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COMMENTS ON THE FUNCTOR EXT†

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LET $\mathscr{C}$ be an abelian category with enough projective objects and C an object in $\mathscr{C}$. Suppose the functor F is a direct summand of the functor $\operatorname{Ext}^1(C, .)$. P. Freyd has shown in [4] that

THEOREM I. *If $\mathscr{C}$ has denumerable sums, then $F \approx \operatorname{Ext}^1(D, .)$ for some D in $\mathscr{C}$.*

In the same publication [1], I established

THEOREM II. *If the projective dimension of C is finite, then $F \approx \operatorname{Ext}^1(D, .)$ for some D in $\mathscr{C}$, regardless of whether $\mathscr{C}$ contains denumerable sums or not.*

The first section of this note is devoted to giving a unified proof of these results. This is followed by showing that there exist abelian categories with an object C such that $\operatorname{Ext}^1(C, .)$ has proper direct summands none of which is isomorphic to $\operatorname{Ext}^1(D, .)$ for any D in $\mathscr{C}$. The rest of the note is devoted to a preliminary investigation of the following question: Suppose M and C are modules over the commutative ring R. What sorts of modules can be submodules of $\operatorname{Ext}^1(M, C)$? It is shown that if M is a finitely generated module and P is a projective submodule of $\operatorname{Ext}^1(M, C)$, then $P = 0$. As a consequence we show that if R is noetherian and N is a finitely generated module of finite projective dimension, then N is isomorphic to a submodule of $\operatorname{Ext}^1(M, C)$ for some finitely generated module M if and only if $\operatorname{Hom}_R(N, R) = 0$.

§1. DIRECT SUMMANDS OF EXT¹(C, .)

We assume throughout this section that $\mathscr{C}$ is an abelian category with enough projective objects. Given two objects C and D in $\mathscr{C}$ we shall denote the abelian group of maps from C to D by (C, D) and the functor $X \mapsto (C, X)$ by $(C, .)$. It is well known that each $f \in (C, D)$ gives rise to a map $\operatorname{Ext}^1(D, .) \to \operatorname{Ext}^1(C, .)$ and that the induced map $(C, D) \to (\operatorname{Ext}^1(D, .), \operatorname{Ext}^1(C, .))$ is an epimorphism whose kernel consists of those maps which can be factored through a projective object (see [5] for instance). Given an $f \in (C, D)$ we shall denote its image in $(\operatorname{Ext}^1(D, .), \operatorname{Ext}^1(C, .))$ by the same symbol f.

Suppose F is a direct summand of the functor $\operatorname{Ext}^1(C, .)$. Then there is a map $f: C \to C$ such that $F \approx \operatorname{Ker}(\operatorname{Ext}^1(C, .) \to \operatorname{Ext}^1(C, .))$. If we let $P \to C$ be an epimorphism with P a projective object then the induced map on the direct sum $P + C \to C$ is an epimorphism which restricted to C gives f. Thus the map $\operatorname{Ext}^1(C, .) \to \operatorname{Ext}^1(P + C, .) = \operatorname{Ext}^1(C, .)$ is our

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original map f. If we denote the Ker$(P + C \to C)$ by A, then we obtain from the exact sequence $0 \to A \to P + C \to C \to 0$, the long exact sequence

$$(1.1)\qquad 0 \to F \to \operatorname{Ext}^1(C,.) \to \operatorname{Ext}^1(C,.) \to \operatorname{Ext}^1(A,.) \to \operatorname{Ext}^2(C,.) \to \cdots$$

Now in [1, see sections 3 and 4] it is shown that there exists a full subcategory $\check{\mathscr{C}}_0$ of the category $\check{\mathscr{C}}$ of all additive covariant functors from $\mathscr{C}$ to abelian groups with the following properties: a) $\operatorname{Ext}^1(C,.) \in \check{\mathscr{C}}_0$ for all $C \in \mathscr{C}$; b) If $G_1 \to G_2$ is in $\check{\mathscr{C}}_0$ then the kernel and cokernel of the map are in $\check{\mathscr{C}}_0$. Thus $\check{\mathscr{C}}_0$ is an abelian category; c) $G \in \check{\mathscr{C}}_0$ is injective in $\check{\mathscr{C}}_0$ if and only if G is half exact. Since F is the Ker$(\operatorname{Ext}^1(C,.) \to \operatorname{Ext}^1(C,.))$ we have that $F \in \check{\mathscr{C}}_0$. Since the $\operatorname{Ext}^i(C,.)$ and $\operatorname{Ext}^i(A,.)$ are half exact functors in $\check{\mathscr{C}}_0$, they are injective objects in $\check{\mathscr{C}}_0$. Thus the exact sequence (1.1) is an injective resolution of F in $\check{\mathscr{C}}_0$. But F being a direct summand of $\operatorname{Ext}^1(C,.)$ is half exact and thus an injective object in $\check{\mathscr{C}}_0$. Therefore (1.1) is an injective resolution of an injective object and thus must split. Applying the following lemma to this long split sequence we obtain the formula:

$$(1.2)\quad \prod_{i\geq 1} Ext^i(C,.) + \prod_{i=2n+1} Ext^i(A,.) = F + \prod_{i\geq 1} Ext^i(C,.) + \prod_{i=2n+2} Ext^i(A,.)\qquad where\ n = 0, 1, \ldots$$

and Π stands for direct product.

LEMMA 1.3 *Let $\mathscr{D}$ be an abelian category with denumerable products and let* $0 \to A_0 \to A_1 \to \cdots \to A_n \to \cdots$ *be an exact sequence in $\mathscr{D}$ which splits. Then* $\prod_{i=2n+1} A_i \approx \prod_{i=2n} A_i$ where $n = 0, 1, 2, \ldots$.

Proof. Let $B_i = \operatorname{Im}(A_i \to A_{i+1})$ for $i > 0$. Then

$$\begin{aligned} A_0 \times \Pi B_i &\approx (A_0 \times B_1) \times (B_2 \times B_3) \times \cdots \times (B_{2n} \times B_{2n+1}) \times \cdots \\ &\approx A_1 \times A_3 \times \cdots \times A_{2n+1} \times \cdots. \end{aligned}$$

But arranging the terms differently, we obtain that

$$\begin{aligned} A_0 \times \Pi B_i &\approx A_0 \times (B_1 \times B_2) \times (B_3 \times B_4) \times \cdots (B_{2n-1} \times B_{2n}) \times \cdots \\ &\approx A_0 \times A_2 \times A_4 \times \cdots \times A_{2n} x \ldots, \end{aligned}$$

which gives our desired result.

Now let X and Y be projective resolutions of C and A respectively. Suppose we denote the coker $(X_{i+1} \to X_i)$ by C_i for $i = 0, \ldots$ and the coker $(Y_{i+1} \to Y_i)$ by A_i for $i = 0, \ldots$. Then $\operatorname{Ext}^{j+1}(C,.) \approx \operatorname{Ext}^1(C_j,.)$ and $\operatorname{Ext}^{j+1}(A,.) \approx \operatorname{Ext}^1(A_j,.)$ for $j = 0, 1, \ldots$. Assume now that the direct sums of the C_i and A_i for i even and for i odd are in $\mathscr{C}$. Then using the fact that $\operatorname{Ext}^1(\sum X_i,.) \approx \Pi \operatorname{Ext}^1(X_i,.)$, the formula (1.2) becomes

$$(1.4)\qquad \operatorname{Ext}^1\left(\sum_{i=0}^{\infty} C_i + \sum_{\substack{i=2n\\ n=o}}^{\infty} A_i,.\right) = F + \operatorname{Ext}^1\left(\sum_{i=0}^{\infty} C_i + \sum_{\substack{i=2n+1\\ n=o}}^{\infty} A_i,.\right).$$

From this it follows from [5] that if $P \to \sum_{i=0}^{\infty} C_i + \sum_{i=2n+9}^{\infty} A_i \to 0$ is exact with P projective, then $F \approx \operatorname{Ext}^1(B,.)$ where B is a direct summand of $\sum_{i=0}^{\infty} C_i + \sum_{\substack{i=2n\\ n=0}}^{\infty} A_i + P$.

Clearly if $\mathscr{C}$ has denumerable direct sums then the direct sums of the C_i and A_i for i even and odd will always exist in $\mathscr{C}$.

Thus we have that if F is a direct summand of $\operatorname{Ext}^1(C,.)$ with C in $\mathscr{C}$, and $\mathscr{C}$ has denumerable direct sums then $F \approx \operatorname{Ext}^1(B,.)$ for some $B \in \mathscr{C}$, the result of P. Freyd cited in the introduction. On the other hand suppose that the $pdC = n < \infty$. Then it follows from the exact sequence $0 \to A \to P + C \to C \to 0$, that the $pdA \leq n < \infty$. Thus we may choose in this case projective resolutions X and Y for C and A such that only a finite number of C_i and A_i are different from zero. Hence the direct sums of the C_i and A_i for i even and for i odd also will exist in $\mathscr{C}$. Therefore if the $pdC < n$, then we have that $F \approx \operatorname{Ext}^1(B,.)$ for some $B \in \mathscr{C}$ regardless of whether $\mathscr{C}$ has denumerable sums or not. Thus our proof of Theorems I and II is completed.

§2. CATEGORIES OF MODULES

The main object of this section is to examine for certain types of rings the following refinement of the problem considered in the previous section: If M is a finitely generated module, then is a direct summand of $\operatorname{Ext}^1(M,.)$ necessarily isomorphic to an $\operatorname{Ext}^1(N,.)$ with N a finitely generated module? It will be shown that even for noetherian rings, the answer to the above question can be no, thus supplying an example of an abelian category with enough projectives such that a direct summand of an $\operatorname{Ext}^1(C,.)$ need not be isomorphic to an $\operatorname{Ext}^1(D,.)$ for some D in the category. Our discussion of the ring situation will be based on the following general observations.

Let $\mathscr{C}$ be an abelian category with enough projective objects. Let $\mathscr{C}^*$ be the full subcategory consisting of those objects C such that $f\colon C \to C$ is an isomorphism if and only if the induced map $\operatorname{Ext}^1(C,.) \to \operatorname{Ext}^1(C,.)$ is an isomorphism.

LEMMA 2.1. *The category $\mathscr{C}^*$ has the following properties:*

a) *If $C \in \mathscr{C}^*$, then every direct summand of C is in $\mathscr{C}^*$.*

b) *If $V \in \mathscr{C}^*$, then zero is the only projective direct summand of C.*

c) *$C \in \mathscr{C}^*$ if and only if $\mathfrak{a}(C)$ is in the radical of* End(C), *where $\mathfrak{a}(C)$ is the two sided ideal in the endomorphism ring of C consisting of those endomorphisms which factor through projective objects.*

Proof. a) Suppose the direct sum $C_1 + C_2$ is in $\mathscr{C}^*$ and $f\colon C_1 \to C_1$ induces an isomorphism on $\operatorname{Ext}^1(C_1,.) \to \operatorname{Ext}^1(C_1,.)$. Then extending f to $C_1 + C_2$ by defining the map on C_2 to be the identity on C_2, we have that $\operatorname{Ext}^1(C_1,.) + \operatorname{Ext}^1(C_2.) \to \operatorname{Ext}^1(C_1,.) + \operatorname{Ext}^1(C_2,.)$ is an isomorphism, from which it follows that $C_1 + C_2 \to C_1 + C_2$ is an isomorphism. Thus f is an isomorphism.

b) Clearly (0) is the only projective object in $\mathscr{C}^*$. Thus b) follows from a).

c) It is well known that the natural map $\operatorname{End}(C) \to \operatorname{End}(\operatorname{Ext}^1(C,.))$ is an anti-epimorphism (reverses multiplication) with kernel $\mathfrak{a}(C)$. Thus $\operatorname{End}(C)/\mathfrak{a}(C)$ is isomorphic to the opposite ring of $\operatorname{End}(\operatorname{Ext}^1(C,.))$. Suppose $C \in \mathscr{C}^*$ and $f \in \mathfrak{a}(C)$. Then $1 + f\colon C \to C$ induces the identity on $\operatorname{Ext}^1(C,.)$. Thus $1+f$ is an isomorphism or a unit in $\operatorname{End}(C)$. Therefore we have that $\mathfrak{a}(C) \subset \operatorname{rad}(\operatorname{End}(C))$.

On the other hand suppose $\mathfrak{a}(C) \subset \operatorname{rad}(\operatorname{End}(C))$. If $f\colon C \to C$ induces an isomorphism on $\operatorname{Ext}^1(C,.)$, then the image of f in $\operatorname{End}(C)/\mathfrak{a}(C)$ is a unit. But since $\mathfrak{a}(C) \subset \operatorname{rad}(\operatorname{End}(C))$, it follows that f is a unit in $\operatorname{End}(C)$, i.e. an isomorphism.

The main point of this preliminary discussion is

PROPOSITION 2.2. *Let $\mathscr{D}$ be a full subcategory of $\mathscr{C}$ satisfying* a) *a direct sum $C_1 + C_2$ is in $\mathscr{D}$ if and only if each C_i is in $\mathscr{C}$ and* b) *for each C in $\mathscr{D}$ there exists an object $C^* \in \mathscr{D}^*$ such that* $\operatorname{Ext}^1(C,.) \approx \operatorname{Ext}^1(C^*,.)$. *Then an object X in $\mathscr{D}$ has the property that each direct summand of* $\operatorname{Ext}^1(X,.)$ *is isomorphic to* $\operatorname{Ext}^1(Y,.)$ *for some Y in $\mathscr{D}$ if and only if every idempotent in* $\operatorname{End}(X)/\mathfrak{a}(X)$ *is the image of an idempotent in* $\operatorname{End}(X)$. *Thus, in case such a Y exists it can be chosen to be a direct summand of X.*

Proof. Clearly if every idempotent in $\operatorname{End}(X)/\mathfrak{a}(X)$ can be lifted to $\operatorname{End}(X)$, then every direct summand of $\operatorname{Ext}^1(X,.)$ is isomorphic to $\operatorname{Ext}^1(Y,.)$ with Y a direct summand of X.

Suppose now that each direct summand of $\operatorname{Ext}^1(X,.)$ is isomorphic to $\operatorname{Ext}^1(Y,.)$ with Y in $\mathscr{D}$. Suppose e is an idempotent in $\operatorname{End}(X)/\mathfrak{a}(X)$. Then the kernel F of the induced map $\operatorname{Ext}^1(X,.) \xrightarrow{e} \operatorname{Ext}^1(X,.)$ is direct summand of $\operatorname{Ext}^1(X,.)$. Therefore we know that $F \approx \operatorname{Ext}^1(Y,.)$ for some $Y \in \mathscr{D}$ which, in view of the hypothesis on $\mathscr{D}$, we can assume to be in $\mathscr{D}^*$. Thus we have maps $\operatorname{Ext}^1(Y,.) \xrightarrow{u} \operatorname{Ext}^1(X,.)$ and $\operatorname{Ext}^1(X,.) \xrightarrow{v} \operatorname{Ext}^1(Y,.)$ such that $vu =$ identity and $uv = e$. Now let $f\colon X \to Y$ and $g\colon Y \to X$ be maps which induce u and v respectively. Then $fg\colon Y \to Y$ is an isomorphism since Y is in $\mathscr{D}^*$ and the induced map vu is the identity on $\operatorname{Ext}^1(Y,.)$. Let $z = fg$. Then $z^{-1}f$ induces u on $\operatorname{Ext}^1(Y,.) \to \operatorname{Ext}^1(X,.)$ since z induces the identity on $\operatorname{Ext}^1(Y,.)$. Since $(gz^{-1}f)(gz^{-1}f) = gz^{-1}(fg)z^{-1}f = gz^{-1}f$, and the image of $gz^{-1}f$ in $\operatorname{End}(X)/\mathfrak{a}(X)$ is e, we have found our desired idempotent. The last part of the proposition follows trivially.

We now turn our attention to the following situation. Λ is a ring with radical $\mathfrak{r}$ such that the semi-simple ring $\Lambda/\mathfrak{r}$ has minimum condition. $\mathscr{C}$ is to be the category of (left) Λ-modules and $\mathscr{D}$ the category of finitely generated modules. Our first aim is to show that if every idempotent in $\Lambda/\mathfrak{r}$ is the image of an idempotent in Λ, then $\mathscr{D}$ satisfies the hypothesis of Proposition 2.2. From now on we shall assume that all our rings Λ have the property that $\Lambda/\mathfrak{r}$ has minimum condition where $\mathfrak{r}$ is the radical of Λ and that all Λ-modules are finitely generated. We begin with a sketch of essentially well known results, stated in a form particularly well suited to our present purposes.

LEMMA 2.3. *Let P_1 and P_2 be projective Λ-modules. A map $f\colon P_1 \to P_2$ obviously induces a map $\bar{f}\colon P_1/\mathfrak{r}P_1 \to P_2/\mathfrak{r}P_2$. The homomorphism* $\operatorname{Hom}_\Lambda(P_1, P_2) \to \operatorname{Hom}_{\Lambda/\mathfrak{r}}(P_1/\mathfrak{r}P_1\ \ P_2/\mathfrak{r}P_2)$ *defined by $f \to \bar{f}$ has the following properties:*

a) *It is an epimorphism;*

b) *If $\bar{f}\colon P_1/\mathfrak{r}P_1 \to P_2/\mathfrak{r}P_2$ is an epimorphism, then $f\colon P_1 \to P_2$ is an epimorphism, which splits, i.e. the exact sequence* $0 \to \operatorname{Ker} f \to P_1 \xrightarrow{f} P_2 \to 0$ *splits;*

c) *If $\bar{f}\colon P_1/\mathfrak{r}P_1 \to P_2/\mathfrak{r}P_2$ is an isomorphism, then $f\colon P_1 \to P_2$ is an isomorphism;*

d) *If* $\bar{f}: P_1/\mathfrak{r}P_1 \to P_2/\mathfrak{r}P_2$ *is a monomorphism, then* $f: P_1 \to P_2$ *is a monomorphism which splits, i.e. the exact sequence* $0 \to P_1 \xrightarrow{f} P_2 \to \operatorname{Cok} f \to 0$ *splits.*

Proof. a) Trivial consequence of P_1 being projective.

b) and c) Since $\bar{f}: P_1/\mathfrak{r}P_1 \to P_2/\mathfrak{r}P_2$ is an epimorphism, we have that the composite $P_1 \to P_2 \to P_2/\mathfrak{r}P_2$ is an epimorphism. Since P_2 is finitely generated, it follows by Nakayama's lemma that $P_1 \to P_2$ is an epimorphism. Since P_2 is projective, we have that the exact sequence $0 \to K \to P_1 \xrightarrow{f} P_2 \to 0$ splits where $K = \operatorname{Ker} f$. Thus K is a finitely generated projective module and the sequence $0 \to K/\mathfrak{r}K \to P_1/\mathfrak{r}P_1 \xrightarrow{\bar{f}} P_2/\mathfrak{r}P_2 \to 0$ is exact. Thus if $\bar{f}$ is an isomorphism, then $K/\mathfrak{r}K = 0$. Since K is finitely generated we have by Nakayama's lemma that $K = 0$. Thus if $\bar{f}$ is an isomorphism we have that f is an isomorphism.

d) Suppose $f: P_1 \to P_2$ has the property that $\bar{f}: P_1/\mathfrak{r}P_1 \to P_2/\mathfrak{r}P_2$ is a monomorphism. Since $\Lambda/\mathfrak{r}$ is a semi-simple artin ring, the monomorphism $\bar{f}$ splits, i.e. there is a map $h: P_2/\mathfrak{r}P_2 \to P_1/\mathfrak{r}P_1$ such that $h\bar{f} =$ identity. By a) we know there is a map $g: P_1 \to P_2$ such that $\bar{g} = h$. Thus by *c*) we have that the composite $gf: P_1 \to P_1$ is an isomorphism, since $\overline{gf} = \bar{g}\bar{f}$ is the identity on $P/\mathfrak{r}P$. Thus the map $f: P_1 \to P_2$ is a monomorphism which splits.

PROPOSITION 2.4. *The following are equivalent statements about* Λ.

a) *Every idempotent in* $\Lambda/\mathfrak{r}$ *is the image of an idempotent in* Λ .

b) *Given any* $\Lambda/\mathfrak{r}$-*module* M, *there is a projective* Λ-*module* P *such that* $P/\mathfrak{r}P \approx M$.

c) *Let* P *be a projective* Λ-*module and* M *a submodule of* P *such that* $M \not\subset \mathfrak{r}P$. *Then* M *contains a non-zero projective module* P' *which is a direct summand of* P *and thus of* M.

d) *A projective module* P *has no proper direct summands if and only if* $P/\mathfrak{r}P$ *is a simple module.*

Proof. a) $\Rightarrow$ *b*). Essentially a) implies that given any simple $\Lambda/\mathfrak{r}$-module M, there is a projective module P such that $P/\mathfrak{r}P \approx M$, since the simple modules are of the form $(\Lambda/\mathfrak{r})\, e$ for some idempotent e in $\Lambda/\mathfrak{r}$ and e can be lifted to an idempotent in Λ. But all $\Lambda/\mathfrak{r}$-modules are direct sums of simple modules, so we are done.

b) $\Rightarrow$ c). Since $M \not\subset \mathfrak{r}P$ we know that $M/\mathfrak{r}P \cap \mathrm{M} \subset P/\mathfrak{r}P$ is a nontrivial semisimple module. Thus there is a projective module P' such that $P'/\mathfrak{r}P' \approx M/\mathfrak{r}P \cap M$. Since P' is projective, there is a map $P' \to M$ such that the induced map $P'/\mathfrak{r}P' \to M/\mathfrak{r}P \cap M$ is our given isomorphism. Now the induced map $P'/\mathfrak{r}P' \to P/\mathfrak{r}P$ of the composite map $P' \to M \to P$ is the composition of the monomorphisms $P'/\mathfrak{r}P' \to M/\mathfrak{r}P \cap M \to P/\mathfrak{r}P$. Thus $P'/\mathfrak{r}P' \to P/\mathfrak{r}P$ is a monomorphism. Therefore the map $P' \to P$ is a monomorphism which splits (see Lemma 2.3 part *c*)), which gives us our desired result.

c) $\Rightarrow$ d) Suppose P is a projective module with no proper direct summands. Let $x \in P - \mathfrak{r}P$. Then let $M = \Lambda x$.

Since $M \not\subset rP$, we know that M contains a nontrivial projective module P' which is a direct summand of P. Thus $P' = M = P$, since P contains no proper direct summands. Therefore we have that P is generated by any $x \in P - \mathfrak{r}P$. Thus $P/\mathfrak{r}P$ is generated by any non-zero element which means that $P/\mathfrak{r}P$ is simple.

d) $\Rightarrow$ a). We first observe that if a module M is the direct sum $M_1 + M_2 + \cdots + M_n$ then each M_i is finitely generated and thus the number of non-zero M_i is at most equal to the length of the finitely generated semi-simple module $M/\mathfrak{r}M$. Thus M satisfies both chain conditions for submodules which are direct summands. Therefore M can be written as a direct sum of modules with no proper direct summands.

Suppose $M_1 + \cdots + M_n$ is a direct sum decomposition of Λ where each M_i has no proper direct summands. Then by d) we know that each $M_i/\mathfrak{r}M_i$ is a simple module. Thus $M_1/\mathfrak{r}M_1 + \cdots + M_n/\mathfrak{r}M_n$ is a direct sum decomposition of $\Lambda/\mathfrak{r}$ into simple modules. Since, up to isomorphism, such a decomposition is unique, we know that given any simple module it is isomorphic to $M_i/\mathfrak{r}M_i$ for some i. From this it follows that given any semi-simple module N there is a projective Λ-module P such that $P/\mathfrak{r}P \approx N$ (see a)$\Rightarrow$ b)).

Suppose now that e is an idempotent in $\Lambda/\mathfrak{r}$. Then $\Lambda/\mathfrak{r} = (\Lambda/\mathfrak{r})(1-e) + (\Lambda/\mathfrak{r})e$. Let P_1 and P_2 be projective modules such that we have isomorphisms $P_1/\mathfrak{r}P_1 \approx (\Lambda/\mathfrak{r})$e and $P_2/\mathfrak{r}P_2 \approx (\Lambda/\mathfrak{r})(1-e)$. Then there exists an isomorphism (see Lemma 2.3) $P_1 + P_2 \approx \Lambda$ such that the induced map $P_1/\mathfrak{r}P_1 + P_2/\mathfrak{r}P_2 \approx (\Lambda/\mathfrak{r})e + (\Lambda/\mathfrak{r})(1-e)$ is the direct sum of our original isomorphisms. If we denote by J_1 and J_2 the images of P_1 and P_2 in Λ, then we have that $\Lambda = J_1 + J_2$ and $J_1/\mathfrak{r}J_1 = (\Lambda/\mathfrak{r})$e and $J_2/\mathfrak{r}J_2 = (\Lambda/\mathfrak{r})(1-e)$. Let $1 = f_1 + f_2$ with the $f_i \in J_i$. Then the f_i are idempotent elements. Since $1 = \bar{f}_1 + \bar{f}_2$ mod $\mathfrak{r}$ (where, $\bar{f}_i$ are the images of $f \in \Lambda/\mathfrak{r}$) and $\bar{f}_1 e(\Lambda/\mathfrak{r})e$ and $f_2 \in (\Lambda/\mathfrak{r})(1-e)$, it follows that $\bar{f}_1 = e$ and $\bar{f}_2 = 1-e$ in $\Lambda/\mathfrak{r}$, which finishes the proof.

We shall say that a ring Λ is an S.B.I. ring (suitable for building idempotents) if Λ satisfies any of the conditions given in Proposition 2.4.

Given a module M we shall denote by $\mathfrak{t}(M)$, the trace ideal of M, the two sided ideal in Λ consisting of the image of all maps $f: M \to \Lambda$. Then we have

PROPOSITION 2.5. *Suppose Λ is a S.B.I. ring. For a Λ-module M, the following statements are equivalent:*

a) $\mathfrak{t}(M) \subset \mathfrak{r}$

b) $\mathfrak{a}(M) \subset \mathrm{rad}(\mathrm{End}(M))$, *where $\mathfrak{a}(M)$ is the two sided ideal consisting of all endomorphisms which factor through projectives.*

c) *M has no non-trivial projective direct summands.*

Proof. a) $\Rightarrow$ b). Let $f \in \mathfrak{a}(M)$. Then if $F \xrightarrow{h} M \to 0$ is an epimorphism with F free, we know that there is a $g: M \to F$ such that $f = hg$. But the image of g must be contained in $\mathfrak{r}F$ since $\mathfrak{t}(M) \subset \mathfrak{r}$. Therefore the image of f is contained in $\mathfrak{r}M$. Thus $1 + f: M \to M$ is an epimorphism, since $\overline{1+f}: M/\mathfrak{r}M \to M/\mathfrak{r}M$ is the identity (by Nakayama's lemma). From the exact sequence $0 \to K \to M \xrightarrow{1+f} M \to 0$, we deduce the exact sequence of functors $0 \to (M,.) \to (M,.) \to (K,.) \to \mathrm{Ext}^1(M,.) \to \mathrm{Ext}^1(M,.) \to \cdots$.

Since $f: M \to M$ factors through a projective module, we have that $1 + f: \mathrm{Ext}^i(M,.) \to \mathrm{Ext}^i(M,.)$ is the identity for $i > 0$. Thus $\mathrm{Ext}^1(K,.) = 0$ and the map $(M,.) \to (K,.)$ is an epimorphism. From this it follows that the sequence $0 \to K \to M \to M \to 0$ splits and K is projective. If $K \neq 0$, then $K/\mathfrak{r}K \neq 0$ since K is finitely generated (being a direct summand of M). Thus we have a non-trivial map $K/\mathfrak{r}K \to \Lambda/\mathfrak{r}$ which can be lifted to a map of $K \to \Lambda$ whose image will not be in $\mathfrak{r}$. Since K is a direct summand of M, this map $K \to \Lambda$ can be extended to $M \to \Lambda$, which would contradict the fact that $\mathfrak{t}(M) \subset \mathfrak{r}$. Thus $K = 0$ or $1 + f$ is an isomorphism. Since this holds for any $f \in \mathfrak{a}(M)$ we have that $\mathfrak{a}(M) \subset \text{rad}(\text{End}(M))$.

b) $\Rightarrow$ c) See Lemma 2.1

c) $\Rightarrow$ a) Suppose we have a map $f : \text{M} \to \Lambda$ with $f(M) \not\subset \mathfrak{r}$. Then by Proposition 2.4, we know there exists a non-trivial projective direct summand P of $f(M)$. Thus we have that there is an epimorphism $M \to P \to 0$, which means that M contains P as a direct summand. Thus if M contains no non-trivial projective direct summands, then $\mathfrak{t}(M) \subset \mathfrak{r}$.

Note. In the proofs that a) $\Rightarrow$ b) $\Rightarrow$ c) we did not use the fact that Λ was an S.B.I. ring. In the proof c) $\Rightarrow$ a) something slightly weaker than Λ being an S.B.I. ring was used, namely if J is an ideal in Λ not contained in $\mathfrak{r}$, then J contains a non-trivial projective direct summand rather than Λ contains a non-trivial direct summand in J. This condition is satisfied automatically if every ideal in Λ is projective (i.e. gl. dim $\Lambda \leq 1$) even if Λ is not an S.B.I. ring. It would be interesting to know if there are examples of rings satisfying this weaker condition other than those already given.

Suppose that Λ is an S.B.I. ring, $\mathscr{C}$ the category of all Λ-modules (finitely generated or not) and $\mathscr{D}$ the category of all finitely generated Λ-modules. Clearly $\mathscr{D}$ satisfies the first condition of Proposition 2.2, namely $M_1 + M_2$ is in $\mathscr{D}$ if and only if M_1 and M_2 are in $\mathscr{D}$. Suppose $M \in \mathscr{D}$. Since M satisfies the descending chain on direct summands, we see that $M = M^* + P$ where M^* is a module without any non-trivial projective direct summands and P is projective. Thus we have that $\text{Ext}^1(M, .) \approx \text{Ext}^1(M^*, .)$. But by Proposition 2.5 it follows that since M has no non-trivial projective direct summands, $M \in \mathscr{D}^*$ (i.e. $\mathfrak{a}(M) \subset \text{rad}(\text{End}(M))$). Thus $\mathscr{D}$ satisfies the hypothesis of Proposition 2.2. We therefore obtain the following reformulation of Proposition 2.2.

THEOREM 2.6. *Let Λ be an S.B.I. ring. Then a (finitely generated) module M has the property that each direct summand of Ext^1(M, .) is isomorphic to* $\text{Ext}^1(N, .)$ *for some finitely generated module N if and only if every idempotent in* $\text{End}(M)/\mathfrak{a}(M)$ *is the image of an idempotent in* $\text{End}(M)$. *In case such an N exists, it can be chosen to be a direct summand of M.*

An important class of rings Λ which have the property that $\Lambda/\mathfrak{r}$ satisfies the minimum condition can be constructed as follows. Suppose that R is a commutative, noetherian semi-local ring (only a finite number of maximal ideals), then every R-algebra which is a finitely generated R-module will have this property. Further, if R is complete in its radical-topology and Λ is an R-algebra which is a finitely generated R-module, then Λ is complete in its $\mathfrak{r}$-adic topology and thus an S.B.I. ring. Finally it should be observed that if M is a finitely generated Λ-module, then $\text{End}_\Lambda(M)$ is a finitely generated R-module, since $\text{End}_\Lambda(M)$ is an R-submodule of the finitely generated R-module $\text{End}_R(M)$. From now on we will assume that R is a commutative, semi-local, noetherian ring and all R-algebras are finitely generated R-modules.

As an easy consequence of Theorem 2.6, we have

PROPOSITION 2.7. *Suppose Λ is an R-algebra and R is complete. If M is a Λ-module (finitely generated of course), and a functor F is a direct summand of* $\operatorname{Ext}^1(M, .)$, *then* $F \approx \operatorname{Ext}^1(N, .)$ *for some direct summand N of M.*

Proof. Without any loss in generality we may assume that M has no non-trivial projective direct summands. Since M is a finitely generated Λ-module, we know by our previous remarks, that $\operatorname{End}_\Lambda(M)$ is an R-algebra which is a finitely generated R-module and thus an S.B.I. ring since R is assumed complete. Thus every idempotent in $\operatorname{End}_\Lambda(M)/\mathfrak{a}(M)$ can be lifted to $\operatorname{Enp}_\Lambda(M)$ since $\mathfrak{a}(M) \subset \operatorname{rad} \operatorname{End}_\Lambda(M)$. Applying Theorem 2.6 gives us the desired conclusion.

When R is not complete, things are not quite so simple as can be seen from the following result.

PROPOSITION 2.8. *Suppose Λ is an R-algebra with R local which is an S.B.I. ring. Let M be an Λ-module with the property that $M_\mathfrak{p} = M \otimes_R R_\mathfrak{p}$ be $\Lambda_\mathfrak{p}$-projective for all non-maximal prime ideals p in R and has no proper projective direct summands. Then the following are equivalent*

a) *$\hat{M}$ has no proper $\hat{\Lambda}$ direct summands where $\hat{M}$ and $\hat{\Lambda}$ are the completions of M and Λ.*

b) $\operatorname{End}(M)/\operatorname{rad}(\operatorname{End}(M))$ *is a division ring.*

c) $\operatorname{Ext}^1(M, .)$ *has no proper direct summands.*

Proof. If M is projective, then it is well known (and follows easily from Proposition 2.4) that $M \approx \Lambda e$ with e a primitive idempotent since $\Lambda e/\mathfrak{r}\Lambda e$ is simple. Under these circumstances it is well known that $\operatorname{End}(M) \approx e\Lambda e$ and that $e\Lambda e$ modulo its radical is a division ring (see [7, p. 57] for example). Assume now that M is not projective.

Since all modules are finitely generated over noetherian rings, one can see, using standard localization arguments, that $M_\mathfrak{p}$ is $\Lambda_\mathfrak{p}$-projective for $\mathfrak{p}$ not maximal if and only if $\operatorname{End}(M)/\mathfrak{a}(M)$ has minimum condition. Thus we know that $\operatorname{End}(M)/\mathfrak{a}(M)$ is an S.B.I. ring. Since $\operatorname{End}_\Lambda(M)/\mathfrak{a}(M)$ is isomorphic to the opposite ring of $\operatorname{End}(\operatorname{Ext}^1(M, .))$, we know that $\operatorname{Ext}^1(M, .)$ has no proper direct summands if and only if $\operatorname{End}_\Lambda(M)/\mathfrak{a}(M)$ has no non-trivial idempotents. Since M has no non-trivial projective direct summands, we know that $\mathfrak{a}(M) \subset \operatorname{rad}(\operatorname{End}(M))$. Since $\operatorname{End}_\Lambda(M)/\mathfrak{a}(M)$ has minimum condition, it follows that $\operatorname{End}_\Lambda(M)/\mathfrak{a}(M)$ has no non-trivial idempotents if and only if $\operatorname{Enf}_\Lambda(M)/\operatorname{rad}(\operatorname{End}(M))$ is a division ring. Thus we have established the equivalence of b) and c).

Since $\operatorname{End}_\Lambda(M)/\mathfrak{a}(M)$ has minimum condition we know that $\operatorname{End}_{\hat{\Lambda}}(\hat{M})/\mathfrak{a}(\hat{M}) \approx \operatorname{End}_\Lambda(M)/\mathfrak{a}(M)$. Since $\operatorname{End}_{\hat{\Lambda}}(\hat{M})$ is an S.B.I. ring, it follows that $\hat{M}$ has no proper direct summands if and only if $\operatorname{End}_{\hat{\Lambda}}(\hat{M})/\mathfrak{a}(\hat{M})$ has no proper idempotents. Thus $\hat{M}$ has no proper $\hat{\Lambda}$ direct summands if and only if $\operatorname{End}_\Lambda(M)/\mathfrak{a}(M)$ has no proper idempotents, thus establishing the equivalence of a) and b).

Suppose now that R is a one-dimensional local ring whose integral closure S is a finite R-module and S is not local. For example, the local ring of a branch point, on an algebraic curve. Since S is in the field of quotients of R, we know that S contains no proper direct summands. On the other hand $\hat{S}$ is the direct sum of local rings, one for each maximal ideal. Since (0) is the only non-maximal prime ideal in R, we have that $S_{\mathfrak{p}}$ is $R_{\mathfrak{p}}$ free for all non-maximal prime ideals $\mathfrak{p}$. Thus we have by the above proposition that $\mathrm{Ext}_R{}^1(S, .)$ has direct summands even though S has no proper direct summands. Since R is local, it is an S.B.I. ring. Thus by theorem 2.6, none of the direct summands of $\mathrm{Ext}_R{}^1(S, .)$ are isomorphic to an $\mathrm{Ext}_R{}^1(N, .)$ with N finitely generated. Therefore the category of finitely generated R-modules is an example of an abelian category with the property that there are direct summands of $\mathrm{Ext}^1(C, .)$ for some object C in the category which are not isomorphic to $\mathrm{Ext}^1(D, .)$ for any D in the category.

We now end this section with the following generalization of a result of Horrock [6].

PROPOSITION 2.9. *Let R be a local ring and Λ an R-algebra which is an S.B.I. ring. Let $\mathscr{E}$ be the full subcategory of the category of (finitely generated) Λ-modules which satisfy* a) *$M_{\mathfrak{p}}$ is $\Lambda_{\mathfrak{p}}$-projective for each non-maximal prime ideal $\mathfrak{p}$ in R and* b) *every direct summand of* $\mathrm{Ext}^1(M, .)$ *is isomorphic to* $\mathrm{Ext}^1(N, .)$ *for some finitely generated Λ-module N. Then $\mathscr{E}$ has the following properties.*

1) *If the $pd_\Lambda M < \infty$ and M satisfies condition* a) *above then M is in $\mathscr{E}$. In particular if M is projective, then M is in $\mathscr{E}$.*

2) *If $M_1 + M_2$ is in $\mathscr{E}$ then each of the M_i is in E.*

3) *If M in $\mathscr{E}$ has no proper direct summands then* $\mathrm{End}(M)/\mathrm{rad}(\mathrm{End}(M))$ *is a divison ring.*

4) *The Krull–Schmidt theorem holds in $\mathscr{E}$, i.e. each M in $\mathscr{E}$ can be written as a direct sum of modules $M_1 + \cdots + M_n$ where each M_i has no proper direct summands and if $N_1 + \cdots + N_s$ is another such representation, then $s = n$ and there is an automorphism σ of M and a permutation τ of $[1, \ldots, n]$ such that $\sigma(M_i) \approx N_{\tau(i)}$ for all i in $[1, \ldots, n]$.*

Proof. 1) By Theorem II, we know that if the $\mathrm{pd}_\Lambda M < \infty$ then M satisfies condition b) since the category of finitely generated Λ-modules has enough projective objects.

2) Follows trivially from the fact that localization permutes with direct sums and that a direct summand of $\mathrm{Ext}^1(M_i, .)$ is also direct summand of $\mathrm{Ext}^1(M, .)$.

3) Consequence of Proposition 2.8.

4) Since M has the descending chain condition on direct summands, we know that $M \approx M_1 + \cdots + M_n$ with the M_i having no proper direct summands. It is classical that 3) implies the rest of the proposition (see [7, p. 58]).

§3. SUBMODULES OF $\mathrm{EXT}_\Lambda{}^1(M, \Lambda)$.

Throughout this section we assume Λ is a ring which is noetherian on both the left and right. Also, unless otherwise specified, we assume that all modules are finitely generated. Our aim is to look at the problem of which modules can be submodules of $\mathrm{Ext}^1(M, \Lambda)$ for some M. The main result is that for a certain type of ring Λ, if a projective module P is a submodule of $\mathrm{Ext}^1(M, \Lambda)$, then $P = 0$.

If M is a left module, then we know that the right operation of Λ on Λ makes $\mathrm{Ext}^1(M, \Lambda)$ a right Λ-module. Similarly if M is a right module, then $\mathrm{Ext}^1(M, \Lambda)$ has a left module structure. We shall denote by ${}_\Lambda\mathcal{T}$ and $\mathcal{T}_\Lambda$ the categories Λ left and right modules which are submodules of $\mathrm{Ext}^1(M, \Lambda)$ for some module M. We begin by developing some of the formal properties of ${}_\Lambda\mathcal{T}$ and $\mathcal{T}_\Lambda$.

Let $F_1 \to F_0 \to M \to 0$ be an exact sequence of left modules with the F_i free R-modules. Denoting the $\mathrm{Coker}(F_0{}^* \to F_1{}^*)$ by $D(M)$ (where $X^* = \mathrm{Hom}(X, \Lambda)$), we have by [2] that there is an exact sequence $0 \to \mathrm{Ext}^1(D(M), R) \to M \to M^{**}$ where $M \to M^{**}$ is the usual map φ given by $\varphi(m)(f) = f(m)$ for all $m \in M$ and $f \in M^*$. On the other hand, applying the functor $\mathrm{Hom}(\ , \Lambda)$ to the sequence $F_0{}^* \to F_1{}^* \to D(M) \to 0$, we obtain our original sequence $F_1 \to F_0 \to M$. Thus we have an exact sequence $0 \to \mathrm{Ext}^1(M, \Lambda) \to D(M) \to D(M)^{**}$. Thus ${}_\Lambda\mathcal{T}$ consists of submodules of the $\mathrm{Ker}(M \to M^{**})$ as M runs through all finitely generated left Λ-modules. We shall denote the $\mathrm{Ker}(M \to M^{**})$ by $t(M)$ and we shall denote by M_0 the image of M in M^{**}. Thus $M/t(M) \approx M_0$. We shall say that M is torsion free, if $t(M) = 0$.

Now the map $M \to M^{**}$ gives rise to a map $M^{***} \to M^*$ and we also have the map $M^* \to M^{***}$. It is well known and easily seen that the composite $M^* \to M^{***} \to M^*$ is the identity. Thus in particular we have that $M^{**} \to M^*$ is an epimorphism. This yields

LEMMA 3.1. *The epimorphism $M \to M_0 \to 0$ induces an isomorphism $M_0{}^* \to M^*$, or equivalently the map $M^* \to t(M)^*$ induced by the inclusion map $t(M) \to M$ is the zero map. Also M_0 is torsion free.*

Proof. The exact sequence $0 \to t(M) \to M \to M_0 \to 0$ gives the exact sequence $0 \to M_0{}^* \to M^* \to t(M)^*$. Since $M \to M^{**}$ factors as $M \to M_0 \to M^{**}$, we have that the map $M^{***} \to M^*$ factors as $M^{***} \to (M_0)^* \to M^*$. Since $M^{***} \to M^*$ is an epimorphism, it follows that the monomorphism $(M_0)^* \to M^*$ is also an epimorphism and thus an isomorphism. The fact that the map $M^* \to t(M)^*$ is the zero map follows trivially.

From the fact that $M_0{}^* \to M^*$ is an isomorphism we have that $M^{**} \to (M_0)^{**}$ is also an isomorphism. Now the composite map $M_0 \to M^{**} \to M_0{}^{**}$ is the same as the natural map $M_0 \to (M_0)^{**}$. Since $M_0 \to M^{**}$ is a monomorphism it follows that $M_0 \to (M_0)^{**}$ is a monomorphism, i.e. M_0 is torsion free.

We now establish our main criterion for when a module is in ${}_\Lambda\mathcal{T}$.

PROPOSITION 3.2. *Let N be a left module. Then*

a) *$N \in {}_\Lambda\mathcal{T}$ if and only if there is an exact sequence $0 \to N \to M$ with $M^* \to N^*$ the zero map. If $0 \to N \to M$ is such that $M^* \to N^*$ is zero, then the* $\mathrm{Im}(N \to M) \subset t(M)$.

b) *$N \approx \mathrm{Ext}^1(X, \Lambda)$ for some X if and only if there exists an exact sequence $0 \to N \to M$ such that $M^* \to N^*$ is the zero map and M/N is torsion free. If such a sequence $0 \to N \to M$ exists, then* $\mathrm{Im}(N \to M) = t(M)$.

Proof. a) If N is in ${}_\Lambda\mathcal{T}$, then we know there is a module M such that there exists a monomorphism $N \to t(M)$. Since by Lemma 3.1, the map $M^* \to t(M)^*$ is the zero map, it follows that the map $M^* \to N^*$ induced by the monomorphism $N \to M$ is the zero map.

Suppose we have an exact sequence $0 \to N \to M \to L \to 0$ such that $M^* \to N^*$ is the zero map. Then $L^* \to M^*$ is an isomorphism. Thus we obtain the commutative diagram

(*)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & M & \longrightarrow & L & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & M^{**} & \approx & L^{**}, & & \end{array}$$

from which it follows that the image of N in M is contained in $t(M)$.

b) If $N \approx \mathrm{Ext}^1(X, \Lambda)$. Then there exists a module M such that $N \approx t(M)$. Since $0 \to t(M) \to M \to M_0 \to 0$ is exact with M_0 torsion free and $M^* \to t(M)^*$ the zero map, we have shown *b*) holds in one direction.

Suppose we have an exact $0 \to N \to M \to L \to 0$ such that L is torsion free and the induced map $M^* \to N^*$ the zero map. Then the diagram (*) holds with the added feature that the map $L \to L^{**}$ is a monomorphism. It then follows that $N = t(M)$. This finishes the proof since $t(M) = \mathrm{Ext}^1(D(M), \Lambda)$ where $D(M)$ is as defined above.

PROPOSITION 3.3. *The category* ${}_\Lambda\mathscr{T}$ *has the following properties*:

a) *If* M_1 and M_2 *are in* ${}_\Lambda\mathscr{T}$, *then* $M_1 + M_2$ *is in* ${}_\Lambda\mathscr{T}$.

b) *If* $M \in {}_\Lambda\mathscr{T}$, *then every submodule and factor module of* M *is in* ${}_\Lambda\mathscr{T}$.

c) *If* $M \in {}_\Lambda\mathscr{T}$ *is isomorphic to* $\mathrm{Ext}^1(X, \Lambda)$, *then each factor module of* M *is isomorphic* $\mathrm{Ext}^1(Y, \Lambda)$ *for some* Y. *Thus if* $M = M_1 + M_2$, *then* $M \approx \mathrm{Ext}^1(X, \Lambda)$ *for some* X *if and only if each* $M_i \approx \mathrm{Ext}^1(Y_i, \Lambda)$ *for some* Y_i.

d) *If* M *is a module and* $M_1, \ldots, M_n$ *are a finite family of submodules of* M *such that each* M_i *is in* ${}_\Lambda\mathscr{T}$, *then the submodule of* M *generated by the* M_i *is in* ${}_\Lambda\mathscr{T}$. *Further if each of the* $M_i \approx \mathrm{Ext}^1(X_i, \Lambda)$, *then the submodule generated by them is isomorphic to* $\mathrm{Ext}^1(Y, \Lambda)$ *for some* Y.

e) *If* $M \in {}_\Lambda\mathscr{T}$, *then* $M^* \in \mathscr{T}_\Lambda$ *and thus* $M^{**} \in {}_\Lambda\mathscr{T}$.

Proof. a) We know that there are exact sequences $0 \to M_i \to L_i$ with $L_i^* \to M_i^*$ the zero map. Then $0 \to M_1 + M_2 \to L_1 + L_2$ is exact and $(L_1 + L_2)^* \to (M_1 + M_2)^*$ is the zero map since that functor $(\ \ )^*$ commutes with finite direct sums.

b) and c) Clearly if $M \in {}_\Lambda\mathscr{T}$, then every submodule of $M \in {}_\Lambda\mathscr{T}$. Suppose M' is a submodule of M and we have an exact sequence $0 \to M \to L \to K \to 0$ chosen in such a way that $L^* \to M^*$ is zero and K is torsion free if $M = \mathrm{Ext}^1(X, \Lambda)$ for some X (by Proposition 3.2 we know this can always be done). Then we have the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & L & \longrightarrow & K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & M/M' & \longrightarrow & L/M' & \longrightarrow & K & \longrightarrow & 0. \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

Then we have the commutative diagram with exact columns

$$\begin{array}{ccc} 0 & & 0 \\ \downarrow & & \downarrow \\ (L/M')^* & \longrightarrow & (M/M')^* \\ \downarrow & & \downarrow \\ L^* & \longrightarrow & M^*. \end{array}$$

Since $L^* \to M^*$ is the zero map, it follows that $(L/M')^* \to (M/M')^*$ is the zero map. Thus the exact sequence $0 \to M/M' \to L/M' \to K \to 0$ has all the properties to show that $M/M' \in {}_\Lambda\mathscr{T}$ and that $M/M' \approx \operatorname{Ext}^1(Y, \Lambda)$ for some Y if $M \approx \operatorname{Ext}^1(X, \Lambda)$ for some X.

The last part of c) follows trivially from a) and what has already been established.

d) Since the submodule generated by the M_i is a homomorphic image of the direct sum $M_1 + \cdots + M_n$, part d) follows from a), b) and c) trivially.

e) Since M is in ${}_\Lambda\mathscr{T}$, we know there is an exact sequence $0 \to M \to L \to K \to 0$ such that the exact sequence $0 \to K^* \to L^* \to M^* \to \operatorname{Ext}^1(K, \Lambda)$ has the property that $0 \to M^* \to \operatorname{Ext}^1(K, \Lambda)$ is exact. This shows that M^* is in $\mathscr{T}_\Lambda$. By symmetry, it then follows that $M^{**} \in {}_\Lambda\mathscr{T}$.

Remark. It would be interesting to know if each $M \in {}_\Lambda\mathscr{T}$ is isomorphic to $\operatorname{Ext}^1(X, \Lambda)$ for some X.

Our first non-formal result concerning ${}_\Lambda\mathscr{T}$ is

LEMMA 3.4. *Let Λ be a ring with minimum condition and radical $\mathfrak{r}$ such that $\Lambda/\mathfrak{r}$ is simple. If P is a projective module in ${}_\Lambda\mathscr{T}$, then $P = 0$.*

Proof. Suppose P is a non-zero projective module in ${}_\Lambda\mathscr{T}$. Since all the simple Λ modules are isomorphic, we have that for n sufficiently large the projective module $Q = \sum_{i=1}^{n} P_i$ where each $P_i = P$ has the property that $Q/\mathfrak{r}Q$ contains $\Lambda/\mathfrak{r}$ as a direct summand. It then follows from Lemma 2.3 that Q contains Λ as a direct summand. Since $P \in {}_\Lambda\mathscr{T}$, we have that $Q \in {}_\Lambda\mathscr{T}$ and thus that $\Lambda \in {}_\Lambda\mathscr{T}$ (see Proposition 3.3a) and b)). Therefore to prove the lemma, it suffices to show that Λ is not in ${}_\Lambda\mathscr{T}$.

If Λ is semi-simple, then the lemma is trivially true. Suppose that Λ is not semi-simple and that n is the smallest integer i such that $\mathfrak{r}^i = 0$. If a module M contains Λ, then Λ is not contained in $\mathfrak{r}M$. For if Λ were contained in $\mathfrak{r}M$ we would have that $\mathfrak{r}^{n-1}(\mathfrak{r}M) = 0$ and thus that $\mathfrak{r}^{n-1}\Lambda = 0$, which is a contradiction since $n > 0$ (remember $\mathfrak{r} \neq 0$ since Λ is not semi-simple). Thus we have that Λ is not contained in $\mathfrak{r}M$. Therefore $\Lambda/\mathfrak{r}M \cap \Lambda \subset M/\mathfrak{r}M$ is not zero and is a direct summand of the semi-simple module $M/\mathfrak{r}M$. Since all the simple Λ-modules are isomorphic and Λ has minimum condition, it follows that Λ contains a copy of the unique simple Λ-module. Thus there exists a map $M/\mathfrak{r}M \to \Lambda$ which is not zero on $\Lambda/\mathfrak{r}M \cap \Lambda$. The composite $M \to M/\mathfrak{r}M \to \Lambda$ gives us a map of $M \to \Lambda$ which is not zero on the submodule Λ of M. Thus if $\Lambda \subset M$, then $M^* \to \Lambda^*$ is not zero, which shows by Proposition 3.2, that Λ is not in ${}_\Lambda\mathscr{T}$.

Assume now that Λ is a ring whose center C is a noetherian ring such that Λ is a finitely generated C-module. Then if $\mathfrak{p}$ is a prime ideal in C, we have that $C_\mathfrak{p} \subset \Lambda_\mathfrak{p}$ and $\Lambda_\mathfrak{p}$ is a finitely generated $C_\mathfrak{p}$ module. Thus $\mathfrak{p}\Lambda_\mathfrak{p} \neq \Lambda_\mathfrak{p}$ (by Nakayama's lemma), therefore it follows that $\mathfrak{p}\Lambda \neq \Lambda$. Consequently, we have that the map from the prime ideals in Λ into the prime ideals in R given by $\mathfrak{P} \to \mathfrak{P} \cap C$ is onto. Also it is not difficult to see that $\mathfrak{P}$ in Λ is maximal if and only if $\mathfrak{P} \cap C$ is maximal in C. We can now state and prove the main result of this section.

Proposition 3.5. *Let Λ be a ring whose center C is a noetherian ring such that Λ is a finitely generated C-module. Suppose further, that the map $\mathfrak{P} \to \mathfrak{P} \cap C$ of the prime ideals in Λ to the prime ideals in C is one to one as well as onto. Then if P is a projective Λ-module in ${}_\Lambda\mathcal{T}$, then $P = 0$.*

Proof. Let $\mathfrak{m}$ be a maximal ideal in C, then $P\mathfrak{m}$ is a projective $\Lambda\mathfrak{m}$-module. Suppose $P\mathfrak{m} \neq 0$. Since there existed only one maximal ideal (two-sided ideal) in Λ lying over $\mathfrak{m}$, we have that the $\mathfrak{r} = \operatorname{rad}(\Lambda\mathfrak{m})$ is a maximal two-sided ideal in $\Lambda\mathfrak{m}$. Thus $\Lambda\mathfrak{m}/\mathfrak{r}$ is a simple ring. Therefore all the simple Λ-modules are isomorphic. Consequently, as in Lemma 3.4, we know that a direct sum of copies of $P\mathfrak{m}$ contains $\Lambda\mathfrak{m}$ as a direct summand. Now the usual localization arguments show that if $M \in {}_\Lambda\mathcal{T}$, then $M_S \in {}_{\Lambda_S}\mathcal{T}$ for any multiplicative set S in C. Thus $P\mathfrak{m}$ is in ${}_\Lambda\mathfrak{m}\mathcal{T}$ and therefore so is $\Lambda\mathfrak{m}$.

Now let $\mathfrak{p}$ be a minimal prime ideal in C contained in $\mathfrak{m}$. Then $\Lambda_\mathfrak{p}$ is contained in ${}_{\Lambda_\mathfrak{p}}\mathcal{T}$ since $\Lambda\mathfrak{m}$ is in ${}_\Lambda\mathfrak{m}\mathcal{T}$. Since $C_\mathfrak{p}$ has minimum chain condition, it follows that $\Lambda_\mathfrak{p}$ also has minimum chain condition since it is finitely generated over $C_\mathfrak{p}$. Since there exists only one prime ideal in Λ lying over $\mathfrak{p}$, it follows that $\Lambda_\mathfrak{p}$ modulo its radical is a simple ring. Thus by Lemma 3.4 it follows that $\Lambda_\mathfrak{p}$ is not in ${}_{\Lambda_\mathfrak{p}}\mathcal{T}$, which is a contradiction. Therefore we have that $P_\mathfrak{M} = 0$ for all maximal ideal $\mathfrak{m}$ in C. Thus we have that $P = 0$, our desired result. As an immediate consequence of Proposition 3.5 we have

Corollary 3.6. *If Λ is a commutative ring and P is a projective module in ${}_\Lambda\mathcal{T}$, then $P = 0$.*

Remark. A. Zaks and D. Zelinsky have communicated examples of finite dimensional algebras over arbitrary fields for which Corollary 3.6 is false.

As a consequence of Corollary 3.6 we have

Proposition 3.7. *If Λ is a commutative ring and M a Λ-module of finite projective dimension, then $M \in {}_\Lambda\mathcal{T}$ if and only if $M^* = 0$.*

Proof. Suppose $M \in {}_\Lambda\mathcal{T}$ with $pd_\Lambda M < \infty$. Let $\mathfrak{p}$ be a prime ideal in $\operatorname{Ass}(\Lambda)$. Then all the units in $\Lambda_\mathfrak{p}$ are zero divisors. But it is well known that if the $pd_{\Lambda_\mathfrak{p}} M_\mathfrak{p} < \infty$, then $M_\mathfrak{p}$ is a $\Lambda_\mathfrak{p}$ free module. Thus we must have by Corollary 3.6, that $M_\mathfrak{p} = 0$ for each prime ideal $\mathfrak{p}$ in $\operatorname{Ass}(\Lambda)$, since $M_\mathfrak{p} \in {}_{\Lambda_\mathfrak{p}}\mathcal{T}$ for all prime ideals $\mathfrak{p}$ in Λ. Thus we have that if $\mathfrak{p} \in \operatorname{Ass}(\Lambda)$, then $\mathfrak{p} \notin \operatorname{Supp}(M)$. Therefore if $M \neq 0$, then the annihilator of M contains a non-zero divisor. From this it follows easily that $M^* = 0$, which is our desired result.

We end this section with the following generalization of Corollary 3.6

PROPOSITION 3.8. *Let* Λ *be a commutative ring and* M *a* Λ*-module. If* M *is in* ${}_{\Lambda}\mathscr{T}$, *then* $\mathfrak{t}(M)$ *is nilpotent.*

Proof. Let $\mathfrak{p}$ be a minimal prime ideal in Λ. Since M is in ${}_{\Lambda}\mathscr{T}$, we know that $M_{\mathfrak{p}}$ is in ${}_{\Lambda_{\mathfrak{p}}}\mathscr{T}$. Now $t(M)$ is the image of the map $\varphi : M^* \otimes_{\Lambda} M \to R$ given by $f \otimes x \mapsto f(x)$. Thus localizing we have that $t(M)_{\mathfrak{p}} = t(M_{\mathfrak{p}})$. Thus if $\mathfrak{t}(M)$ is not contained in $\mathfrak{p}$, then we have that $\mathfrak{t}(M_{\mathfrak{p}}) = \Lambda_{\mathfrak{p}}$. Since $\Lambda_{\mathfrak{p}}$ is a local ring, it follows that there is an epimorphism $M_{\mathfrak{p}} \to \Lambda_{\mathfrak{p}}$ and thus $M_{\mathfrak{p}}$ contains a copy of $\Lambda_{\mathfrak{p}}$. Thus $\Lambda_{\mathfrak{p}}$ is in ${}_{\Lambda_{\mathfrak{p}}}\mathscr{T}$, which is impossible in view of Corollary 3.6. Therefore we have that $\mathfrak{t}(M) \subset p$ for all minimal prime ideals $\mathfrak{p}$ in Λ. Thus $\mathfrak{t}(M)$ is nilpotent.

§4. SUBMODULES OF $\text{EXT}_R{}^1(M, C)$.

Throughout this section we assume that R is an arbitrary commutative ring unless stated to the contrary. Our main results here are the following generalizations of Corollary 3.6 and Proposition 3.7.

THEOREM 4.1. *Let* M *be a finitely generated* R*-module and* C *an arbitrary module* (*not necessarily finitely generated*). *If* P *is a projective submodule of* $\text{Ext}_R{}^1(M, C)$, *then* $P = 0$.

Proof. Let $0 \to C \to E(C) \to D \to 0$ be exact with $E(C)$ an injective envelope of C. Then we have the exact sequence

$$0 \to \text{Hom}_R(M, C) \to \text{Hom}_R(M, E(C)) \to \text{Hom}_R(M, D) \to \text{Ext}^1(M, C) \to 0$$

Suppose P is a projective submodule of $\text{Ext}^1(M, C)$. Since $\text{Hom}_R(M, D) \to \text{Ext}^1(M, C) \to 0$ is exact, there is a submodule of $\text{Hom}_R(M, D)$ which gets mapped onto P. But since P is projective, this submodule of $\text{Hom}_R(M, D)$ which gets mapped onto P will contain P as a direct summand. Thus $\text{Hom}_R(M, D)$ contains a submodule isomorphic to P.

Since M is finitely generated, we can find an epimorphism $F \to M$ with F a finitely generated free R-module. This map gives us a monomorphism $0 \to \text{Hom}(M, D) \to \text{Hom}_R(F, D)$. Thus $\text{Hom}_R(F, D)$ contains a copy of P. Now the sequence $0 \to \text{Hom}_R(F, C) \to \text{Hom}_R(F, E(C)) \to \text{Hom}_R(F, D) \to 0$ is exact since F is a free module. Letting N be the preimage of P in $\text{Hom}_R(F, E(C))$, we have that $N = \text{Hom}_R(F, C) + P$ (direct sum). Now $\text{Hom}_R(F, C) \approx \sum C$ and $\text{Hom}_R(F, E(C)) \approx \sum E(C)$ where the sums are finite direct sums and the map of $\sum C \to \sum E(C)$ corresponding to the map $\text{Hom}_R(F, C) \to \text{Hom}_R(F, E(C))$ is the obvious map given by the inclusion map $C \to E(C)$. Since the finite direct sum of essential extensions is essential, we have that $\text{Hom}_R(F, E(C))$ is an essential extension of $\text{Hom}_R(F, C)$. But the submodule N of $\text{Hom}_R(F, C)$ is the direct sum $\text{Hom}_R(F, C) + P$. Thus P must be zero, or else $\text{Hom}_R(F, E(C))$ would not be an essential extension of $\text{Hom}_R(F, C)$ which completes the proof.

THEOREM 4.2. *Let* R *be a noetherian ring and* N *a finitely generated module of finite projective dimension. Then* N *is isomorphic to a submodule of* $\text{Ext}^1(M, C)$ *for some finitely generated module* M *and some* (*arbitrary*) C *if and only if* $N^* = 0$.

Proof. If $N^* = 0$, then by the results of §3 we know that N is a submodule of

$\mathrm{Ext}^1(M, R)$ for some finitely generated R-module M. The proof in the other direction is the same argument given in Proposition 3.7.

We also have the following generalization of Proposition 3.8.

COROLLARY 4.2. *Let M be a finitely presented R-module and C an arbitrary module. If N is a submodule of $\mathrm{Ext}^1(M, C)$, then $\mathrm{t}(N)$ is a nil ideal (i.e. every element is nilpotent).*

Proof. Let $0 \to K \to F \to M \to 0$ be exact with F a finitely generated free module. Suppose $\mathfrak{p}$ is a prime ideal in R. Then we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{Hom}(M, C)_{\mathfrak{p}} & \longrightarrow & \mathrm{Hom}(F, C)_{\mathfrak{p}} & \longrightarrow & \mathrm{Hom}(K, C)_{\mathfrak{p}} & \longrightarrow & \mathrm{Ext}^1(M, C)_{\mathfrak{p}} & \longrightarrow 0 \\ & \downarrow & & \downarrow & & \downarrow & & \searrow & \\ 0 \longrightarrow & \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, C_{\mathfrak{p}}) & \longrightarrow & \mathrm{Hom}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, C_{\mathfrak{p}}) & \longrightarrow & \mathrm{Hom}_{R_{\mathfrak{p}}}(K_{\mathfrak{p}}, C_{\mathfrak{p}}) & \longrightarrow & & \mathrm{Ext}^1_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, C_{\mathfrak{p}}) \end{array}$$

Counting from the left, we know that the first two vertical maps are isomorphisms since M and F are finitely presented. The third vertical map is a monomorphism since K is finitely generated (remember that M is finitely presented). Then it follows from diagram chasing, that the last vertical map is a monomorphism. Thus if N is a submodule of Ext^1 (M, C), then $N_{\mathfrak{p}}$ is a submodule of $\mathrm{Ext}_{R_{\mathfrak{p}}}{}^1(M_{\mathfrak{p}}, C_{\mathfrak{p}})$. Now suppose there exists a map $f : N \to R$ such that $f(n) \notin \mathfrak{p}$ for some $n \in N$. Then the induced map $f_{\mathfrak{p}} : N_{\mathfrak{p}} \to R_{\mathfrak{p}}$ defined by $f_{\mathfrak{p}}(n/s) = f(n)/s$ has the property that $f_{\mathfrak{p}}(n/f(n) = 1$. Thus $f_{\mathfrak{p}} : N_{\mathfrak{p}} \to R_{\mathfrak{p}}$ is onto. Thus $N_{\mathfrak{p}}$ contains a copy of $R_{\mathfrak{p}}$, which by Theorem 4.1 is impossible. Thus for each minimal prime ideal $\mathfrak{p}$ we have that each map $f : N \to R$ has its image in $\mathfrak{p}$. Thus $\mathrm{t}(M) \subset \mathfrak{p}$ for all prime ideals in $\mathfrak{p}$ in R. But the intersection of all the prime ideals in R consists precisely of all the nilpotent elements. Thus $\mathrm{t}(N)$ is a nil ideal.

We end this paper by showing that the conclusion of Theorem 4.1 need not hold if M is not finitely generated. Before presenting such an example we need the following observation concerning commutative, noetherian local rings which are complete.

Suppose R is a commutative, noetherian local ring which is complete and E is an injective envelope of $R/\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of R. Then it is a result of Matlis [8] that if M is a finitely generated R-module, then the natural map $M \to \mathrm{Hom}_R(\mathrm{Hom}_R$ $(M, E), E)$ is an isomorphism.

PROPOSITION 4.3. *Let R be as above and M a finitely generated R-module. Then we have an isomorphism of functors $\mathrm{Ext}_R{}^i(\ \ , M) \approx \mathrm{Hom}_R(\mathrm{Tor}_i R(\ \ , \mathrm{Hom}(M, E)), E)$ (for all i). If $\{A_k\}_k$ is a direct limit system of modules then the natural map $\mathrm{Ext}_R{}^i(\varinjlim A_k, M) \to \varprojlim$ $\mathrm{Ext}^i(A_k, M)$ is an isomorphism (for all i).*

Proof. Since E is injective we know by [3, V, Prop. 5.1] that there is an isomorphism of functors $\mathrm{Ext}_R{}^i(\ \ , \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E)) \cong \mathrm{Hom}_R(\mathrm{Tor}_i{}^R(\ \ , \mathrm{Hom}_R(M, E)), E)$. The fact that M is finitely generated implies that $M \approx \mathrm{Hom}_R(\mathrm{Hom}_R(M, E), E)$. Thus we obtain the first result.

Now let $\{A_k\}$ be a direct limit system. Since the functor Tor commutes with direct limits, we have that $\mathrm{Tor}_i{}^R(\varinjlim A_k, \mathrm{Hom}_R(M, E)) \approx \varinjlim \mathrm{Tor}_i{}^R(A_k, \mathrm{Hom}_R(M, E))$. Since

$\text{Hom}(\varinjlim X_k, B) \approx \varprojlim(X_k, B)$ for any direct limit family $\{X_k\}$ and any B, we have that $\text{Hom}(\text{Tor}_i^R(\varinjlim A_k, \text{Hom}_R(M, E))E) \approx \varprojlim \text{Hom}(\text{Tor}_i^R(A_k, \text{Hom}_R(M, E)), E)$. Applying the isomorphism established, gives the desired result.

Suppose R is a commutative domain which is a complete, noetherian local ring with field of quotients $K \neq R$. Then K is a direct limit of free modules, so we have that $\text{Ext}_R^i(K, R) = 0$ for all $i > 0$ by Proposition 4.3. Also since $R \neq K$, R contains no non-trivial divisible submodules. Thus $\text{Hom}(K, R) = 0$, since K is divisible and every homorphic image of a divisible module is divisible. Now from the short exact sequence $0 \to R \to K \to K/R \to 0$, we deduce the exact sequence

$$\text{Hom}_R(K, R) \to \text{Hom}_R(R, R) \to \text{Ext}_R^1(K/R, R) \to \text{Ext}_R^1(K, R)$$

which, in view of our remarks above, shows us that $R \approx \text{Ext}_R^1(K/R, R)$. Thus K/R is an example of a module such that $\text{Ext}_R^1(K/R, R)$ is projective.

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