JOURNAL OF FUNCTIONAL ANALYSIS **133**, 129–137 (1995)

Continuous Functions That Are Locally Constant on Dense Sets

MARY ELLEN RUDIN AND WALTER RUDIN

University of Wisconsin-Madison, Madison, Wisconsin 53706

Communicated by D. Sarason

Received August 8, 1994

A compact Hausdorff space T is constructed on which the constant functions are the only real-valued continuous ones that are locally constant on a dense subset of T. © 1995 Academic Press, Inc.

If X is a compact Hausdorff space and $C(X)$ is the space of all real-valued continuous functions on X, we associate to each $f \in C(X)$ the set $\Omega(f)$ consisting of those $x \in X$ that have neighborhoods on which f is constant. This is the (obviously open) set on which f is locally constant.

Following the notation used in [1] and [2], we define

$$E_0(X) = \{f \in C(X) : \Omega(f) \text{ is dense in } X\}.$$

At a recent conference on Function Spaces, at the University of Southern Illinois, Alain Bernard and Stuart Sidney gave lectures in which $E_0(X)$ played a role. When X is compact, metric, and has no isolated points, then $E_0(X)$ is a first category subspace of $C(X)$ (with respect to the sup-norm topology) which nevertheless has some of the properties (such as Banach–Steinhaus and closed graph) which usually occur in spaces of the second category. The question whether $E_0(I)$ (where $I = [0, 1]$) can operate on some function space was also discussed.

In his lecture, Sidney raised

QUESTION 1. *Does $E_0(X)$ always separate points on X?*

One may of course also ask

QUESTION 2. *Does $E_0(X)$ always contain a nonconstant function?*

In the present paper we construct two spaces, S and T, which show that the answer is *no* in both cases. Even though it would of course be enough

0022-1236/95 $12.00

to do this for Question 2, it seems best to start with S. That space is much easier to describe and much easier to visualize than T. We have arranged the proofs, as far as possible, to follow the same pattern in the two examples. The main difference is that S has just two "layers", whereas T has infinitely many.

In order to set the stage, we begin with some (very easy) positive results. The following notations will be used throughout this paper:

$I = [0, 1]$ is the closed unit interval.

The letter c denotes the cardinality of the continuum.

$\mathscr{K}$ is the collection of all Cantor sets $K \subset I$.

PROPOSITION. *$E_0(X)$ separates points on X, hence is dense in $C(X)$, if*

(i) *X is totally disconnected, or*

(ii) *the set of isolated points of X is dense in X, or*

(iii) *the set of all $p \in X$ at which X is locally connected is dense in X, or*

(iv) *X does not contain c pairwise disjoint open sets.*

Of these, (i) and (ii) are totally trivial. In fact, $E_0(X) = C(X)$ in case (ii).

(iv) covers all compact metric spaces, as well as all compact groups, of arbitrary cardinality.

The proofs of (iii) and (iv) use the well-known Cantor functions φ_K that are associated to Cantor sets $K \subset I$. These are continuous on I, nondecreasing, constant on every component of $I \backslash K$, but nonconstant on every open interval that intersects K. Clearly, $\Omega(\varphi_K) = I \backslash K$, and $E_0(I)$ separates points on I.

To prove (iii) and (iv), pick $p, q \in X, p \neq q$, and choose $h \in C(X)$ so that $h(p) = 0$, $h(q) = 1$, $h(X) \subset I$.

For (iii), pick some K, put $f = \varphi_K \circ h$, let x_0 be a point at which X is locally connected, and let V be a connected neighborhood of x_0. Either h is constant on V, in which case $V \subset \Omega(f)$, or $h(V)$ contains an open interval $J \subset I \backslash K$, in which case f is constant on the open set $h^{-1}(J) \subset V$. Thus $\Omega(f)$ intersects every neighborhood of x_0, so that x_0 is a limit point of $\Omega(f)$.

For (iv), let $H \subset I$ be a Cantor set, observe that H is homeomorphic to $H \times H$ and that H is therefore the union of c pairwise disjoint Cantor sets K. For one of these, $h^{-1}(K)$ has empty interior, hence $h^{-1}(I \backslash K)$ is dense in X. With that K, define $f = \varphi_K \circ h$, and observe that $\Omega(f) \supset h^{-1}(I \backslash K)$.

Finally, $f(p) \neq f(q)$ in (iii) and (iv) because $\varphi_K(0) \neq \varphi_K(1)$.

Our examples will show that these sufficient conditions cannot be weakened very much: The space S has just one nontrivial connected component, namely I, the rest is totally disconnected. Neither S nor T contain more than c pairwise disjoint open sets. To give a negative answer to Question 2, T must of course be connected; it is, in fact, path-connected.

As regards (iii), [2] contains this, but with a stronger hypothesis, namely: X should contain a dense open set which is locally connected. The proof of (iii) shows, in function-algebraic terminology, that $E_0(I)$ "operates" from $C(X)$ to $E_0(X)$.

EXAMPLE 1. There is a compact Hausdorff space S on which $E_0(S)$ does not separate points.

1.1. *Description of S.* Let A be an uncountable index set, of cardinality $\leqslant c$. The points of S are:

(a) all $x \in I$.

(b) all (x, K, α) with $x \in K \in \mathscr{K}$ and $\alpha \in A$.

To describe the topology of S, put

$$V(x, \delta) = \{ y \in I : |y - x| < \delta \},$$

for $x \in I$, $\delta > 0$. Use these intervals to help define basic open neighborhoods of points in S:

(a) If $x \in I$, then for every $\delta > 0$ and every finite set F of pairs (K, α), with $K \in \mathscr{K}$, $\alpha \in A$, put

$$B_{\delta, F}(x) = V(x, \delta) \cup \{ (y, K, \alpha) : y \in K \cap V(x, \delta) \text{ and } (K, \alpha) \notin F \}.$$

(b) If $p = (x, K, \alpha)$, then for every $\delta > 0$

$$B_\delta(p) = \{ (y, K, \alpha) : y \in K \cap V(x, \delta) \}.$$

We regard (K, α) as a subset of S, namely

$$(K, \alpha) = \{ (x, K, \alpha) : x \in K \}.$$

1.2. *Topological properties of S.* The above named sets $B_{\delta, F}(x)$ and $B_\delta(p)$ form a base for a topology τ on X. It is easy to see that τ has the following properties:

(i) The restriction of τ to I is the standard topology of I.

(ii) Each (K, α) is homeomorphic to K, hence is compact and also open in S.

(iii) If $x \in I$ and $(K, \alpha) \in \mathscr{K} \times A$ then $B_{\delta, F}(x)$ intersects (K, α) only if $(K, \alpha) \notin F$.

Hence x has a neighborhood which fails to intersect (K, α). This is all that is needed to see that τ is Hausdorff.

(iv) τ is compact.

To prove this, let $\mathscr{G}$ be an open cover of S. The compactness of I shows that there are finitely many neighborhoods $B_{\delta, F}(x)$, each lying in some member of $\mathscr{G}$, whose union covers I and therefore covers all but finitely many of the compact sets (K, α). This proves (iv).

We now fix some $f \in E_0(S)$ and some $K \in \mathscr{K}$, and prove:

1.3. *There is an open interval J which intersects K, such that f is constant on $K \cap J$.* For every $\alpha \in A$, (K, α) is open in S, and $\Omega(f)$ is open and dense in S. Every (K, α) contains therefore an open set on which f is constant. Thus there are real numbers $r(\alpha)$ and open intervals $J(\alpha)$, with rational endpoints, which intersect K, such that

$$f(x, K, \alpha) = r(\alpha) \tag{1}$$

for all $x \in K \cap J(\alpha)$ and all $\alpha \in A$.

Since the set of all $J(\alpha)$'s is countable and A is uncountable, there is an open interval J such that

$$A_0 = \{\alpha \in A : J(\alpha) = J\} \tag{2}$$

is uncountable.

Hence (1) holds now for all $x \in K \cap J$ and all $\alpha \in A_0$.

Since A_0 is uncountable, there is a real number r such that the sets

$$A_0(\varepsilon) = \{\alpha \in A_0 : |r(\alpha) - r| < \varepsilon\} \tag{3}$$

are infinite, for every $\varepsilon > 0$.

Now pick $x \in K \cap J$ and some $B_{\delta, F}(x)$ with $V(x, \delta) \subset J$, and pick $\varepsilon > 0$. Since $A_0(\varepsilon)$ is infinite, $B_{\delta, F}(x)$ intersects some (in fact, infinitely many) sets (K, α) with $\alpha \in A_0(\varepsilon)$, in points p at which $|f(p) - r| < \varepsilon$.

In other words, for every $\varepsilon > 0$ every neighborhood of x contains points p at which $|f(p) - r| < \varepsilon$. Since f is continuous at x, it follows that $f(x) = r$.

Since r was chosen before x, we conclude that

$$f(x) = r \qquad \text{for every} \quad x \in K \cap J. \tag{4}$$

This proves (1.3).

1.4. *If* $g \in C(I)$ *is not constant then* g *is one-to-one on some* $K \in \mathscr{K}$. Assume, without loss of generality, that $g(a) = 0$, $g(b) = 1$, $g([a, b]) = I$, for some $a < b$. If

$$\varphi(t) = \max\{x \in [a, b]: g(x) = t\}$$

for $t \in I$, then $\varphi: I \to [a, b]$ is strictly increasing, and is continuous on a set D such that $I \backslash D$ is at most countable. Hence D contains some $H \in \mathscr{K}$, and $K = \varphi(H)$ has the desired property.

1.5. *Every* $f \in E_0(S)$ *is constant on* I. This is an immediate consequence of 1.3 and 1.4.

Note, If p and q are distinct points of S, and if at least one of them is not in I, then some $f \in E_0(S)$ has $f(p) \neq f(q)$. The points of I are thus the only ones that are not separated by $E_0(S)$.

EXAMPLE 2. There is a compact Hausdorff space T on which the constant functions are the only members of $E_0(T)$.

2.1. *Description of* T. Let A be an uncountable index set, of cardinality $\leqslant c$. As usual, $\mathbb{N}$ will denote the set of all positive integers. The letters H and K will denote nonempty compact subsets of a cube I^n for some $n \in \mathbb{N}$.

For all $n \in \mathbb{N}$, $\mathscr{K}_n$ denotes the set of all sequences $(K_1, K_2, ..., K_n)$ such that

(i) $K_1 \in \mathscr{K} = \mathscr{K}_1$

(ii) $K_i \subset I^i$ for all $i \leqslant n$

(iii) $K_{i+1} \subset K_i \times (0, 1]$ for all $i < n$, and

(iv) the projection into K_i of any nonempty open subset of K_{i+1} contains a nonempty open subset of K_i.

The points of T are

(a) finite strings of the form

$$(x_1, K_1, \alpha_1, x_2, K_2, \alpha_2, ..., x_{n-1}, K_{n-1}, \alpha_{n-1}, x_n)$$

and

(b) infinite strings of the form

$$(x_1, K_1, \alpha_1, x_2, K_2, \alpha_2, ...)$$

subject to the following conditions:

When $n = 1$, (a) reduces to the singleton $x_1 \in I$.

When $n > 1$, $n \in \mathbb{N}$, then

$$\begin{cases} \bar{K} = (K_1, K_2, ..., K_{n-1}) \in \mathscr{K}_{n-1} \\ \bar{x} = (x_1, x_2, ..., x_n) \in K_{n-1} \times (0, 1] \\ \bar{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_{n-1}) \in A^{n-1}. \end{cases} \qquad (*)$$

The strings in (b) have to satisfy (*) for all $n \in \mathbb{N}$.

For brevity, we shall write

$$(\bar{x}, \bar{K}, \bar{\alpha})_n \qquad \text{and} \qquad (\bar{x}, \bar{K}, \bar{\alpha})_\infty$$

in place of the strings in (a) and (b). Whenever these symbols appear, it will be understood that (*) holds.

If $1 \leqslant n \leqslant \infty$, T_n is the set of all $(\bar{x}, \bar{K}, \bar{\alpha})_n$. Thus

$$T = T_1 \cup T_2 \cup \cdots \cup T_\infty.$$

In order to describe basic open neighborhoods of points in T we need one further bit of notation:

$$\mathscr{F}_n = \{(K, \alpha) : K \subset I^n, \alpha \in A\}.$$

(a) If $p = (\bar{x}, \bar{K}, \bar{\alpha})_n$ and $n \in \mathbb{N}$, then for every $\delta > 0$ and every finite set $F \in \mathscr{F}_n$, $B_{\delta, F}(p)$ is the set of all $(\bar{y}, \bar{H}, \bar{\beta})_m$ such that $n \leqslant m \leqslant \infty$,

$$(H_i, \beta_i) = (K_i, \alpha_i) \quad \text{for all} \quad i < n, \qquad |y_i - x_i| < \delta \quad \text{for all} \quad i \leqslant n,$$

and, when $m > n$, either $y_{n+1} < \delta$ or $(K_n, \alpha_n) \notin F$.

Note that every neighborhood of every point of T_n intersects not only T_n, but also $T_{n+1}, T_{n+2}, ..., T_\infty$.

(b) If $p = (\bar{x}, \bar{K}, \bar{\alpha})_\infty$ then for every $\delta > 0$ and every $n \in \mathbb{N}$, $B_{\delta, n}(p)$ is the set of all $(\bar{y}, \bar{H}, \bar{\beta})_m$ such that $n < m \leqslant \infty$ and, for all $i \leqslant n$,

$$(H_i, \beta_i) = (K_i, \alpha_i) \qquad \text{and} \qquad |y_i - x_i| < \delta.$$

2.2. *Topological properties of T.* The sets $B_{\delta, F}(p)$ and $B_{\delta, n}(p)$ form a base for a topology τ on T which has the following properties.

(i) T_1 is homeomorphic to I.

(ii) If $n \in \mathbb{N}$ and $p = (\bar{x}, \bar{K}, \bar{\alpha})_{n+1}$ then the set of all $(\bar{y}, \bar{K}, \bar{\alpha})_{n+1}$ is homeomorphic to $K_n \times (0, 1]$.

(iii) τ is a Hausdorff topology. This is proved as in Example 1, except that one has to consider a number of cases.

(iv) τ is compact.

This is harder than it was in Example 1. Let

$$Y_n = T_1 \cup T_2 \cup \cdots \cup T_n.$$

Define π_n: $Y_{n+1} \to Y_n$ as follows. If $q \in Y_n$ then $\pi_n(q) = q$. If $q \in T_{n+1}$ then $\pi_n(q)$ is obtained from

$$q = (x_1, K_1, \alpha_1, ..., K_{n-1}, \alpha_{n-1}, x_n, K_n, \alpha_n, x_{n+1})$$

by dropping the last three entries. Note that π_n is continuous and that $\pi_n(T_{n+1}) = T_n$.

The way in which neighborhoods of points in T_∞ are defined shows that T is homeomorphic to the inverse limit of the Y_n's, with the π_n's as bonding maps. It suffices therefore to prove that each Y_n is compact.

Since $Y_1 = T_1$, 2.2(i) shows that Y_1 is compact. Suppose $n \geqslant 1$ and Y_n is compact. Let $\mathscr{G}$ be an open cover of Y_{n+1}. There is then a finite set $P \subset Y_n$ and neighborhoods $B_p = B_{\delta, F}(p)$ of all points $p \in P$ (here $\delta = \delta(p)$, $F = F(p)$) such that each B_p lies in some member of $\mathscr{G}$ and their union covers Y_n.

Suppose $q = (\bar{y}, \bar{H}, \bar{\beta})_{n+1} \in Y_{n+1} \setminus \bigcup B_p$. Then there is a $p \in P$ such that $\pi_n(q) \in B_p$. By the definition of $B_{\delta, F}(p)$, if this p is in T_m for some $m < n$, then $q \in B_{\delta, F}(p)$. So $p = (\bar{x}, \bar{K}, \bar{\alpha})_n \in T_n$. Note that $(H_i, \beta_i) = (K_i, \alpha_i)$ for all $i < n$ (since $\pi_n(q) \in B_p$) and that $(H_n, \beta_n) \in F(p)$ because $q \notin B_p$. For the same reason, $y_{n+1} \geqslant \delta(p)$. Thus q lies in the set

$$W_p(\bar{H}, \bar{\beta}) = \{(\bar{z}, \bar{H}, \bar{\beta})_{n+1} : z_{n+1} \geqslant \delta(p)\}$$

which, by 2.2 (ii), is homeomorphic to $H_n \times [\delta_p, 1]$ and is thus compact.

Since P is finite and each $F(p)$ is finite, there are only finitely many possibilities for these W's. Their union is thus compact, and it covers $Y_{n+1} \setminus \bigcup B_p$.

(v) Suppose $n \in \mathbb{N}$ and $p = (\bar{x}, \bar{K}, \bar{\alpha})_{n+1}$. For $0 < t \leqslant 1$, let $p_t = (\bar{z}, \bar{K}, \bar{\alpha})_{n+1}$, where $z_i = x_i$ for all $i \leqslant n$ and $z_{n+1} = t$, and let $p_0 = \pi_n(p)$. *Then* $\{p_t : 0 \leqslant t \leqslant 1\}$ *is homeomorphic to* I.

Every point of T_{n+1} thus lies on a half-open interval (also in T_{n+1}) which is glued to T_n by its missing end-point, which lies in T_n. This makes T path-connected.

2.3. We now fix $(K_1, ..., K_n) \in \mathscr{K}_n$, $\bar{\alpha} = (\alpha_1, ..., \alpha_{n-1}) \in A^{n-1}$, and put $\bar{K} = (K_1, ..., K_{n-1})$ as in 2.1.

Claim. If $f \in C(T)$ and $\Omega(f)$ is dense in T, then there is a nonempty open set V in K_n such that f is constant on $\{(\bar{x}, \bar{K}, \bar{\alpha})_n : \bar{x} \in V\}$.

Let $\mathscr{V}$ be a countable base for K_n. For every $\alpha \in A$, let W_α be the set of all points $(\bar{w}, \bar{H}, \bar{\beta})_m$ with $n < m \leqslant \infty$ that satisfy

$$H_i = K_i \quad \text{for all} \quad i \leqslant n, \qquad \beta_i = \alpha_i \quad \text{for all} \quad i < n,$$

and $\beta_n = \alpha$.

Then W_α is open, and since $\Omega(f)$ is dense in T, $W_\alpha \cap \Omega(f)$ is a nonempty open set. Since T_∞ has empty interior, there is an m, $n < m < \infty$, a point $q = (\bar{w}, \bar{H}, \bar{\beta})_m$ in $W_\alpha \cap \Omega(f) \cap T_m$ and a neighborhood $B_{\delta, F}(q)$ on which f is constant. Say $f = r(\alpha)$ on this neighborhood.

The intersection of $B_{\delta, F}(q)$ with T_m is the set

$$Z_\alpha = \{(\bar{z}, \bar{H}, \bar{\beta})_m : |z_i - w_i| < \delta \text{ for all } i \leqslant m\}.$$

If

$$U_\alpha = \{\bar{u} \in H_{m-1} : |z_i - w_i| < \delta \text{ for all } i < m\}$$

then U_α is a nonempty open set in H_{m-1}. Since $(H_1, ..., H_{m-1}) \in \mathscr{K}_{m-1}$, $n \leqslant m-1$, and $H_n = K_n$, the projection of U_α into K_n contains a nonempty $V_\alpha \in \mathscr{V}$ (see 2.1 (iv)). So:

> If $\bar{x} \in V_\alpha$, $p = (\bar{x}, \bar{K}, \bar{\alpha})_n$, and $B_{\delta, F}(p)$ is one of p's basic neighborhoods, with $(K_n, \alpha) \notin F$, then $B_{\delta, F}(p)$ intersects Z_α in a point p' at which $f(p') = r(\alpha)$. (**)

Since $\mathscr{V}$ is countable and A is uncountable, there is a $V \in \mathscr{V}$ that the set

$$A_0 = \{\alpha \in A : V_\alpha = V\}$$

is uncountable, and there is a real number r such that the sets

$$A_0(\varepsilon) = \{\alpha \in A_0 : |r(\alpha) - r| < \varepsilon\}$$

are infinite, for every $\varepsilon > 0$.

If $\bar{x} \in V$, $p = (\bar{x}, \bar{K}, \bar{\alpha})_n$, and $\varepsilon > 0$, the fact that $A_0(\varepsilon)$ is infinite shows now, by (**) that every neighborhood of p contains points p' at which $|f(p') - r| < \varepsilon$. Since f is continuous at p, it follows that $f(p) = r$. This proves the claim.

2.4. We fix $\bar{K} = (K_1, ..., K_{n-1})$ and $\bar{\alpha} = (\alpha_1, ..., \alpha_{n-1})$ as in 2.3, and claim: If $g \in C(T)$, g is not constant, and n is the smallest integer such that g is not constant on T_n, then there exists a nonempty $K_n \subset K_{n-1} \times (0, 1]$ such that

(a) $(K_1, ..., K_{n-1}, K_n) \in \mathscr{K}_n$, and

(b) if V is a nonempty open subset of K_n then g is not constant on $\{(\bar{x}, \bar{K}, \alpha)_n : \bar{x} \in V\}$.

Note first that if g is not constant on T, then (because T_ω has empty interior) g fails to be constant on some T_n, and hence there *is* a smallest n for which this is true.

If $n = 1$, we refer to 1.4.

Suppose $n > 1$, and, without loss of generality, $g = 0$ on $T_1 \cup \cdots \cup T_{n-1}$. There exists then a point $p \in T_n$ at which $g(p) \neq 0$. In view of 2.2 (v), there is a closed interval $I_0 \subset (0, 1]$ such that g is not constant on $\{p_t : t \in I_0\}$, because $g(p_0) = 0$. Another application of 1.4 gives us a Cantor set C in $\{p_t : t \in I_0\}$ such that g is one-to-one on C.

Associate to every open set $J \subset I_0$ the closed set $L(J)$ of all $\bar{z} \in K_{n-1}$ for which g is constant on the set

$$\{((\bar{z}, t), \bar{K}, \bar{\alpha})_n : t \in J\}.$$

(The constant depends on $\bar{z}$ and J.)

Let K_n be $K_{n-1} \times I_0$, minus the union of all sets of the form

$$(\text{int } L(J)) \times J.$$

These are open, hence K_n is compact, and $K_n \neq \emptyset$ because $C \subset K_n$. Our choice of $L(J)$ makes (b) clear.

To check (a) we show that every point of K_n has a neighborhood in K_n whose projection into K_{n-1} has nonempty interior.

So fix $\bar{y} \in K_n$, $\varepsilon > 0$, $J_0 = (y_n - \varepsilon, y_n + \varepsilon) \cap I_0$, and put

$$U = \{\bar{u} \in K_{n-1} : |u_i - y_i| < \varepsilon \text{ for all } i < n\}.$$

Then $(U \times J_0) \cap K_n$ is an open neighborhood of $\bar{y}$ in K_n.

Since $\bar{y} \in K_n$, $V = U \backslash L(J_0) \neq \emptyset$, and V is open.

If $\bar{v} \in V$ then g is not constant on $\{\bar{v}\} \times J_0$, hence (using 1.4 one more time) there is a Cantor set $C(\bar{v}) \subset \{\bar{v}\} \times J_0$ on which g is one-to-one. So $C(\bar{v})$ is in K_n, hence $C(\bar{v}) \subset (U \times J_0) \cap K_n$. Since $C(\bar{v})$ projects to $\{\bar{v}\}$, it follows that $(U \times J_0) \cap K_n$ projects on a set containing V.

2.5. *If $f \in E_0(T)$ then f is constant.* This is an immediate consequence of 2.3 and 2.4.

References

1. A. Bernard, Une fonction non Lipschitzienne peut-elle opérer sur un espace de Banach non trivial? *J. Funct. Anal.* **122** (1994), 451–477.

2. A. Bernard and S. J. Sidney, Banach like normed linear spaces, preprint.