z)\,dz^2$ ending at ξ that is contained in $\mathbb{H}$ and also, the intervals $(\xi-\delta,\xi)$ and $(\xi,\xi+\delta)$ are ϕ-trajectory arcs of $\widehat{Q}(z)\,dz^2$ for sufficiently small $\delta > 0$. Hence, by definition, $\deg_{\widehat{Q},\mathbb{H}}(\xi) = 2$. Using Lemma 2.1, this implies that we must also have $\deg_Q(\xi) = 2$, i.e. $M = 2$. Thus it only remains to determine μ^- and μ^+.

Note that since ξ_0 has degree N with respect to $Q(z)\,dz^2$, we can determine (from the fact that there are $N+2$ ϕ-trajectories in $\mathbb{C}$ ending at ξ_0 forming equal angles with each other) that the interior angle between $\gamma(0,T)$ and $f((c^-,\xi))$ at ξ_0 is

$$\pi\psi^- = \pi\frac{\deg_{H,Q}(f(c^-))+2}{N+2},$$

where $H = \mathbb{H}\setminus\gamma(0,T]$. Similarly, the interior angle between $\gamma(0,T)$ and $f((\xi,c^+))$ at ξ_0 is

$$\pi\psi^+ = \pi\frac{\deg_{H,Q}(f(c^+))+2}{N+2}.$$

Hence, by the Schwarz reflection principle, the function

$$F(z) = \big(f(z)-\xi_0\big)^{1/\psi^-}$$

extends to a conformal mapping on a neighbourhood of c^-. Thus in a neighbourhood of $z = c^-$, we can write

$$f(z) = \xi_0 + (z-c^-)^{\psi^-}h(z)^{\psi^-}, \tag{10}$$

where h is analytic in a neighbourhood of $z = c^-$ with $h(c^-) \neq 0$. Now

$$\frac{Q_f'(z)}{Q_f(z)} = \frac{Q'(f(z))f'(z)^2}{Q(f(z))} + 2\frac{f''(z)}{f'(z)}.$$

The residue at $z = c^-$ of the left-hand side of the equation is μ^-, and we can use (10) to determine the residue at $z = c^-$ of the right-hand side. Thus we get

$$\mu^- = \deg_{H,Q}\big(f(c^-)\big).$$

We apply the same method to c^+ to get μ^+. $\square$

3. Domains slit by θ-trajectory arcs in $\mathbb{H}$

Let $Q(w)\,dw^2$ be an algebraic quadratic differential on $\mathbb{H}$ with $\deg_{\mathbb{H},Q}(\xi_0) = N \in \{0,1,\dots\}$ for some $\xi_0 \in \mathbb{R}$. Then by Theorem 1.1 and Lemma 2.1,

$$Q(w) = R(w-\xi_0)^N\prod_{j=1}^n (w-a_j)^{\alpha_j},$$

where $\alpha_j \in \mathbb{R}$, $a_j \in \mathbb{C}$ for $j = 1,\dots,n$ and R is a non-zero constant. Now suppose that $\gamma : [0,T) \mapsto \overline{\mathbb{H}}$ is a simple curve such that $\gamma(0) = \xi_0$ and $\gamma(0,T)$ is a ϕ-trajectory of $Q(z)\,dz^2$ in $\mathbb{H}$ ($\phi = [0,\pi)$) that is parameterized by half-plane capacity. As mentioned in the introduction, there exist conformal maps $f_t : \mathbb{H} \mapsto H_t = \mathbb{H}\setminus\gamma(0,t]$ satisfying the hydrodynamic normalization. Then by restricting $Q(w)\,dw^2$ to a quadratic differential on H_t we can induce via f_t and (7), a quadratic differential on $\mathbb{H}$:

$$Q_t(z)\,dz^2 = Q\big(f_t(z)\big)f_t'(z)^2\,dz^2. \tag{11}$$

We now use Corollary 2.2 and (11) to prove Theorem 1.2.

Proof of Theorem 1.2. Note that by the Schwarz reflection principle, each f_t can be extended to a conformal map on $\widehat{\mathbb{C}} \setminus [C^-(t), C^+(t)]$. Since $f_t(z)$ satisfies the hydrodynamic normalization, this implies that

$$f_t'(z)^2 = 1 + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty.$$

So by (11),

$$\frac{Q_t(z)}{Q(f_t(z))} = 1 + O\left(\frac{1}{z^2}\right) \quad \text{as } z \to \infty.$$

If we let $\zeta = 1/z$, then we get

$$\frac{Q_t(1/\zeta)}{Q(f_t(1/\zeta))} = 1 + O\big(\zeta^2\big) \quad \text{as } \zeta \to 0. \tag{12}$$

Since f_t is analytic in a neighbourhood of infinity, (12) is a Taylor series expansion and hence we can look at the Taylor series coefficients, in particular:

$$\int\limits_{C(0,\epsilon)} \frac{f'(1/\zeta)^2}{\zeta^2}\, d\zeta = \int\limits_{C(0,\epsilon)} \frac{Q_t(1/\zeta)}{\zeta^2 Q(f_t(1/\zeta))}\, d\zeta = 0$$

for small enough $\epsilon > 0$ where $C(0,\epsilon)$ is the anticlockwise contour about the circle with centre at zero and radius $\epsilon > 0$. Then by Theorem 1.1 and Corollary 2.2, we can write

$$Q(w) = R(w - \xi_0)^N \prod_{j=1}^{n} (w - a_j)^{\alpha_j},$$

and

$$Q_t(z) = R(t)(z - \xi)^2 \big(z - C^-(t)\big)^{\mu^-} \big(z - C^+(t)\big)^{\mu^+} \prod_{j=1}^{n} \big(z - A_j(t)\big)^{\alpha_j}.$$

Note that (12) implies that $R(t) = R$. Then by the residue theorem (since $f_t(1/\zeta) = 1/\zeta + \cdots$ as $\zeta \to 0$), we have

$$R\left[2\xi(t) + \mu^- C^-(t) + \mu^+ C^+(t) + \left(\sum_{k=1}^{n} \alpha_k A_k(t)\right)\right] - R\left[N\xi_0 + \left(\sum_{k=1}^{n} \alpha_k a_k\right) - \left(\sum_{l=1}^{m} \beta_l b_l\right)\right] = 0.$$

This implies (4). To get (5), note that f_t satisfies the chordal Loewner differential equation (1) and hence if we let $G_t = f_t^{-1} \circ f_s$ for some $s \in (0,T)$ and $t > s$, then the chain rule implies that G_t satisfies the differential equation

$$\frac{\partial G_t}{\partial t}(z) = \frac{2}{G_t(z) - \xi(t)}.$$

Then for some s sufficiently close to t, we can write each $A_j(t) = G_t(w_j)$ for some $w_j \in \mathbb{C}$ for $j = 1, \ldots, n$. Thus

$$\dot{A}_j(t) = \frac{2}{A_j(t) - \xi(t)}.$$

Similarly, we get

$$\dot{C}^\pm(t) = \frac{2}{C^\pm(t) - \xi(t)}.$$

Hence we get (5) from differentiating (4). $\square$

We remark that the proof of Theorem 1.2 is very similar to some of the considerations in [2].

Extension. We can extend Theorem 1.2 to the case when γ is made up of different θ_k-trajectory arcs of some algebraic quadratic differential $Q(z)\,dz^2$. Let $\gamma : [0,T) \mapsto \overline{\mathbb{H}}$ be a curve with $\gamma(0) \in \mathbb{R}$ and $\gamma(0,T) \subset \mathbb{H}$ such that there is a partition

$$0 = t_0 < t_1 < \cdots < t_r = T$$

such that $\gamma(t_{k-1}, t_k)$ is a θ_k-trajectory arc of $Q(z)\,dz^2$ and $\gamma(t_k)$ is an ordinary point of $Q(z)\,dz^2$ for $k = 1, \ldots, r$. Then we can find the driving function $\xi(t)$ of γ as follows: first apply Theorem 1.2 to the θ_1-trajectory arc $\gamma(0, t_1)$ to get the driving function $\xi_1(t)$, and then apply Theorem 1.2 inductively to each $f_{t_k}^{-1}(\gamma(t_k, t_{k+1}))$ (which is a θ_{k+1}-trajectory arc of the quadratic differential $Q_{t_k}(z)\,dz^2 = Q(f_{t_k}(z)) f'_{t_k}(z)^2\,dz^2$) to get $\xi_k(t)$. Then

$$\xi(t) = \xi_k(t - t_{k-1}) \quad \text{for } t \in [t_{k-1}, t_k).$$

We also have the following corollary to Theorem 1.2.

Corollary 3.1. *Suppose that $Q(w)\,dw^2$ and γ are as defined in Theorem 1.2. Then the driving function ξ and A_j, C^-, C^+ as defined in Theorem 1.2 are in $C^\infty(0, T)$. Moreover, we can write any derivative of ξ, C^-, C^+, A_j explicitly in terms of ξ, C^-, C^+, A_j and the exponents μ^-, μ^+, α_j.*

Proof. Recall that in the proof of Theorem 1.2 we had the formulae

$$\dot{A}_j(t) = \frac{2}{A_j(t) - \xi(t)}, \qquad \dot{C}^\pm(t) = \frac{2}{C^\pm(t) - \xi(t)}.$$

This implies that each term in (5) is differentiable so we can write the second derivative of ξ in terms of $\xi(t)$, $A_j(t)$, $C^\pm(t)$ and the exponents μ^-, μ^+, α_j. This in turn implies that we can write the third derivative of ξ in terms of $\xi(t)$, $A_j(t)$, $C^\pm(t)$ and the exponents μ^-, μ^+, α_j. Continuing inductively, we have showed that every derivative of ξ exists and can be expressed in terms of $\xi(t)$, $A_j(t)$, $C^\pm(t)$ and the exponents μ^-, μ^+, α_j. Note that each derivative of ξ is finite for $t \in (0, T)$ since

$$\big|A_j(t) - \xi(t)\big|, \big|C^\pm(t) - \xi(t)\big| > 0.$$

Then the fact that ξ is smooth implies that $A_j(t)$, $C^\pm(t)$ are also smooth. $\square$

We can now also prove Proposition 1.3.

Proof of Proposition 1.3. Let $Q_t(z) = Q(f_t(z)) f'_t(z)^2$. Then for each $t \in (0, T)$, there exist real numbers $a_t < b_t$ with $\xi(t) \in (a_t, b_t)$ such that the intervals $(a_t, \xi(t))$ and $(\xi(t), b_t)$ are ϕ-trajectory arcs of $Q_t(z)\,dz^2$. Thus by reflection, we can extend the quadratic differential $Q_t(z)\,dz^2$ to a full neighbourhood of $\xi(t)$. As in the proof of Corollary 2.2, we can see that $\deg_Q(\xi(t)) = 2$ for all $t \in (0, T)$. Hence $\xi(t)$ is a removable singularity of $\Phi_t(z)$ and $\Phi_t(\xi(t)) \neq 0, \infty$. Also, the right-hand side of (6) always exists since, by definition, γ avoids poles and zeroes of $Q(w)\,dw^2$.

Then from the definition of θ-trajectory arcs, $\dot{\gamma}$ always exists and is never 0 and recall that $f_t(\xi(t)) = \gamma(t)$. This implies that

$$\dot{\gamma}(t) = \dot{f}_t\big(\xi(t)\big) + f'_t\big(\xi(t)\big)\dot{\xi}(t).$$

Combining the Loewner differential equation (1) with the definition of $Q_t(z)$, we have

$$\dot{f}_t(z) = -\frac{2}{z - \xi(t)}\sqrt{\frac{Q_t(z)}{Q(f_t(z))}} = -2\sqrt{\frac{\Phi_t(z)}{Q(f_t(z))}} \quad \Rightarrow \quad \dot{f}_t\big(\xi(t)\big) = -2\sqrt{\frac{\Phi_t(\xi(t))}{Q(\gamma(t))}}.$$

Thus we have

$$\dot{\gamma}(t) = -2\sqrt{\frac{\Phi_t(\xi(t))}{Q(\gamma(t))}} + \sqrt{\frac{Q_t(\xi(t))}{Q(\gamma(t))}}\dot{\xi}(t) = -2\sqrt{\frac{\Phi_t(\xi(t))}{Q(\gamma(t))}},$$

since $Q_t(\xi(t)) = 0$ for all $t \in (0, T)$. $\square$

4. Applying Theorem 1.2

In practice, understanding $\xi(t)$ via (4) is often not possible: it is difficult to calculate the positions of the zeroes and poles of Q_t because the information we have on them is all relative to $\xi(t)$ (which we are trying to find). On the other hand, (5) is more useful in applications. In this section, we will demonstrate how we can use (5) to calculate numerically the driving function of a given slit that consists of θ_k-trajectory arcs of a given quadratic differential. The method is basically a modified version of Euler's method.

Firstly, for any smooth function h on $(0, T)$, Taylor's theorem implies that for all $M = 1, 2, \ldots,$

$$\left|h\left(t + \frac{1}{K}\right) - \left(h(t) + \sum_{m=1}^{M-1} \frac{1}{m!K^m}\frac{d^m h}{dt^m}(t)\right)\right| \leqslant \frac{1}{M!K^M} \sup_{s \in (t, t+\frac{1}{K})} \left|\frac{d^M h}{dt^M}(s)\right| \tag{13}$$

for $t, t + 1/K \in (0, T)$. We will apply (13) to the functions ξ, A_k and $C^\pm$ (as defined in Theorem 1.2) noting that, by Corollary 3.1, they are smooth and all of their derivatives can be expressed in terms of $\xi(t)$, $A_k(t)$ and $C^\pm(t)$. Thus if we know $\xi(s)$, $A_k(s)$ and $C^{\pm}(s)$ we can use (13) to obtain an approximate formula for $\xi(s+K^{-1})$, $A_k(s+K^{-1})$ and $C^{\pm}(s+K^{-1})$ (choosing K^{-1} to be small and/or M to be large so that the right-hand side of (13) is small); then we can apply (13) to $\xi(s+K^{-1})$, $A_k(s+K^{-1})$ and $C^{\pm}(s+K^{-1})$ to find $\xi(s+2K^{-1})$, $A_k(s+2K^{-1})$ and $C^{\pm}(s+2K^{-1})$. Continuing like this, we obtain an approximation of ξ at the points $\{s+nK^{-1}\}$.

So clearly what we need to do now is find the starting values $\xi(s)$, $A_k(s)$ and $C^{\pm}(s)$ so we can apply the above method. But because ξ is not differentiable at 0, we cannot use (13) with $t=0$. The way around this is to note that if $\deg_{\mathbb{H},Q}(\xi_0)=N\in\{0,1,2,\ldots\}$ then, since we know $\deg_{\mathbb{H},Q_t}(C^+(t))$, we can calculate the angle that the trajectory makes with the line $[\xi_0,\infty)$ (as in the proof of Corollary 2.2). Then we find that the angle is $\pi\psi$ where

$$\psi=\frac{2\deg_{\mathbb{H},Q_t}(C^+(t))+2}{N+2}.$$

So if we choose s small enough, we have

$$f_s\approx F_s^{\psi,\xi_0},$$

where F_s^{ψ,ξ_0} is the conformal map that maps $\mathbb{H}$ conformally onto H_s^{ψ} that is hydrodynamically normalized where H_s^{ψ} is the upper half-plane slit by the straight line arc starting at ξ_0 making an angle $\pi\psi$ with $[\xi_0,\infty)$ and with half-plane capacity $2s$. Then we also have

$$A_k(s)\approx\left(F_s^{\psi,\xi_0}\right)^{-1}(a_k),$$

and also, $C^-(s)$, $C^+(s)$ are approximately the two preimages of ξ_0 under F_s^{ψ,ξ_0}. Then we can use (4) to calculate $\xi(s)$ approximately. We can then insert this into the formula (13) as described above.

Note that $F_t^{\psi,x}$ can be found using the fact that

$$F_{\lambda t}^{p,0}(z)=\left(z-(1-2p)\sqrt{t}-p\sqrt{t}\right)^p\left(z-(1-2p)\sqrt{t}+(1-p)\sqrt{t}\right)^{1-p} \tag{14}$$

for some $\lambda>0$. Then we reparameterize to remove the λ and then translate the point 0 to ξ_0. Unfortunately, inverting this function cannot be done explicitly but it can be done numerically very efficiently using Newton's method. Alternatively, by selecting a small s, we can assume that

$$A_k(s)\approx a_k$$

for all k. Then we note that the 2 preimages of ξ_0 under $F_s^{\psi,x}$ can be determined explicitly (see [12]). This obviates the need to numerically invert $F_s^{\psi,x}$.

Another difficulty is that, in general, given a slit, we cannot parameterize it by half-plane capacity so it would be difficult, for example, to know at which t one should stop. Most formulae for calculating half-plane capacity of some compact set K rely on knowing the conformal map f_K of $\mathbb{H}$ onto $\mathbb{H}\setminus K$ (normalized hydrodynamically). One possibility would be to use the probabilistic definitions of half-plane capacity given in [8]. We will use the fact that Proposition 1.3 and Corollary 3.1 imply that we can give all derivatives of $\gamma(t)$ in terms of $\xi(t)$, $A_k(t)$, $C^-(t)$, $C^+(t)$ and the exponents μ^-, μ^+, α_k. Thus if we know these, we can also use (13) to approximate γ. This in turn allows us to calculate the length of the slit γ. Hence if we know beforehand the length of our slit, we can calculate at what value of t we stop.

We now have everything we need in order to use (13) to calculate the driving function numerically of any slit that is made up of θ_k-trajectory arcs of a quadratic differential $Q(w)\,dw^2$. We will demonstrate how this is done in the following example.

Example. Suppose that $\gamma:(0,T)\to\mathbb{H}$ is a piecewise linear arc parameterized by half-plane capacity that satisfies

- $\gamma(0)=0$;
- from $t=0$ to $t=t_1$, γ is the straight line arc from 0 to i (call this Γ_1);
- from $t=t_1$ to $t=t_2$, γ is the straight line arc from i to $2+i$ (call this Γ_2);
- from $t=t_2$ to $t=t_3=T$, γ is the straight line arc from $2+i$ to $2+2i$ (call this Γ_3).

First note that γ is made up of alternating $(\pi/2)$- and 0-trajectory arcs of the quadratic differential $1\,dw^2$ in $\mathbb{H}$ and hence we can use Theorem 1.2 (or more specifically the extension of Theorem 1.2 detailed in Section 3) to calculate $\dot{\xi}$. As mentioned previously, there is no easy way to know beforehand what $t_1,\ldots,t_3$ are. For simplicity, we will only use $M=1$ in (13), i.e.

$$f\left(t+\frac{1}{K}\right)\approx f(t)+\frac{\dot{f}(t)}{K},$$

and fix a large K. Obviously Γ_1 forms a right angle with real line; so we can use (14) to determine the function

$$f_{t_1}=F_{t_1}^{1/2,0}(z)=\sqrt{z^2-4t_1}.$$

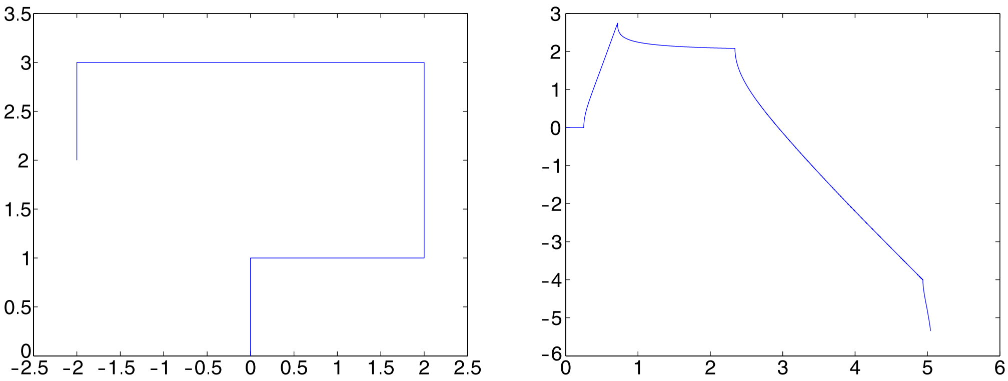

Fig. 3. The example path in the upper half-plane (left) and a plot of its driving function on the y-axis against time on the x-axis (right).

It is easy to see that in this case, $t_1 = 1/4$ and ξ is constantly 0 for $t \in (0, t_1]$. This induces the quadratic differential using (11)

$$Q_{t_1}(z)\,dz^2 = \frac{z^2\,dz^2}{(z+1)(z-1)}.$$

Hence, we let $A_1(t_1) = -1$, $A_2(t_1) = 1$. Also $f_{t_1}^{-1}(\Gamma_2)$ is a 0-trajectory arc of $Q_1(z)\,dz^2$ starting from $\xi(t_1) = 0$ on $\mathbb{R}$ (by the conformal invariance of trajectories). Now note that $f_{t_1}^{-1}(\Gamma_2)$ makes an angle of $\pi/4$ with $(0, \infty)$ and hence

$$f_{t_1+K^{-1}} \approx F_{K^{-1}}^{1/4,0}(z)$$

since K is large. We can then use Newton's method to find the preimages under the above approximation of $f_{t_1+K^{-1}}$ of the points $A_1(t_1)$, $A_2(t_1)$ and the 2 preimages of zero to get the points $A_1(t_1 + K^{-1})$, $A_2(t_1 + K^{-1})$, $C^-(t_1 + K^{-1})$, $C^+(t_1 + K^{-1})$ and hence, using (4), we can find $\xi(t_1 + K^{-1})$. Then inserting this into (13), as detailed above we can also find $\xi(t_1 + nK^{-1})$ and $A_1(t_1 + nK^{-1})$, $A_2(t_1 + nK^{-1})$, $C^-(t_1 + nK^{-1})$, $C^+(t_1 + nK^{-1})$. Also, using Proposition 1.3, we can find $|\dot{\gamma}(t_1 + nK^{-1})|$. If we let

$$n_2(K) = \inf\left\{n\colon \sum_{j=1}^{n} \frac{1}{K}\left|\dot{\gamma}\left(t_1 + nK^{-1}\right)\right| > (\text{length of } \Gamma_2) = 2\right\},$$

then $t_1 + n_2(K)K^{-1} \approx t_2$ for K large. So we just assume that $t_2 = t_1 + n_2(K)K^{-1}$. Let $A_3(t_2) = C^-(t_2)$ and $A_4(t_2) = C^+(t_2)$. Hence by (11),

$$Q_{t_2}(z)\,dz^2 = \frac{(z - \xi(t_2))^2 (z - A_3(t_2))\,dz^2}{(z - A_1(t_2))(z - A_2(t_2))(z - A_4(t_2))}.$$

Then, by the conformal invariance of trajectories, $f_{t_2}^{-1}(\Gamma_3)$ is a $\pi/2$-trajectory of $Q_{t_2}(z)\,dz^2$ and also, $f_{t_2}^{-1}(\Gamma_3)$, makes an angle $3\pi/4$ with $(\xi(t_2), \infty)$ and so

$$f_{t_2+K^{-1}} \approx F_{K^{-1}}^{3/4,\xi(t_2)}(z).$$

Then, as before, we can use Newton's method to find the preimages under the above approximation of $f_{t_2+K^{-1}}$ of the points $A_1(t_2), \ldots, A_4(t_2)$ and the 2 preimages of $\xi(t_2)$ to get the points $A_1(t_2 + K^{-1}), \ldots, A_4(t_2 + K^{-1})$, $C^-(t_2 + K^{-1})$, $C^+(t_2 + K^{-1})$ and then use (4) to get $\xi(t_2 + K^{-1})$. We insert these into (13) iteratively to get $\xi(t_2 + nK^{-1})$ and $A_1(t_2 + nK^{-1}), \ldots, A_4(t_2 + nK^{-1})$, $C^-(t_2 + nK^{-1})$, $C^+(t_2 + nK^{-1})$ until $t_2 + nK^{-1} \approx T$. Thus the end result is that we have numerically approximated the driving function of γ. This is the first 3 steps of the slit given in Fig. 3. Of course, our calculation of ξ will be more accurate by taking larger K.

For example, we can use the above method to calculate the driving function of any path on the square/triangle/hexagonal lattice on $\mathbb{H}$ starting from some point in $\mathbb{R}$. In fact we can calculate the driving function of a path on the square/triangle/hexagonal lattice in any polygon D by mapping the half-plane conformally onto D and pulling back the quadratic differential $1\,dw^2$ on D to $Q(z)\,dz^2$ on $\mathbb{H}$ using the transformation law. Also, note that in general, any curve γ can be approximated by a curve γ_δ which lies on the square lattice $\delta\mathbb{Z}^2$. Then it can be shown that

$$\xi_\delta \to \xi \quad \text{uniformly as } \delta \searrow 0,$$

where ξ_δ is the driving function of γ_δ and ξ is the driving function of γ. Hence, we can use the above method to calculate ξ_δ then take the limit as $\delta \searrow 0$ to obtain ξ.

We end this section by looking at what happens when the slit approaches the boundary.

Proposition 4.1. *Suppose that $\gamma : [0, T) \mapsto \overline{\mathbb{H}}$ is a simple curve such that $\gamma(0) \in \mathbb{R}$, $\gamma(0, T) \subset \mathbb{H}$ and $\gamma(0, T)$ is a θ-trajectory arc of some quadratic differential $Q(z)\,dz^2$. Then let ξ be the driving function of γ. If*

$$\lim_{t \uparrow T} \gamma(t) \in \mathbb{R} \cup \gamma(0, T),$$

i.e. γ makes a loop at time T, then

$$\left| \frac{d^n \xi}{d^n t}(t) \right| \to \infty \quad \text{as } t \nearrow T$$

for all $n = 1, \ldots$.

Proof. For $t \in (0, T)$, we define

$$\Gamma_t = \{\gamma(s)\colon\ s \in (t, T)\}.$$

Then Γ_t is a θ-trajectory arc in $H_t = \mathbb{H} \setminus \gamma(0, t]$ of $Q(w)\,dw^2$ and it is also a crosscut in H_t (see [16]). Then by the conformal invariance of θ-trajectories, $f_t^{-1}(\Gamma_t) \subset \mathbb{H}$ is a θ-trajectory arc of $Q_t(z)\,dz^2$. Moreover, $f_t^{-1}(\Gamma_t)$ is a crosscut of $\mathbb{H}$ with one end point at $\xi(t)$ and the other end point in $\mathbb{R}$ such that either $C^+(t)$ or $C^-(t)$ is contained in the closure of the bounded component of $\mathbb{H} \setminus f_t^{-1}(\Gamma_t)$. Without loss of generality, assume it is $C^+(t)$. Then since $\operatorname{diam}(f_t^{-1}(\Gamma_t)) \to 0$ as $t \nearrow T$, we must have $\xi(t) = C^+(T)$ and hence by (5), $\dot{\xi}(t) \to \infty$ as $t \nearrow T$. Similarly, we differentiate the formula given in (5) as mentioned in Corollary 3.1 to obtain the result for higher order derivatives. □

This means that as γ gets closer and closer to making a loop, the approximation by (13) stops working no matter what M we choose. This phenomenon can be observed in Fig. 3, as we turn the last corner in γ, we can see that ξ decreases faster even though the slit is not yet that close to the boundary.

5. Generalizing Theorem 1.2

5.1. Multiple slits

Suppose that $\gamma_k : [0, T) \to \overline{\mathbb{H}}$ for $k = 1, \ldots, N$ are disjoint simple curves with $\gamma_k(0) \in \mathbb{R}$ and $\gamma_k(0, T) \subset \mathbb{H}$. By the Riemann mapping theorem, there exists unique f_t that map $\mathbb{H}$ conformally onto $H_t = \mathbb{H} \setminus \bigcup_{k=1}^N \gamma_k(0, t]$ that satisfies the hydrodynamic normalization. We can reparameterize such that

$$\bigcup_{k=1}^{N} \gamma_k(0, t]$$

has half-plane capacity $2t$. Then f_t satisfies

$$\dot{f}_t(z) = -2 f_t'(z) \sum_{k=1}^{N} \frac{b_k(t)}{z - \xi_k(t)},$$

where

$$\sum_{k=1}^{N} b_k(t) = 1,$$

and $\xi_k(t) = f_t^{-1}(\gamma_k(t))$. See [14] for more details.

Theorem 5.1. *Suppose that $Q(w)\,dw^2$ is an algebraic quadratic differential on $\mathbb{H}$ such that the points $\xi_k(0) \in \mathbb{R}$ satisfy*

$$\deg_{\mathbb{H}, Q}\big(\xi_k(0)\big) = \beta_k \in \{0, 1, 2, \ldots\}$$

for all k. Then we can write

$$Q(w) = R \left(\prod_{k=1}^{N} \big(w - \xi_k(0)\big)^{\beta_k} \right) \left(\prod_{j=1}^{n} (w - a_j)^{\alpha_j} \right)$$

with $a_j \in \mathbb{C}$, $\alpha_j \in \mathbb{R}$ for $j = 1, \ldots, n$ and R is some non-zero constant. Suppose that $\gamma_k : [0, T) \to \overline{\mathbb{H}}$ for $k = 1, \ldots, N$ are disjoint simple curves such that $\gamma_k(0) \in \mathbb{R}$, $\gamma_k(0, T) \subset \mathbb{H}$ that are parameterized as above. Suppose that the functions f_t map $\mathbb{H}$ conformally onto

$$H_t = \mathbb{H} \setminus \bigcup_{k=1}^{N} \gamma_k(0, t]$$

and are hydrodynamically normalized. Then

$$2\sum_{k=1}^{N}\xi_k=-\left(\sum_{k=1}^{N}\left(\mu_k^-C_k^-(t)+\mu_k^+C_k^+(t)\right)\right)-\left(\sum_{j=1}^{n}\alpha_jA_j(t)\right)+\Sigma_0, \tag{15}$$

and

$$\dot{\xi}_l(t)=2\left(\sum_{k=1,\,k\neq l}^{N}\frac{b_k(t)}{\xi_l(t)-\xi_k(t)}\right)-\left(\sum_{k=1}^{N}\frac{\mu_k^-b_l(t)}{C_k^-(t)-\xi_l(t)}+\frac{\mu_k^+b_l(t)}{C_k^+(t)-\xi_l(t)}\right)-\left(\sum_{j=1}^{n}\frac{\alpha_jb_l(t)}{A_j(t)-\xi_l(t)}\right) \tag{16}$$

for all $l\in\{1,\dots,N\}$. Where $C_k^-(t)$ and $C_k^+(t)$ are the two preimages of $\xi_k(0)$ under f_t satisfying $C_k^-(t)<C_k^+(t)$;

$$\mu_k^{\pm}=\deg_{H_t,Q}\left(f_t\left(C_k^{\pm}(t)\right)\right);$$

$A_j(t)=f_t^{-1}(a_j)$; and

$$\Sigma_0=\left(\sum_{k=1}^{N}\beta_k\xi_k(0)\right)+\left(\sum_{j=1}^{n}\alpha_ja_j\right).$$

Proof. By Theorem 1.1 and Lemma 2.1, we can write

$$Q(w)=R\left(\prod_{k=1}^{N}\left(w-\xi_k(0)\right)^{\beta_k}\right)\left(\prod_{j=1}^{n}(w-a_j)^{\alpha_j}\right).$$

Then either by modifying the proof of Corollary 2.2 or iterating N slit functions and applying Corollary 2.2 N times, it is not too difficult to see that if we define $Q_t(z)$ by (11), then

$$Q_t(z)=R\left(\prod_{k=1}^{N}\left(z-\xi_k(t)\right)^2\left(z-C_k^-(t)\right)^{\mu_k^-}\left(z-C_k^+(t)\right)^{\mu_k^+}\right)\left(\prod_{j=1}^{n}\left(z-A_j(t)\right)^{\alpha_j}\right). \tag{17}$$

Then the proof of (15) is exactly the same as the proof of (4) in Theorem 1.2. To prove (16), we consider the logarithmic derivative of $Q_t(z)$ with respect to z and t separately using the definition of $Q_t(z)$ given by (11) to get

$$\frac{\dot{Q}_t(z)}{Q_t(z)}=2\frac{Q_t'(z)}{Q_t(z)}\left(\sum_{k=1}^{N}\frac{b_k(t)}{z-\xi_k(t)}\right)-4\left(\sum_{k=1}^{N}\frac{b_k(t)}{(z-\xi_k(t))^2}\right).$$

Then comparing the residue at $z=\xi_l(t)$ of both sides, we find that this is exactly (16). □

Similarly, we can prove a version of Proposition 1.3 and Corollary 3.1 for multiple slits. This means that we can use the method detailed in Section 4 with (16) to calculate the driving function for multiple θ_k-trajectory arc slits.

5.2. The radial case

The chordal Loewner differential equation was introduced because the upper half-plane was an easier domain to work with for many applications but the original setting of the Loewner differential equation is in the unit disc $\mathbb{D}=\{z\in\mathbb{C}\colon |z|<1\}$. Suppose that $\gamma:[0,T)\mapsto\overline{\mathbb{D}}$ is a simple curve such that $\gamma(0)\in\mathbb{T}=\{z\colon |z|=1\}$ and $\gamma(0,T)\subset\mathbb{D}\setminus\{0\}$. Then $D_t=\mathbb{D}\setminus\gamma(0,t]$ is simply-connected and $0\in D_t$ for all $t\in(0,T)$. Hence the Riemann mapping theorem implies that there is a unique conformal map f_t mapping $\mathbb{D}$ conformally onto D_t such that $f_t(0)=0$ and $f_t'(0)>0$. Then Schwarz's lemma and the Carathéodory kernel theorem implies that $f_t'(0)$ is strictly decreasing and continuous so we can reparameterize such that $f_t'(0)=e^{-t}$. $f_t'(0)$ is sometimes called the *conformal radius* of D_t and hence we are *parameterizing by conformal radius*. Then the functions f_t satisfy the radial Loewner differential equation

$$\dot{f}_t(z)=-zf_t'(z)\frac{z+e^{i\xi(t)}}{z-e^{i\xi(t)}},$$

where $e^{i\xi(t)}=f_t^{-1}(\gamma(t))$. See [10] for more details.

Theorem 5.2. *Suppose that $Q(w)\,dw^2$ is an algebraic quadratic differential on $\mathbb{D}$ such that $\deg_Q(0)=K\in\mathbb{Z}$ and $e^{i\xi_0}\in\mathbb{T}=\{|z|=1\}$ satisfies*

$$\deg_{\mathbb{D},Q}\left(e^{i\xi_0}\right)=N\in\{0,1,2,\dots\}.$$

Then we have

$$Q(w) = Rw^K\left(w - e^{i\xi_0}\right)^N \prod_{j=1}^{n} (w - a_j)^{\alpha_j},$$

where $a_j \in \mathbb{C}$, $\alpha_j \in \mathbb{R}$ *for* $j = 1, \dots, n$ *and* R *is a non-zero constant. Suppose that* $\gamma : [0, T) \mapsto \overline{\mathbb{D}}$ *is a simple curve such that* $\gamma(0) = e^{i\xi_0}$ *and* $\gamma(0, T) \subset \mathbb{D} \setminus \{0\}$ *is a* ϕ*-trajectory arc of* $Q(w)\,dw^2$ *in* $\mathbb{D}$ *that is parameterized by conformal radius. Suppose that the functions* f_t *map* $\mathbb{D}$ *conformally onto*

$$D_t = \mathbb{D} \setminus \gamma(0, t]$$

and are normalized as above. Then we have

$$e^{2i\xi(t)} = e^{Kt}\Pi_0 C^-(t)^{-\mu^-} C^+(t)^{-\mu^+} \prod_{j=1}^{n} A_j(t)^{-\alpha_j}, \tag{18}$$

and

$$\dot{\xi}(t) = -\frac{1}{2i}\left(\mu^- \frac{C^-(t) + e^{i\xi(t)}}{C^-(t) - e^{i\xi(t)}} + \mu^+ \frac{C^+(t) + e^{i\xi(t)}}{C^+(t) - e^{i\xi(t)}} + \sum_{j=1}^{n} \alpha_j \frac{A_j(t) + e^{i\xi(t)}}{A_j(t) - e^{i\xi(t)}} + K\right), \tag{19}$$

where, as usual, the functions $A_j(t)$ *are defined by*

$$A_j(t) = f_t^{-1}(a_j) \quad \text{for } j = 1, \dots, n;$$

and

$$\mu^\pm = \deg_{D_t, Q}\left(f_t\left(C^\pm(t)\right)\right);$$

$C^+(t) > C^-(t)$ *are the two preimages of* $e^{i\xi_0}$ *under* f_t*; and also,*

$$\Pi_0 = e^{iN\xi_0} \prod_{j=1}^{n} a_j^{\alpha_j}.$$

Proof. The formula for $Q(w)$ can be obtained from Theorem 1.1 using the transformation law. We then define

$$Q_t(z) = Q\left(f_t(z)\right) f_t'(z)^2 \tag{20}$$

as usual. Since the point 0 is fixed by f_t, this implies that the $\deg_{Q_t}(0) = \deg_Q(0) = K$ (by the conformal invariance of $\deg_Q$). Thus we can apply Corollary 2.2 (again, using the transformation law) to get

$$Q_t(z) = R(t) z^K\left(z - e^{i\xi(t)}\right)^2 \left(z - C^-(t)\right)^{\mu^-} \left(z - C^+(t)\right)^{\mu^+} \prod_{j=1}^{n} \left(z - A_j(t)\right)^{\alpha_j}$$

for some $R(t) \neq 0$. Note that by symmetry, we must have $\deg_Q(\infty) = \deg_{Q_t}(\infty) = K$. Thus by considering the expansion as $z \to \infty$ of both sides of (20), we get $R(t) = Re^{-(K+2)t}$. This immediately implies (18) by multiplying both sides of (20) by z^K and considering the limit as $z \to 0$ (using the fact that $f_t'(z) = e^{-t} + O(z)$ as $z \to 0$). Then we get (19) in the same way as we get (5) in the proof of Theorem 1.2. □

As in the case of multiple slits, a version of Proposition 1.3 and Corollary 3.1 holds for this case.

5.3. Other versions of the Loewner differential equation

There are several other versions of the Loewner differential equation for simply-connected domains in the literature; the methods in this paper should work in those cases as well and the proofs should be similar to the proofs of Theorem 1.2, etc. Also, [5,6] generalizes the Loewner differential equation to multiply-connected domains and again, some of the methods should work in these cases possibly using methods in [1] to extend Theorem 1.1 to multiply-connected domains. Finally, even if we consider general 2-dimensional growth processes given by the Loewner–Kufarev differential equation (see Chapter 6 of [15]), some of the methods in this paper should still be applicable.

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References

[1] D. Crowdy, The Schwarz–Christoffel mapping to bounded multiply connected polygonal domains, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 461 (2005) 2653–2678.
[2] S. Friedland, M. Schiffer, On coefficient regions of univalent functions, J. Anal. Math. 31 (1977) 125–168.
[3] W. Kager, B. Nienhuis, L.P. Kadanoff, Exact solutions for Loewner evolutions, J. Stat. Phys. 115 (2004) 805–822.
[4] T. Kennedy, Computing the Loewner driving process of random curves in the half plane, arXiv:math/0702071v1.
[5] Y. Komatu, Untersuchungen über konforme Abbildung von zweifach zusammenhängenden Gebieten, Proc. Phys.-Math. Soc. Japan (3) 25 (1943) 1–42.
[6] Y. Komatu, On conformal slit mapping of multiply-connected domains, Proc. Japan Acad. 26 (1950) 26–31.
[7] R. Kühnau, Über die analytische Darstellung von Abbildungsfunktionen insbesondere von Extremalfunktionen der Theorie der konformen Abbildung, J. Reine Angew. Math. 228 (1967) 93–132.
[8] G.F. Lawler, Conformally Invariant Processes in the Plane, American Mathematical Society, USA, 2005.
[9] J.R. Lind, A sharp condition for the Loewner equation to generate slits, Ann. Acad. Sci. Fenn. Math. 30 (2005) 143–158.
[10] K. Löwner, Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I, Math. Ann. 89 (1923).
[11] D.E. Marshall, S. Rohde, The Loewner differential equation and slit mappings, J. Amer. Math. Soc. 18 (2005) 763–778.
[12] D.E. Marshall, S. Rohde, Convergence of a variant of the Zipper algorithm for conformal mapping, SIAM J. Numer. Anal. 45 (2007).
[13] Z. Nehari, Conformal Mapping, Dover Publications, New York, 1982.
[14] E. Peschl, Zur Theorie der schlichten Funktionen, J. Reine Angew. Math. 176 (1937) 61–94.
[15] C. Pommerenke, Univalent Functions, Vandenhoeck & Ruprecht, Göttingen, 1975.
[16] C. Pommerenke, Boundary Behaviour of Conformal Maps, Springer-Verlag, Berlin, 1992.
[17] A.C. Schaeffer, D.C. Spencer, Coefficient Regions for Slit Functions, American Mathematical Society, USA, 1950.
[18] M. Schiffer, Sur l'equation differentielle de M. Loewner, C. R. Acad. Sci. Paris 221 (1945) 359–371.
[19] O. Schramm, Scaling limits of loop-erased random walks and uniform spanning trees, Israel J. Math. 118 (2000) 221–288.
[20] K. Strebel, Quadratic Differentials, Springer-Verlag, Berlin, 1984.