Increasing directionally convex orderings of random vectors having the same copula, and their use in comparing ordered data

Narayanaswamy Balakrishnan\textsuperscript{a}, Félix Belzunce\textsuperscript{b}, Miguel A. Sordo\textsuperscript{c}, Alfonso Suárez-Llorens\textsuperscript{c,*}

\textsuperscript{a} Department of Mathematics and Statistics, McMaster University, Hamilton, Ontario, Canada L8S 4K1
\textsuperscript{b} Dpto. Estadística e Investigación Operativa, Universidad de Murcia, 30100 Espinardo (Murcia), Spain
\textsuperscript{c} Dpto. Estadística e Investigación Operativa, Universidad de Cádiz, 11002 Cádiz, Spain

\begin{abstract}
In this paper, we establish some results for the increasing convex comparisons of generalized order statistics. First, we prove that if the minimum of two sets of generalized order statistics are ordered in the increasing convex order, then the remaining generalized order statistics are also ordered in the increasing convex order. This result is extended to the increasing directionally convex comparisons of random vectors of generalized order statistics. For establishing this general result, we first prove a new result in that two random vectors with a common conditionally increasing copula are ordered in the increasing directionally convex order if the marginals are ordered in the increasing convex order. This latter result is, of course, of interest in its own right.
\end{abstract}

1. Introduction

The basic aim of this paper is to compare random vectors of generalized order statistics. For this purpose, we consider the increasing directional convex ordering of two vectors of generalized order statistics. The paper by Müller and Scarsini [36] comparing two random vectors with the same copula gives us a key result in this direction. In their paper, Müller and Scarsini [36] present several results in which, for two random vectors with the same copula and under some dependence assumptions about the copula, the stochastic comparison of marginals is enough for the multivariate comparison of the corresponding random vectors. Since two random vectors of generalized order statistics with the same set of parameters have the same copula, we consider here their comparison in the increasing directionally convex order of random vectors with the same copula. Müller and Scarsini [36] provide a result for the directional convex order, and we extend their result in this paper to the increasing directionally convex order.

Hence, the purpose of this paper is two-fold. In the first part, we provide a new result for the increasing directionally convex order comparison of two random vectors having the same copula, (see [7] for an early application of the increasing directionally convex order to queuing systems). Then, in the second part, we provide conditions for the increasing directionally convex ordering of two random vectors of generalized order statistics.

The stochastic comparison of ordered data has received considerable attention during the last two decades. The stochastic comparisons of the usual order statistics are especially of great interest since order statistics appear in many applied problems including reliability theory. Details of results on these stochastic orderings associated with order statistics can be

* Corresponding author.

E-mail addresses: bala@mcmaster.ca (N. Balakrishnan), belzunce@um.es (F. Belzunce), mangel.sordo@uca.es (M.A. Sordo), alfonso.suarez@uca.es (A. Suárez-Llorens).

© 2011 Elsevier Inc. All rights reserved.

doi:10.1016/j.jmva.2011.08.017
found in Boland et al. [16,15]. Some additional results concerning the multivariate comparison of vectors of order statistics can be found in Belzunce et al. [12], and more recently in Belzunce et al. [9]. A review of analogous results for record values can be found in Belzunce et al. [10]. As a natural generalization, one could develop similar results under the more general setting of generalized order statistics; see, for example, Franco et al. [18], Belzunce et al. [11], Khaledi [25], Hu and Zhuang [20,21], Khaledi and Kochar [28], Guoxin and Jinshan [19], Qiu and Wu [40], Xie and Hu [47], and Balakrishnan et al. [4]. In particular, Belzunce et al. [11] have established several results for the comparison of random vectors of generalized order statistics, and additional results in this direction can be found in Fang et al. [17], Belzunce et al. [13], Zhuang and Hu [51,52], and Xie and Hu [48]. In the recent work of Balakrishnan et al. [4], the univariate and multivariate likelihood ratio orderings of the generalized order statistics and some associated conditional variables have been established. In this paper, we develop some new results for the increasing convex order comparison of generalized order statistics and their extension to the multivariate increasing directionally convex order comparison.

The rest of this paper is organized as follows. In Section 2, we provide the main definitions and results pertaining to convex orders that are essential for subsequent developments, and especially the main result about the increasing directionally convex order. In Section 3, we use this result to establish a general result about increasing directionally convex ordering of random vectors of generalized order statistics. Finally, in Section 4, we present some concluding remarks.

In this paper, for any random variable \( X \) and an event \( A \), we use \( \{ X \mid A \} \) to denote the random variable whose distribution is the conditional distribution of \( X \) given \( A \). Expected values are assumed to exist whenever they are mentioned. We use \( =_{d} \) to denote equality in law. Given two random variables \( X \) and \( Y \), we will say that \( X \preceq_{\alpha} Y \) if \( E(\phi(X)) \leq E(\phi(Y)) \) for all increasing functions \( \phi \) for which the involved expectations exist. Given a random variable \( X \) with distribution function \( F \), we define the quantile function as \( F_{X}^{-1}(p) = \inf\{ x : F_{X}(x) \geq p \} \), for all real values \( p \in (0, 1) \), and we shall denote by \( \bar{F} \equiv 1 - F \) the corresponding survival function.

2. Results about increasing convex orders

When comparing the variability of two random variables, one could consider several different criteria. Two of the most important ones in this regard are the convex and increasing convex orders, which are defined as follows.

**Definition 2.1.** Given two random variables \( X \) and \( Y \), we say that \( X \) is less than \( Y \) in the convex [increasing convex] order, denoted by \( X \preceq_{\alpha} Y \), if

\[
E[\phi(X)] \leq E[\phi(Y)],
\]

for all convex [increasing convex] functions \( \phi \), for which the involved expectations exist.

The convex order implies the equality of means, i.e., if \( X \preceq_{\alpha} Y \), then \( E[X] = E[Y] \), and so by taking \( \phi(x) = x^{2} \), we obtain that if \( X \preceq_{\alpha} Y \), then \( \text{Var}[X] \leq \text{Var}[Y] \). Also, in case \( E[X] = E[Y] \), then \( X \preceq_{\alpha} Y \iff X \preceq_{\leq_{\alpha}} Y \). The following result about the increasing convex order will be used in our subsequent developments (see Lemma 2.1 of Sordo and Ramos [45]).

**Lemma 2.2.** Let \( X \) and \( Y \) be two random variables with distribution functions \( F \) and \( G \), respectively. Then,

\[
\int_{0}^{1} F^{-1}(t)dt \leq \int_{0}^{1} G^{-1}(t)dt \quad \text{for all } p \in (0, 1).
\]

An additional result, which will be used later on, is as follows.

**Lemma 2.3.** Let \( X \) and \( Y \) be two continuous random variables with interval supports and with distribution functions \( F \) and \( G \), respectively. Let \( h = F^{-1}G \). Then, \( X \preceq_{\alpha} Y \) implies

\[
E \{ \phi( h(Y)) \mid Y > x \} \leq E \{ \phi( Y) \mid Y > x \}
\]

for all \( x \in \mathbb{R} \) and for any increasing convex function \( \phi \).

**Proof.** Suppose \( X \preceq_{\alpha} Y \) and \( \phi \) is an increasing convex function. Then, it is well known that \( \phi(X) \preceq_{\alpha} \phi(Y) \). By Lemma 2.2, this is equivalent to saying

\[
\int_{0}^{1} F^{-1}_{\phi}(t)dt \leq \int_{0}^{1} G^{-1}_{\phi}(t)dt \quad \text{for all } p \in (0, 1) \text{ and for all } \phi \text{ increasing and convex},
\]

where \( F^{-1}_{\phi}(t) = \phi( F^{-1}(t)) \) and \( G^{-1}_{\phi}(t) = \phi( G^{-1}(t)) \) are the quantile functions of \( \phi(X) \) and \( \phi(Y) \), respectively. Evidently, (2.2) is equivalent to

\[
\int_{G(x)} F^{-1}_{\phi}(t)dt \leq \int_{G(x)} G^{-1}_{\phi}(t)dt \quad \text{for all } x \in \mathbb{R} \text{ and for all increasing convex } \phi.
\]

(2.3)
Since
\[ \frac{\int_p^1 F^{-1}(t) \, dt}{1 - p} = E \left[ X \mid X > F^{-1}(p) \right]. \]
(2.3) is equivalent to
\[ E \left[ \phi(X) \mid \phi(X) > F^{-1}_p(G(x)) \right] \leq E \left[ \phi(Y) \mid \phi(Y) > G^{-1}_p(G(x)) \right] \]
for all \( x \in \mathbb{R} \) and for all increasing convex \( \phi \).
(2.4)

Due to the fact that \( X \equiv_U h(Y) \), where \( h = F^{-1}G \), (2.4) can be readily rewritten as (2.1), which completes the proof of the lemma.

In the multivariate situation, there are several possible ways of extending this concept, depending on the kind of convexity that we consider.

Given two random vectors \( \mathbf{X} \) and \( \mathbf{Y} \), we say that \( \mathbf{X} \) is less than \( \mathbf{Y} \) in the multivariate convex [increasing convex] order, denoted by \( \mathbf{X} \leq_{\text{CI}} \mathbf{Y} \), if
\[ E[\phi(\mathbf{X})] \leq E[\phi(\mathbf{Y})], \]
for all convex [increasing convex] functions \( \phi : \mathbb{R}^n \to \mathbb{R} \), for which the involved expectations exist.

Some other suitable classes of functions defined on \( \mathbb{R}^n \) can also be considered to extend convex orders to the multivariate case by means of a difference operator. To be specific, let \( \Delta_i^h \) be the \( h \)th difference operator defined for a function \( \phi : \mathbb{R}^n \to \mathbb{R} \) as
\[ \Delta_i^h \phi(x) = \phi(x + \epsilon e_i) - \phi(x), \]
where \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \). A function \( \phi \) is said to be directionally convex if \( \Delta_i^h \Delta_j^k \phi(x) \geq 0 \) for all \( 1 \leq i \leq j \leq n \) and \( \epsilon, \delta \geq 0 \). We observe that directionally convex functions are also known as ultramodular functions; see, for example, Marinacci and Montrucchio [34]. A function \( \phi \) is said to be supermodular if \( \Delta_i^h \Delta_j^k \phi(x) \geq 0 \) for all \( 1 \leq i \leq j \leq n \) and \( \epsilon, \delta \geq 0 \). If \( \phi \) is twice differentiable, then it is directionally convex if \( \partial^2 \phi/\partial x_i \partial x_j \geq 0 \) for every \( 1 \leq i \leq j \leq n \), and it is supermodular if \( \partial^2 \phi/\partial x_i \partial x_j \geq 0 \) for every \( 1 \leq i < j \leq n \). Clearly, a function \( \phi \) is directionally convex if it is supermodular and it is componentwise convex.

When we consider directionally convex [increasing directionally convex] functions in (2.5), then we say that \( \mathbf{X} \) is less than \( \mathbf{Y} \) in the directionally convex [increasing directionally convex] order, denoted by \( \mathbf{X} \leq_{\text{dir} \rightarrow \text{CI}} \mathbf{Y} \). The directionally convex orders not only compare the dependence structures of two random vectors, but also the variability of the marginals.

The main result of this section is regarding the increasing directional convex order for random vectors with the same copula. A copula \( C \) is a cumulative distribution function with uniform marginals on \([0, 1]\). Furthermore, it has been shown that if \( H \) is a \( n \)-dimensional distribution function with marginal distribution functions \( F_1, \ldots, F_n \), then there exists a \( n \)-copula \( C \) such that, for all \( (x_1, \ldots, x_n) \in \mathbb{R}^n \), we have \( H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)) \). Moreover, if \( F_1, \ldots, F_n \) are continuous, then \( C \) is unique; for elaborate details on various properties of copulas, interested readers may refer to Nelsen [39]. The copula contains information about the dependence of the random vector separated from the behavior of the marginal distributions.

Some copulas with a specific dependence structure will be considered here; in particular, we will focus on CI copulas. In general, given a random vector \( (X_1, \ldots, X_n) \), we say that \( (X_1, \ldots, X_n) \) is conditionally increasing in sequence (CIS) if (see [8]) \( X_i \uparrow_{\text{CI}} (X_1, \ldots, X_{i-1}) \), \( i = 2, \ldots, n \), i.e., if
\[ [X_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] \leq_{\text{CI}} [X_i | X_1 = x_1', \ldots, X_{i-1} = x_{i-1}'] \]
whenever \( x_j \leq x_j' \), \( j = 1, 2, \ldots, i - 1 \).

We say that the random vector \( (X_1, \ldots, X_n) \) is conditionally increasing (CI) if and only if the random vector \( \mathbf{X}_\pi := (X_{\pi(1)}, \ldots, X_{\pi(n)}) \) is CIS for all permutations \( \pi \) of \( \{1, 2, \ldots, n\} \).

We shall now present the main result of this section.

**Theorem 2.4.** Let \( \mathbf{X} \) and \( \mathbf{Y} \) be random vectors with a common CI copula \( C \) and assume that \( X_i \leq_{\text{CI}} Y_i \) for all \( i = 1, \ldots, n \). Then, \( \mathbf{X} \leq_{\text{dir} \rightarrow \text{CI}} \mathbf{Y} \).

**Proof.** Let \( X \) and \( Y \) be two univariate random variables. It is well known that \( X \leq_{\text{CI}} Y \) holds if and only if there exists a random variable \( Z \) such that \( X \leq_{\text{CI}} Z \leq_{\text{CI}} Y \); see [38], Theorem 1.5.14. Therefore, by the assumption made, we can consider a random vector \( \mathbf{Z} \) with the same CI copula \( C \), such that \( X_i \leq_{\text{CI}} Z_i \leq_{\text{CI}} Y_i \), for \( i = 1, \ldots, n \). Now using Theorem 4.5 in Müller and Scarsini [36] for a CI copula, we obtain that the \( \leq_{\text{CI}} \) order of the marginal distributions implies \( \leq_{\text{dir} \rightarrow \text{CI}} \) for the vectors. Hence,
In addition, for two random vectors sharing a common copula, the $\preceq_{\text{dir}}$ order of the marginal distributions implies the $\preceq_{\text{dir}}$ order for the vectors, $Z \preceq_{\text{dir}} Y$, (see [38]). Hence, $E[\phi(Z)] \leq E[\phi(Y)]$ for all increasing real functions $\phi : \mathbb{R}^n \mapsto \mathbb{R}$, for which the involved expectations exist. Then, the required result follows easily by transitivity. □

**Remark 2.5.** We would like to mention here that the first proof we presented for Theorem 2.4 replicated all details of Theorem 4.5 in Müller and Scarsini [36]. We started by introducing the concept of local spread due to Müller and Scarsini [36] which is stronger than the one introduced earlier by Rothschild and Stiglitz [41]. For a univariate discrete distribution $F$, on the finite set of points $x_1 < x_2 < \cdots < x_n$, basically a local spread removes the probability from a point $x_i \in \mathbb{R}$ and spreads it to the preceding and posterior points in the support (see [37] for a recent work on this topic). We defined a local spread as mean increasing if the new discrete distribution that differs from $F$ by a local spread has a larger mean. Based on the previous definition, we then proved parallel results to those of Müller and Scarsini [36] and Müller and Stoyan [38] for both $\preceq_{\text{dir}}$ and $\preceq_{\text{dir} \rightarrow \infty}$ orderings. The first proof was rather technical and long, and for this reason one of the referees suggested simply to delete it. This referee also inspired the actual proof provided here. Moreover, this shorter proof would allow the reader to focus more on the main results of this paper that deal with the increasing convex comparisons of generalized order statistics. It is also worth noting that comparisons similar to those in Theorem 2.4 can be found in Rüschendorf [42], who uses some other interesting techniques for his proof.

Although the main application of this result here is in the comparison of the random vectors of GOS’s, the result can be useful in some other situations as well. As a consequence of Theorem 2.4, we have the following result.

**Theorem 2.6.** Let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ be two $n$-dimensional random vectors with a common CI copula. If $X_i \preceq_{\text{dir}} Y_i$ for all $i = 1, \ldots, n$, then

$$\sum_{i=1}^{n} \phi_i(X_i) \leq \sum_{i=1}^{n} \phi_i(Y_i),$$

where $\phi_i$ is increasing convex for $i = 1, 2, \ldots, n$.

**Proof.** If we consider the random vectors $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ with the same CI copula, we have, from Theorem 2.4, that $(X_1, \ldots, X_n) \preceq_{\text{dir}} (Y_1, \ldots, Y_n)$. Hence, $E[\phi(X)] \leq E[\phi(Y)]$ for any increasing directional convex function $\phi : \mathbb{R}^n \mapsto \mathbb{R}$ for which the involved expectations exist. It is well known that if $\psi : \mathbb{R} \mapsto \mathbb{R}$ is increasing convex and $\phi : \mathbb{R}^n \mapsto \mathbb{R}$ is increasing directionally convex, then the composition $\psi(\phi)$ is increasing directionally convex, and therefore $\phi(X) \preceq_{\text{dir}} \phi(Y)$. In particular, the function $\phi(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} \phi_i(X_i)$, where $\phi_i$ is increasing convex for $i = 1, 2, \ldots, n$, is increasing directionally convex (see Proposition 4.4 in Marinacci and Montrucchio [34]). Consequently, we have $\sum_{i=1}^{n} \phi_i(X_i) \preceq_{\text{dir}} \sum_{i=1}^{n} \phi_i(Y_i)$. □

A special case of the result when $\phi_i(x) = x$, for all $i = 1, \ldots, n$, readily yields the following corollary for convolutions.

**Corollary 2.7.** Let $X = (X_1, \ldots, X_n)$ and $Y = (Y_1, \ldots, Y_n)$ be two $n$-dimensional random vectors with a common CI copula. If $X_i \preceq_{\text{dir}} Y_i$ for all $i = 1, \ldots, n$, then

$$\sum_{i=1}^{n} X_i \preceq_{\text{dir}} \sum_{i=1}^{n} Y_i.$$

From Corollary 2.7, we can compare expected utilities for increasing convex utility functions of convolutions of random variables not necessarily independent. For example, we consider a particular case of the previous results.

Let us consider two gamma distributed random variables $X$ and $Y$ with density functions $f(x) = x^{n-1} \lambda^\gamma \exp(-\lambda x) / \Gamma(\gamma)$ and $g(x) = x^{n-1} \mu^\beta \exp(-\mu x) / \Gamma(\beta)$ for $x > 0$, respectively (denoted by $X \sim \Gamma(\alpha, \lambda)$ and $Y \sim \Gamma(\beta, \mu)$, with $\alpha, \lambda, \beta, \gamma > 0$. In this case, from Taylor [46], it is known that if $\alpha \geq \beta$ and $\alpha / \lambda \leq \beta / \mu$, then $X \preceq_{\text{dir}} Y$. Now, let $(X_1, \ldots, X_n)$ and $(Y_1, \ldots, Y_n)$ be two $n$-dimensional random vectors with a common CI copula. If $X_i \sim \Gamma(\alpha_i, \lambda_i)$ and $Y_i \sim \Gamma(\beta, \mu)$, and they satisfy the previous condition for the parameters for all $i = 1, \ldots, n$, then $\sum_{i=1}^{n} X_i \preceq_{\text{dir}} \sum_{i=1}^{n} Y_i$.

It is worth noting that convolutions appear naturally in several problems in risk theory, reliability and statistics. For example, consider an insurance company with individual risks $X_1, \ldots, X_n$, in which case the company bears the aggregate risk $S = \sum_{i=1}^{n} X_i$. In reliability theory, convolutions appear when a failed unit is replaced by a new one and the total life is obtained by the addition of the two life lengths. Furthermore, several statistics of interest are linear combinations of random variables. In the literature, one can find several results concerning comparisons of variability of convolutions or linear combinations. Most of these results are given for some parametric models (such as uniform, gamma and Rayleigh distributions) of independent random variables; see, for example, Kochar and Ma [29,30], Korwar [32], Khaledi and Kochar [26,27] Manesh and Khaledi [33], Zhao and Balakrishnan [50], Kochar and Xu [31], Amiri et al. [1], and Xu and Balakrishnan [49]. For dependent components, a general but elegant result is provided by Müller [35], which proves that
the convolution of the components of two random vectors, ordered in the supermodular order, are ordered in the increasing convex order.

3. Increasing convex comparisons of generalized order statistics

Order statistics and record values have found important applications in several fields of science and engineering. An extensive review of theoretical results and applications can be found in the volumes of Balakrishnan and Rao [5, 6]. Due to close similarity between some distributional, structural and dependence properties of order statistics and record values, Kamps [23, 24] introduced the model of generalized order statistics which includes, as special cases, random vectors of order statistics and record values, and in addition some other models of interest such as sequential order statistics and progressively Type-II censored order statistics.

We now present the definition of generalized order statistics, due to Kamps [23, 24].

**Definition 3.8.** Let \( n \in \mathbb{N}, k \geq 1, m_1, \ldots, m_{n-1} \in \mathbb{R}, M_r = \sum_{j=r}^{n-1} m_j, 1 \leq r \leq n-1 \), be parameters such that \( y_r = k + n - r + M_r \geq 1 \) for all \( r \in \{1, \ldots, n-1\} \), and let \( \overline{m} = (m_1, \ldots, m_{n-1}) \) if \( n \geq 2 \) (\( \overline{m} \in \mathbb{R} \) arbitrary, if \( n = 1 \)). Then, the random vector \((U_{(1, n, \overline{m}, k)}, \ldots, U_{(n, n, \overline{m}, k)})\) with joint density function

\[
h(u_1, \ldots, u_n) = k \left( \prod_{j=1}^{n-1} y_j \right) \left( \prod_{j=1}^{n-1} (1 - u_j)^m_j \right) (1 - u_n)^{k-1},
\]

defined over the cone \( 0 \leq u_1 \leq \cdots \leq u_n \leq 1 \), is called the uniform generalized order statistics. Now, for a given distribution function \( F \), the random vector

\[
(X_{(1, n, \overline{m}, k)}, \ldots, X_{(n, n, \overline{m}, k)}) \equiv (F^{-1}(U_{(1, n, \overline{m}, k)}), \ldots, F^{-1}(U_{(n, n, \overline{m}, k)}))
\]
is then called the generalized order statistics (GOS’s) from the distribution \( F \).

Stochastic comparisons of GOS’s have been discussed rather extensively during the past 10 years. Interested readers may refer to Franco et al. [18], Belzunce et al. [11, 13], Khaledi [25], Hu and Zhuang [20, 21, 52], Khaledi and Kochar [28], Fang et al. [17], Guoxin and Jinshan [19], Qiu and Wu [40], Xie and Hu [47, 48], and Balakrishnan et al. [4]. In this section, we establish results for increasing convex and increasing directionally convex orders, and we first need to state some existing results that are necessary for proving our results.

First, we have the following lemma from Barlow and Proschan ([8], p. 120).

**Lemma 3.9.** Let \( W \) be a measure on the interval \((a, b)\), not necessarily nonnegative. Let \( h \) be a nonnegative function defined on \((a, b)\). If \( \int_a^b h(t) dW(t) \geq 0 \) for all \( x \in (a, b) \) and if \( h \) is increasing, then \( \int_a^b h(t) dW(t) \geq 0 \).

We then have the following result concerning the minima from two vectors of generalized order statistics.

**Lemma 3.10.** Let \( X \) and \( Y \) be two continuous random variables with distribution functions \( F \) and \( G \), respectively. Let

\[
X = (X_{(1, n, \overline{m}, k)}, \ldots, X_{(n, n, \overline{m}, k)}) \quad \text{and} \quad Y = (Y_{(1, n, \overline{m}, k)}, \ldots, Y_{(n, n, \overline{m}, k)})
\]

be two random vectors of generalized order statistics from \( F \) to \( G \), respectively, with parameter \( \gamma_1 = k + n - 1 + M_1 \). Similarly, let

\[
X' = (X_{(1, n', \overline{m}', k)}, \ldots, X_{(n', n', \overline{m}', k)}) \quad \text{and} \quad Y' = (Y_{(1, n', \overline{m}', k)}, \ldots, Y_{(n', n', \overline{m}', k)})
\]

be two random vectors of generalized order statistics from \( F \) to \( G \), respectively, with parameter \( \gamma_1' = k + n - 1 + M_1' \). Let \( \gamma_1' \leq \gamma_1 \). If \( X_{(1, n, \overline{m}, k)} \leq_{i.c.x} Y_{(1, n, \overline{m}, k)} \), then \( X_{(1, n', \overline{m}', k)} \leq_{i.c.x} Y_{(1, n', \overline{m}', k)} \).

**Proof.** From Lemma 2.2, the condition

\[
X_{(1, n, \overline{m}, k)} \leq_{i.c.x} Y_{(1, n, \overline{m}, k)}
\]
is equivalent to

\[
\int_p^1 F_{(1, n, \overline{m}, k)}^{-1}(t) dt \leq \int_p^1 G_{(1, n, \overline{m}, k)}^{-1}(t) dt \quad \text{for all} \ p \in (0, 1).
\]

(3.6)

Since

\[
F_{(1, n, \overline{m}, k)}^{-1}(t) = F^{-1}(1 - (1 - t)^{1/\gamma_1}),
\]
Let us consider an increasing convex function $\phi$. Proof. Then, to establish this, let us consider the function $\gamma_i \int_0^1 \left[ G^{-1}(t) - F^{-1}(t) \right] (1 - t)^{\gamma_i} \, dt \geq 0$ for all $p \in (0, 1)$.

Let us now consider the non-negative increasing function $h(t) = \gamma_i (1 - t)^{\gamma_i}$ for $t \in [p, 1)$ and $h(t) = 0$ for $p \in (0, p)$. From Lemma 3.9, we have

$$\gamma_i \int_0^1 \left[ G^{-1}(t) - F^{-1}(t) \right] (1 - t)^{\gamma_i} \, dt \geq 0 \quad \text{for all } p \in (0, 1),$$

or, equivalently,

$$\int_0^1 F^{-1}(t) \, dt \left( 1 - (1 - t)^{\gamma_i} \right) \leq \int_0^1 G^{-1}(t) \, dt \left( 1 - (1 - t)^{\gamma_i} \right) \quad \text{for all } p \in (0, 1).$$

Now, by using some of the previous arguments, we find $X_{(1, n', \tilde{m}', k)} \preceq_{icx} Y_{(1, n', \tilde{m}', k)}$. $\square$

We shall now establish the main results of this section.

**Theorem 3.11.** Let $X$ and $Y$ be two continuous random variables with distribution functions $F$ and $G$, respectively. Let

$$X = (X_{(1, n, \tilde{m}, k)}, \ldots, X_{(n, n, \tilde{m}, k)}) \quad \text{and} \quad Y = (Y_{(1, n, \tilde{m}, k)}, \ldots, Y_{(n, n, \tilde{m}, k)})$$

be two random vectors of generalized order statistics from $F$ to $G$, respectively, with $m_i \geq -1$ for all $i$. If $X_{(1, n, \tilde{m}, k)} \preceq_{icx} Y_{(1, n, \tilde{m}, k)}$, then

$$X_{(r, n, \tilde{m}, k)} \preceq_{icx} Y_{(r, n, \tilde{m}, k)} \quad \text{for } r = 2, \ldots, n.\]$$

**Proof.** Let us consider an increasing convex function $\phi$, and prove that

$$E \left[ \phi \left( X_{(r, n, \tilde{m}, k)} \right) \right] \leq E \left[ \phi \left( Y_{(r, n, \tilde{m}, k)} \right) \right] \quad \text{for } r = 2, \ldots, n.$$

For establishing this, let us consider the function $h = F^{-1}_{(r, n, \tilde{m}, k)} G_{(r, n, \tilde{m}, k)} = F^{-1} G$ for $r = 1, \ldots, n$, so that

$$\phi \left( X_{(r, n, \tilde{m}, k)} \right) \equiv_{as} \phi \left( h \left( Y_{(r, n, \tilde{m}, k)} \right) \right), \quad r = 1, 2, \ldots, n.$$

For $r = 2, \ldots, n$, we have

$$E \left[ \phi \left( X_{(r, n, \tilde{m}, k)} \right) \right] = E \left[ \phi \left( h \left( Y_{(r, n, \tilde{m}, k)} \right) \right) \right] = \int E \left[ \phi \left( h \left( Y_{(r, n, \tilde{m}, k)} \right) \right) \right] \, \gamma_i \cdot g_{(r-1, n, \tilde{m}, k)}(t) \, dt. \quad (3.7)$$

where $g_{(r-1, n, \tilde{m}, k)}(t)$ is the density function of $Y_{(r-1, n, \tilde{m}, k)}$. Now, we claim that

$$\left[ Y_{(r, n, \tilde{m}, k)} \mid Y_{(r-1, n, \tilde{m}, k)} = t \right] \overset{d}{=} \left[ Y_{(1, n-r+1, \tilde{m}', k)} \mid Y_{(1, n-r+1, \tilde{m}', k)} > t \right]. \quad (3.8)$$

where $\tilde{m}' = (m'_1, \ldots, m'_{n-r})$ (recall that $\tilde{m} = (m_1, \ldots, m_{n-1})$) is such that

$$m'_j = m_{n-j} \quad \text{for } j = 1, \ldots, n-r. \quad (3.9)$$

In order to prove this claim, recall from (2.2) in Belzunce et al. [11] that

$$P \left[ Y_{(r, n, \tilde{m}, k)} > x \mid Y_{(r-1, n, \tilde{m}, k)} > t \right] = \left( \frac{F(x)}{F(t)} \right)^{\gamma_i} \quad \text{for } x > t. \quad (3.10)$$

Since $F_{(1, n-r+1, \tilde{m}', k)}(x) = (\bar{F}(x))^{\gamma_i}$, we have

$$P \left[ Y_{(1, n-r+1, \tilde{m}', k)} > x \mid Y_{(1, n-r+1, \tilde{m}', k)} > t \right] = \left( \frac{\bar{F}(x)}{\bar{F}(t)} \right)^{\gamma_i} \quad \text{for } x > t. \quad (3.11)$$
From (3.9), it follows that
\[ \gamma'_1 = k + n - r + M'_r = k + n - r + M_r = \gamma_r \] (3.12)
and then (3.10)–(3.12) together prove the claim. On the other hand, from the assumption that \( m_i \geq -1 \) for all \( i \), it follows that
\[ m_1 + \cdots + m_r - 1 \geq 1 - r, \]
i.e.,
\[ M_1 - M_r \geq 1 - r \]
which implies that
\[ \gamma'_1 = k + n - r + M_r \leq k + n - 1 + M_1 = \gamma_1. \]

We can then use Lemma 3.10 to obtain that the assumption \( X_{(1,n,\tilde{m},k)} \leq_{idir} Y_{(1,n,\tilde{m},k)} \) implies
\[ X_{(1,n-r+1,\tilde{m}',k)} \leq_{idir} Y_{(1,n-r+1,\tilde{m}',k)} \]
which in turn implies, by Lemma 2.3, that
\[ E \left[ \phi(r_Y(Y_{(1,n-r+1,\tilde{m}',k)})) \mid Y_{(1,n-r+1,\tilde{m}',k)} > t \right] \leq E \left[ \phi(r_Y(Y_{(1,n-r+1,\tilde{m},k)})) \mid Y_{(1,n-r+1,\tilde{m},k)} > t \right]. \] (3.13)
where \( h = F_{r-1}^{-1}(1,n-r+1,\tilde{m}',k) = F^{-1}G. \)

Now, taking into account that \( \phi h \) is increasing, it follows from (3.8) to (3.13) that (3.7) is equivalent to
\[ \int \left\{ E \left[ \phi(r_Y(Y_{(1,n-r+1,\tilde{m},k)})) \mid Y_{(1,n-r+1,\tilde{m},k)} > t \right] g_{(r-1,n,\tilde{m},k)}(t) \right\} dt \leq \int \left\{ E \left[ \phi(r_Y(Y_{(1,n-r+1,\tilde{m}',k)})) \mid Y_{(1,n-r+1,\tilde{m}',k)} > t \right] g_{(r-1,n,\tilde{m},k)}(t) \right\} dt. \] (3.14)
By repeating the argument, we see that the RHS of (3.14) is equal to
\[ \int E \left[ \phi(r_Y(Y_{(1,n,\tilde{m},k)})) \mid Y_{(r-1,n,\tilde{m},k)} = t \right] g_{(r-1,n,\tilde{m},k)}(t) dt = E \left[ \phi(r_Y(Y_{(1,n,\tilde{m},k)})) \right], \]
which completes the proof of the theorem. \( \square \)

The preceding theorem can be extended to the comparison of two random vectors of GOS’s in the increasing directionally order as follows.

**Theorem 3.12.** Let \( X \) and \( Y \) be two continuous random variables with distribution functions \( F \) and \( G \), respectively. Let
\[ X = (X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)}) \quad \text{and} \quad Y = (Y_{(1,n,\tilde{m},k)}, \ldots, Y_{(n,n,\tilde{m},k)}) \]
be two random vectors of generalized order statistics from \( F \) and \( G \), respectively, with \( m_i \geq -1 \) for all \( i \). If \( X_{(1,n,\tilde{m},k)} \leq_{idir} Y_{(1,n,\tilde{m},k)} \), then
\[ X = (X_{(1,n,\tilde{m},k)}, \ldots, X_{(n,n,\tilde{m},k)}) \leq_{idir-cx} Y = (Y_{(1,n,\tilde{m},k)}, \ldots, Y_{(n,n,\tilde{m},k)}). \]

**Proof.** First, we observe that two random vectors of GOS’s with the same set of parameters and possibly based on different distributions have the same copula; see [9]. Moreover, any random vector of GOS’s is MTP2, and is therefore CI; see [22] for details. Hence, the required result follows from the preceding theorem and Theorem 2.4. \( \square \)

Therefore, if \( X_{(1,n,\tilde{m},k)} \leq_{idir} Y_{(1,n,\tilde{m},k)} \), then \( E[\phi(X)] \leq E[\phi(Y)] \) for any increasing directional convex function \( \phi : \mathbb{R}^n \rightarrow \mathbb{R} \) for which the involved expectations exist. Again, using Theorem 2.6, if we consider the function \( \phi(X_1, X_2, \ldots, X_n) = \sum_{i=1}^{n} \phi_i(X_i) \), where \( \phi_i \) is increasing convex for \( i = 1, \ldots, n \), then \( \sum_{i=1}^{n} \phi_i(X_{(1,n,\tilde{m},k)}) \leq_{idir} \sum_{i=1}^{n} \phi_i(Y_{(1,n,\tilde{m},k)}). \)

**4. Applications and examples**

In this section, we describe some special cases of GOS’s through an appropriate selection of the parameters, where the results established in the preceding sections can be applied. For a detailed description of these special cases, interested
It is evident that this is a particular case of the joint density function of generalized order statistics based on $F$ and $G$, respectively. If $X_i$ and $Y_i$ are the ordered random lifetimes resulting from $X$ and $Y$ under such a minimal repair policy. Then, the joint density function of $(X_{(1:n)}, \ldots, X_{(n:n)})$ is given by

$$f(t_1, \ldots, t_n) = n! \prod_{j=1}^{n} p_j f(t_j) (F(t_j))^{(n-j)+1} p_{j-1}^{(n-j)p_{j}+1} \quad \text{for } 0 \leq t_1 \leq \cdots \leq t_n.$$ 

It is evident that this is a particular case of the joint density function of generalized order statistics based on $F$ for the choice of parameters $k = p$ and $m_i = (n - j + 1) p_j - (n - j) p_{j+1} - 1$.

In all three cases, Theorem 3.12 states that given two random vectors of order statistics (progressively Type-II censored order statistics or order statistics under multivariate imperfect repair), if the minimums from the random vectors are ordered in the increasing convex order, then the random vectors themselves are ordered in the increasing directionally convex order. Specifically, we have the following result.

**Theorem 4.13.** Let $X$ and $Y$ be two continuous random variables with distribution functions $F$ and $G$, respectively. Let $(X_{(1:n)}, \ldots, X_{(n:n)})$ and $(Y_{(1:n)}, \ldots, Y_{(n:n)})$ be two random vectors of order statistics (progressively Type-II censored order statistics or order statistics under multivariate imperfect repair) based on $F$ and $G$, respectively. If $X_{(1:n)} \leq_{idir} Y_{(1:n)}$, then

$$(X_{(1,n,m,k)}, \ldots, X_{(n,n,m,k)}) \leq_{idircx} Y = (Y_{(1,n,m,k)}, \ldots, Y_{(n,n,m,k)}).$$

### 4.2. Record values and k-record values

Setting $k = 1$ and $m_i = -1$ for all $i = 1, \ldots, n - 1$ in the generalized order statistics model, we obtain the random vector of the first $n$ record values or the first $n$ epoch times of a non-homogeneous Poisson process. Given a sequence of i.i.d. random variables with common distribution $F$, the record times are defined by

$$L(1) = 1,$$

$$L(n) = \min(j > L(n-1)) X_j > X_{(n-1)}, \quad n = 2, 3, \ldots.$$ 

The sequence of record values is then defined as $X(n) = X_{(n)}$, $n = 1, 2, \ldots$.

A generalization of record values is the case in which $k \in \mathbb{N}$, resulting in the so-called $k$-records. Theorem 3.12 states that it is possible to compare two random vectors (or components) of these models when the first components in the two random vectors are ordered in the increasing convex order. Of particular interest is the case of record values. In this case, the first component is equally distributed as the distribution from which the record values are arising from. Consequently, the increasing convex order of the distributions on which the two random vectors are based on is a sufficient condition for the comparison of the vectors and in particular for the increasing convex order of record values from the two populations. This result strengthens Theorem 3.3 of Belzunce and Shaked [14] wherein the increasing convex order of two record values is presented under stronger conditions. In particular, the result is stated under the mean residual life order (which is stronger than the increasing convex order) of the distributions from which the records are arising from and the convexity of the mean residual life function of one of the two distributions. To be more specific, we have the following result.

**Theorem 4.14.** Let $X$ and $Y$ be two continuous random variables with distribution functions $F$ and $G$, respectively. Let $X_{(1)}, X_{(2)}, \ldots$ and $Y_{(1)}, Y_{(2)}, \ldots$ be the sequences of record values arising from $F$ and $G$, respectively. If $X \leq_{id} Y$, then

$$(X_{(1)}, \ldots, X_{(n)}) \leq_{idircx} (Y_{(1)}, \ldots, Y_{(n)}) \quad \text{for all } n = 1, 2, \ldots.$$
4.3. A Parametric example

Here, we provide an example from parametric families to which the results established in the preceding sections can be applied. First, we state the following sufficient condition for \( X_{(1,n,n,k)} \) and \( Y_{(1,n,n,k)} \) to be ordered in the increasing convex order. Let \( S(h(x)) \) denote the number of sign changes of a function \( h(x) \).

**Lemma 4.15.** Let \( X \) and \( Y \) be two absolutely continuous random variables with distribution functions \( F \) and \( G \), respectively. If \( S(F(x) - G(x)) \leq 1 \), with sequence \((-,-)\) when equality holds, and

\[
E \left[ X_{(1,n,n,k)} \right] \leq E \left[ Y_{(1,n,n,k)} \right],
\]

then

\[
X_{(1,n,n,k)} \leq_{icx} Y_{(1,n,n,k)}.
\]

**Proof.** Let \( F_{(1,n,n,k)} \) and \( G_{(1,n,n,k)} \) be the distribution functions of \( X_{(1,n,n,k)} \) and \( Y_{(1,n,n,k)} \), respectively. Since

\[
S(F(x) - G(x)) = S(G^{-1}(F(x)) - x) = S \left( F_{(1,n,n,k)}^{-1}(F(x)) - x \right) = S \left[ F_{(1,n,n,k)}(x) - G_{(1,n,n,k)}(x) \right],
\]

the result follows by applying Theorem 4.A.22(b) of Shaked and Shanthikumar [44] to the random variables \( X_{(1,n,n,k)} \) and \( Y_{(1,n,n,k)} \). \( \square \)

From a practical point of view, Lemma 4.15 can be useful in order to do inference.

**Example 4.16.** Let \( X \) and \( Y \) be two Weibull random variables, \( X \sim W(\alpha, \lambda) \) and \( Y \sim W(\beta, \mu) \), with respective survival functions \( F \) and \( G \), given by \( F(t) = e^{-\lambda t^\alpha} \), \( t \geq 0 \) and \( G(t) = e^{-\mu t^\beta} \), \( t \geq 0 \), respectively. It is easy to see that if \( \alpha \geq \beta \), then \( S(F - G) = 1 \) with sequence \((-,-)\). Now, let \( X_{(1,n,n,k)} \) and \( Y_{(1,n,n,k)} \) be the first GOS's based, respectively, on \( F \) and \( G \). Then, \( X_{(1,n,n,k)} \) and \( Y_{(1,n,n,k)} \) are also Weibull random variables given by

\[
X_{(1,n,n,k)} \sim W(\alpha, \lambda \gamma_k), \quad Y_{(1,n,n,k)} \sim W(\beta, \mu \gamma_k).
\]

Taking into account that

\[
E \left[ X_{(1,n,n,k)} \right] = \gamma_k^{-1/\alpha} E \left[ X \right]
\]

and

\[
E \left[ Y_{(1,n,n,k)} \right] = \gamma_k^{-1/\beta} E \left[ Y \right],
\]

it follows from Lemma 4.15 that

\[
\begin{align*}
X \sim W(\alpha, \lambda), \ Y \sim W(\beta, \mu) & \\
\alpha \geq \beta \quad \text{and} \quad E[X] \leq E[Y] \quad & \implies X_{(1,n,n,k)} \leq_{icx} Y_{(1,n,n,k)}.
\end{align*}
\]

5. Concluding remarks

In this paper, we have established some results about the increasing directionally convex order. We have then utilized these results to prove a general result about increasing directionally convex ordering of random vectors of generalized order statistics. This is a very general and interesting approach since the corresponding results for marginal distributions of generalized order statistics readily follow. It will, of course, be of great interest to see whether the orderings considered here can be extended from the random vector case to the matrix-variate case. It can be envisioned that in the latter case the orderings (such as CIS) could be defined in different ways, but a study of their properties and their implications would be interesting.

**Acknowledgments**

We are thankful to anonymous referees for their valuable comments which have improved the presentation and contents of this paper. The second author was supported by Ministerio de Educación y Ciencia under Grant MTM2009-08311 and Fundación Séneca (CARM 08811/P1/08). The third and fourth authors were supported by Ministerio de Educación y Ciencia under Grant MTM2009-08326 and Consejería de Economía Innovación y Ciencia grant P09-SEJ-4739.
References


