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Comparison theorem for a nonlinear boundary value problem on time scales

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Abstract

We prove a comparison theorem for the lower and upper solutions of a nonlinear two point boundary value problem on time scales. This theorem plays an important role in the development of the method of generalized quasilinearization on time scales. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The study of dynamical systems on time scales is now an active area of research. One of the reasons for this is the fact that the study on time scales unifies the study of both discrete and continuous processes, besides many others. The pioneering works in this direction are [1–3,8].

By a time scale (measure chain) $\mathbf{T}$, we mean a nonempty closed subset of $\mathbb{R}$.

Definition 1. Let $\mathbf{T}$ be a time scale and define the forward jump operator $\sigma(t)$ at t for $t < \sup \mathbf{T}$ by $\sigma(t) = \inf\{\tau > t : \tau \in \mathbf{T}\}$ and the backward jump operator $\rho(t)$ at t, for $t > \inf \mathbf{T}$ by $\rho(t) = \sup\{\tau < t : \tau \in \mathbf{T}\}$ for all $t \in \mathbf{T}$.

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We assume throughout that $\mathbf{T}$ has a topology that it inherits from the standard topology of the real numbers $\mathbb{R}$. If $\sigma(t) > t$, we say that t is *right scattered* (rs), while if $\rho(t) < t$, we say that t is *left scattered* (ls). If $\sigma(t) = t$, we say t is *right dense* (rd), while if $\rho(t) = t$ we say t is *left dense* (ld). For a point $t_0 \in \mathbf{T}$ let $\mu^*(t_0) = \sigma(t_0) - t_0$.

A function $f : \mathbf{T} \to \mathbb{R}$ is said to be rd-continuous provided f is continuous at rd points in $\mathbf{T}$ and the left-hand limit exists and are finite at the ld points in $\mathbf{T}$. Finally, if $\sup \mathbf{T} < \infty$, and $\sup \mathbf{T}$ is ls, we let $\mathbf{T}^\kappa := \mathbf{T} \setminus \{\sup \mathbf{T}\}$. Otherwise $\mathbf{T}^\kappa = \mathbf{T}$.

Throughout this paper we assume that a, b are points in $\mathbf{T}$ such that $a < b$. Define an interval $[a, b]$ in $\mathbf{T}$ as $[a, b] := \{t \in \mathbf{T} \mid \mathbf{a} \leqslant \mathbf{t} \leqslant \mathbf{b}\}$.

We are essentially interested in developing the method of generalized quasilinearization for boundary value problems on time scales. As a prerequisite, we establish here a comparison theorem for the following boundary value problem (BVP) on an arbitrary time scale (measure chain) $\mathbf{T}$:

$$-y^{\Delta\Delta} = f(t, y^\sigma, (y^\Delta)^\sigma), \tag{1.1}$$

$$B_1(u) = \alpha_1 u(a) - \beta_1 u^\Delta(a) = b_1, \tag{1.2}$$

$$B_2(u) = \alpha_2 u(b) - \beta_2 u^\Delta(\rho(b)) = b_2, \tag{1.3}$$

where $\alpha_0,\ \alpha_1 \geqslant 0,\ \beta_0, \beta_1 > 0,\ b_1, b_2 \in \mathbb{R}$.

The organization of the paper is as follows. In Section 2 we present the first and second derivative tests at points of extrema for functions on time scales. The main section of the paper, Section 3 deals with the comparison test for the lower and upper solutions of a second order BVP.

2. Some calculus on time scales

In their seminal works, Hilger [4,5] and Agarwal and Bohner [1], introduced the basic calculus on time scales. We develop here the so-called first and second derivative tests at points of extrema for functions on time scales. These results play a crucial role in our further study of the generalized quasilinearization method on time scales.

We begin with the following well known definition [4].

Definition 2. Let $f : \mathbf{T} \to \mathbb{R}$ be a function. Let $t \in T^\kappa$. Then, the Δ-*derivative* of f, denoted by $f^\Delta(t)$, is defined to be the number (if it exists) such that for every $\varepsilon > 0$ there exists a neighbourhood U of t such that for all $s \in U$ we have

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leqslant \varepsilon |\sigma(t) - s|.$$

It can be shown that if $f : \mathbf{T} \to \mathbb{R}$ is continuous at $t \in T^\kappa$, and t is rs, then

$$f^\Delta(t) = \Delta f(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

For $\mathbf{T}=\mathbb{R}$, $f^{\Delta}(t)$ coincides with the usual derivative, where as if $\mathbf{T}=\mathbf{hZ}$, where $\mathbf{Z}$ is the set of integers, then $f^{\Delta}(t)$ is the forward difference operator

$$f^{\Delta}(t)=\Delta f=\frac{f(t+h)-f(t)}{h}.$$

Certain technical difficulties arise when dealing with the points which are ld and rs, simultaneously. To overcome these difficulties, Erbe and Hilger [3] have introduced the *left sided derivative* of f, denoted by $f^{\Delta_l}(t)$. For rd-continuous differentiable function $f:\mathbf{T}\to\mathbb{R}$, $t\in\mathbf{T}^{\kappa}$, we have [3], $f^{\Delta_l}(t)=\lim_{s\to t^-} f^{\Delta}(s)$. The kth Δ-derivative of a function, when it exists, may be defined at a point $t\in\mathbf{T}^{\kappa}$.

Several results concerning the method of generalized quasilinearization for BVPs require the use of the second derivative test. While the main result of the paper, namely the comparison theorem, is of independent interest it is also very useful in developing the method of generalized quasilinearization, on time scales. We discuss briefly the second derivative test on time scales, a result which is not explicitly stated so far in the time scale literature. The next two lemmas deal with the sign of the first and second Δ-derivatives of a function at a point of local extremum.

Lemma 1. *Let $f:\mathbf{T}\to\mathbb{R}$ be a function which is Δ-differentiable on $\mathbf{T}^{\kappa}$. If f has a local extremum at $t_0\in\mathbf{T}^{\kappa}$, then*

(i) $f^{\Delta}(\rho(t_0))f^{\Delta}(t_0)\leqslant 0$ *if $t_0\in T^{\kappa}$ is not simultaneously ld and rs,*
(ii) $f^{\Delta_l}(t_0)f^{\Delta}(t_0)\leqslant 0$ *if $t_0\in T^{\kappa}$ is simultaneously ld and rs and provided that f is rd-continuously differentiable on T^{κ}.*

The proof follows from the definition of the Δ-derivative and is left to the reader.

It is easy to construct examples to show that the second Δ-derivative of a function f is nonnegative at such points of $\mathbf{T}$ which are simultaneously left dense and right scattered (ld–rs) and f has a maximum at those points [7]. In order to develop a result, analogous to the standard result in usual Calculus, on the sign of the second derivative of a f at an extremum point, we introduce the following notation. Besides, we expect this notation to be useful when it is necessary to deal with ld–rs points of $\mathbf{T}$.

Definition 3. Suppose that $t_0\in\mathbf{T}^{\kappa^2}$ and that the first Δ derivative exists at t_0, and is rd-continuous. Let $f^{\Delta\Delta_T}(t_0)$ denote the difference between the first Δ-derivative of f at t_0 and the left sided derivative of f at t_0. That is,

$$f^{\Delta\Delta_T}(t_0)=f^{\Delta}(t_0)-f^{\Delta_l}(t_0).$$

Remark 1. In general, $f^{\Delta\Delta_T}(t_0)$ does not coincide with $f^{\Delta\Delta_l}(t_0)$. For example, we may consider the function $f:(-\infty,0]\cup[1,\infty)\to\mathbb{R}$ given by $f(t)=t^2$, for this function we have $f^{\Delta\Delta_l}(0)=2\neq 1=f^{\Delta\Delta_T}(0)$. However, we note that if f is a continuously differentiable function on $\mathbf{T}^{\kappa}$, then $f^{\Delta\Delta_T}(t)=0$ for all $t\in\mathbf{T}^{\kappa}$. We are interested in using $f^{\Delta\Delta_T}$ only when dealing with the points of $\mathbf{T}$ which are simultaneously ld and rs.

Lemma 2. *If a function* $f:\mathbf{T}\to\mathbb{R}$ *has a local maximum at a point* $t_0\in\mathbf{T}^{\kappa^2}$, *then*

(i) $f^{\Delta\Delta}(\rho(t_0))\leqslant 0$ *provided that* t_0 *is not simultaneously ld and rs and that* $f^{\Delta\Delta}(\rho(t_0))$ *exists.*
(ii) $f^{\Delta\Delta_T}(t_0)\leqslant 0$ *provided that* t_0 *is simultaneously ld and that rs* f *is rd-continuously differentiable on* $\mathbf{T}^{\kappa}$.

Proof. (i) Let $t=t_0\in\mathbf{T}^{\kappa^2}$ be a point of local maximum for a function $f:\mathbf{T}\to\mathbb{R}$. Suppose that $f^{\Delta\Delta}(\rho(t_0))$ exists. Then,

$$f^{\Delta\Delta}(\rho(t_0))=\begin{cases}\lim\limits_{s\to t_0}\dfrac{f^{\Delta}(t_0)-f^{\Delta}(s)}{t_0-s}, & t_0 \text{ is both ld and rd,}\\[2ex] \dfrac{f^{\Delta}(t_0)-f^{\Delta}(\rho(t_0))}{t_0-\rho(t_0)}, & t_0 \text{ is ls.}\end{cases}$$

The proof follows by noting from Lemma 1 that $f^{\Delta}(t_0)\leqslant 0$, $f^{\Delta}(\rho(t_0))\geqslant 0$ whereas $f^{\Delta}(s)$ is nonnegative or nonpositive according as $s<t_0$ or $s>t_0$.

(ii) For any $s<t_0$, we know that, if f is rd-continuously differentiable, then $f^{\Delta_l}(t_0)=\lim_{s\to t_0} f^{\Delta}(s)\geqslant 0$ which implies that $f^{\Delta\Delta_T}(t_0)\leqslant 0$.

3. Comparison theorem

Consider the following BVP:

$$-u^{\Delta\Delta}=f(t,u(\sigma(.)),u^{\Delta}(\sigma(.)))\quad\text{on } [a,b]^{\kappa^2},\qquad B_1(u)=b_1,\qquad B_2(u)=b_2,$$

where B_1, B_2 are as given in (1.2) and (1.3). We shall assume that the function $f:[a,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is rd-continuous mapping in t; is continuous in the remaining variables.

Here in this paper, we are interested in considering the lower and the upper solutions of the above BVP and wish to establish a comparison result concerning the lower and upper solutions. For a variety of comparison theorems concerning the lower and upper solutions of BVP considered in this paper we refer the reader to [6].

Let $C^2_{\mathrm{rd}}([a,b])=\{f:[a,b]\to\mathbb{R}\mid f^{\Delta}$ is continuous on $[a,b]^{\kappa}$, and $f^{\Delta\Delta}$ is rd-continuous $[a,b]^{\kappa^2}\}$.

Definition 4. We say that a function $v\in C^2_{\mathrm{rd}}([a,b])$ is a lower solution of (3.1)–(3.3) if

$$-v^{\Delta\Delta}\leqslant f(t,v(\sigma(.)),v^{\Delta}(\sigma(.)))\quad\text{on } [a,b]^{\kappa^2}$$

$$\alpha_1 v(a)-\beta_1 v^{\Delta}(a)\leqslant b_1,\qquad \alpha_2 v(b)-\beta_2 v^{\Delta}(\rho(b))\leqslant b_2.$$

A function $w\in C^2_{\mathrm{rd}}([a,b])$ is said to be an *upper solution* of the above BVP if the reversed inequalities hold in the above definition.

The following theorem, which is the main result of the paper, is crucial in the use of monotone iterative techniques for BVPs on time scales and also in the development of the generalized quasilinearization on time scales.

Theorem 1. *Suppose that v, w are lower and upper solutions of the BVP* (1.1)–(1.3), *respectively. Assume that the function f is decreasing in its third argument. That is, for $y_1 \leqslant y_2$, $f(t,x,y_1) \geqslant f(t,x,y_2)$, for $(t,x,y_1),(t,x,y_2) \in [a,b] \times \mathbb{R} \times \mathbb{R}$.*

Further, suppose that for $z \in C^2_{\mathrm{rd}}([a,b])$,

(i) *$z(t) > 0$ on $[a,b]$ and $B_i(z) > 0$, $i = 1,2$.*
(ii) *for every $\lambda > 0$*

$$-\lambda z^{\Delta\Delta}(t) > f(t,(w+\lambda z)(\sigma(t)),(w+\lambda z)^{\Delta}(\sigma(t))) - f(t,w(\sigma(t)),w^{\Delta}(\sigma(t)))$$

for $t \in \mathbf{T}^{\kappa^2}$. Then $v(t) \leqslant w(t)$ on $[a,b]$.

Proof. Suppose that the conclusion of the theorem is false. Then there exists a minimal $\lambda > 0$ such that

$$v(t) \leqslant h(t) \quad \text{where } h(t) = w(t) + \lambda z(t), \quad t \in [a,b].$$

Since the function $v - h \in C^2_{\mathrm{rd}}([a,b])$, it attains its maximum in $[a,b]$. Let $t_0 \in [a,b]^{\kappa}$ be such that

$$v(t_0) = h(t_0) \text{ and } v(t) < h(t) \quad \text{for any } t > t_0. \tag{3.1}$$

We first claim that such a point $t_0 \in \mathbf{T}$ cannot be simultaneously ld and rs.

For, if possible, let $t_0 \in \mathbf{T}$ be simultaneously ld and rs. Let $p(t) = v(t) - h(t)$ $t \in [a,b]$ and we have

$$p(\sigma(t_0)) = p(t_0) + \mu(t_0) p^{\Delta}(t_0).$$

Since, p is continuously differentiable, and t_0 is a maximum point of p, we can easily see that, $p^{\Delta}(t_0) = p^{\Delta_l}(t_0) \geqslant 0$. This, together with the condition (ii) given in Lemma 1 imply that $p^{\Delta}(t_0) = 0$. It then follows that $p(\sigma(t_0)) = p(t_0) = 0$ which is a contradiction to (3.1). Therefore, t_0 cannot be a ld–rs point of $\mathbf{T}$.

Since the point t_0 is not simultaneously ld and rs we have

$$v^{\Delta}(\rho(t_0)) \geqslant h^{\Delta}(\rho(t_0)), \qquad v^{\Delta\Delta}(\rho(t_0)) \leqslant h^{\Delta\Delta}(\rho(t_0)).$$

Under the assumptions of the theorem, we get

$$\begin{aligned}
-v^{\Delta\Delta}(\rho(t_0)) &\geqslant -h^{\Delta\Delta}(\rho(t_0)) \\
&> -w^{\Delta\Delta}(\rho(t_0)) + f(\rho(t_0),h(t_0),h^{\Delta}(t_0)) - f(\rho(t_0),w(t_0),w^{\Delta}(t_0)) \\
&\geqslant f(\rho(t_0),h(t_0),h^{\Delta}(t_0)) \\
&\geqslant f(\rho(t_0),v(t_0),v^{\Delta}(t_0)) \quad \text{(since } f \text{ is decreasing in its third argument)} \\
&= f(\rho(t_0),v^{\sigma}(\rho(t_0)),v^{\Delta^{\sigma}}(\rho(t_0))),
\end{aligned}$$

which is a contradiction.

In the above argument, it is clear that since t_0 is a point which is not both ld and rs, we can easily see that $\sigma(\rho(t_0)) = t_0$.

If $t_0 = a$, we have $v^{\Delta}(a) \leqslant w^{\Delta}(a) + \lambda z^{\Delta}(a)$. From the boundary conditions, this implies $\lambda(\alpha_1 z(a) - \beta_1 z^{\Delta}(a)) \leqslant 0$. Since $\lambda > 0$, this contradicts $B_1(z) > 0$. Similarly, if $t_0 = b$, we can arrive at a contradiction. This completes the proof. □

With the help of the above theorem it is possible to study the problems associated with the well known Lienard type of equations on general time scales.

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