ELSEVIER

Discrete Mathematics 175 (1997) 173–195

DISCRETE MATHEMATICS

On codes satisfying the double chain condition [1]

Torleiv Kløve*

Department of Informatics, University of Bergen, HIB, N-5020 Bergen, Norway

Received 7 July 1995; revised 21 March 1996

Abstract

The double chain condition is described. A number of bounds on the length and weight hierarchy of codes satisfying the double chain condition are given. Constructions of codes satisfying the double chain condition and with trellis complexity 1 or 2 are given.

1. Introduction and notations

We consider binary linear codes. The *support* of a vector $\boldsymbol{x}=(x_1,x_2,\ldots,x_n)$ in $\mathrm{GF}(2)^n$ is defined by

$$\chi(\boldsymbol{x}) = \{i | x_i \neq 0\},$$

and the support of a subset $S \subseteq \mathrm{GF}(2)^n$ is defined by

$$\chi(S) = \bigcup_{\boldsymbol{x}\in S} \chi(\boldsymbol{x}).$$

The *support weight* of S is defined by

$$w_S(S) = |\chi(S)|.$$

Hence, $w_S(S)$ is the number of positions where at least one vector in S is non-zero. The *weight hierarchy* of an $[n,k]$ code C is the sequence $(d_1,d_2,\ldots,d_k)$, where

$$d_r = d_r(C) = \min\{w_S(D) | D \text{ is an } [n,r] \text{ subcode of } C\}.$$

In particular, $d_1 = d$, the minimum distance of C. The parameters $d_1,d_2,\ldots,d_k$ of a code were first introduced by Helleseth et al. [4]. A simple, but important property is

* E-mail: torleiv@ii.uib.no.

[1] This work was supported by The Norwegian Research Council, grants no. 107542/410 and 107623/420. Parts of the results were presented at the 33rd Annual Allerton Conf. on Commun., Control, and Computing, October 4–6, 1995.

PII S0012-365X(96)00149-5

the following, first proved by Helleseth et al. [4, Theorem 6.1]:

$$0 < d_1 < d_2 < \cdots < d_k.$$

Forney [2] called $(d_1, d_2, \ldots, d_k)$ the *length/dimension profile*. The inverse was first studied by Kasami et al. [8] and Vardy and Be'ery [12]. In the notation of Forney [2], the *dimension/length profile* $(k_0, k_1, \ldots, k_n)$ is defined by

$$k_i = r \quad \text{for } d_r \leqslant i < d_{r+1}.$$

In particular, $k_i = 0$ for $i < d$ and $k_n = k$.

Forney [3] introduced the *double chain condition* which can be rephrased as follows. An $[n,k]$ code C is called a DCC (double chain condition) code if it has the following property: there exist two chains of subcodes of C, the *left chain*

$$D_1^{\mathrm{L}} \subset D_2^{\mathrm{L}} \subset \cdots \subset D_k^{\mathrm{L}} = C,$$

and the *right chain*

$$D_1^{\mathrm{R}} \subset D_2^{\mathrm{R}} \subset \cdots \subset D_k^{\mathrm{R}} = C,$$

where, for $1 \leqslant r \leqslant k$, we have

$$\dim(D_r^{\mathrm{L}}) = \dim(D_r^{\mathrm{R}}) = r,$$

$$\chi(D_r^{\mathrm{L}}) = \{1, 2, \ldots, d_r\},$$

$$\chi(D_r^{\mathrm{R}}) = \{n - d_r + 1, n - d_r + 2, \ldots, n\}.$$

A code is said to satisfy the double chain condition if it is equivalent to a DCC code. The same concept in a different notation was first studied by Kasami et al. [8]. They showed that the Reed–Muller codes satisfy the double chain condition. Forney [2,3] proved that several other classes of codes have this property.

Forney [2] defined the *state complexity profile* $(s_0, s_1, \ldots, s_n)$ of an $[n,k]$ code and gave a lower bound on the s_i in terms of the dimension/length profile and what he called the *inverse* dimension/length profile. Codes satisfying the double chain condition are optimal with respect to this bound in the sense that the bound is satisfied with equality for all i, and this is our reason to studying these codes. For these codes the s_i are given by

$$s_i = k - k_i - k_{n-i}$$

for $0 \leqslant i \leqslant k$. Further, the *state complexity* is

$$s = \max\{s_i \mid 0 \leqslant i \leqslant n\}.$$

Sometimes we will include s and d in the notation for an $[n,k]$ code C, and refer to C as an $[n,k,d]$ and $[n,k,s,d]$ code. Further, if C is a DCC code, we will also refer to it as an $[n,k]^{\mathrm{DCC}}$, $[n,k,d]^{\mathrm{DCC}}$, and $[n,k,s,d]^{\mathrm{DCC}}$ code.

The main part of this paper is a determination of the parameters n, k, d for which there exist $[n,k,1,d]^{\mathrm{DCC}}$ and $[n,k,2,d]^{\mathrm{DCC}}$ codes. Further, we give some general bounds on the parameters of DCC codes.

An $[n,k]^{\mathrm{DCC}}$ code C has a basis $\mathscr{G} = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_k\}$ such that

$$\langle \boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_r \rangle = D_r^{\mathrm{L}} \quad \text{for } 1 \leqslant r \leqslant k. \tag{1}$$

Here $\langle \boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_r \rangle$ denotes the vector space spanned by $\{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_r\}$. Similarly, for a vector space D and a vector $\boldsymbol{x}$ we will use the notation $\langle D, \boldsymbol{x} \rangle$ to denote the space spanned by D and $\boldsymbol{x}$, etc. In the following, when we consider an $[n,k]^{\mathrm{DCC}}$ code C we will assume that a basis $\mathscr{G}$ has been chosen such that (1) is satisfied. We note that such a basis is not unique since we may substitute $\boldsymbol{g}_i + \sum_{j=1}^{i-1} \alpha_j \boldsymbol{g}_j$ for $\boldsymbol{g}_i$ without affecting (1). We as usual write $\boldsymbol{g}_r = (g_{r1}, g_{r2}, \ldots, g_{rn})$, and we will refer to these elements without further comments. We note that

$$g_{rd_r} = 1; \qquad g_{ri} = 0 \quad \text{for } d_r < i \leqslant n.$$

Similarly, C has a basis $\mathscr{H} = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_k\}$ such that

$$\langle \boldsymbol{h}_1, \boldsymbol{h}_2, \ldots, \boldsymbol{h}_r \rangle = D_r^{\mathrm{R}} \quad \text{for } 1 \leqslant r \leqslant k. \tag{2}$$

For any vector $\boldsymbol{x} \in C \backslash \{\boldsymbol{0}\}$, let

$$l(\boldsymbol{x}) = \min \chi(\boldsymbol{x}) \quad \text{and} \quad u(\boldsymbol{x}) = \max \chi(\boldsymbol{x}),$$

that is, $l(\boldsymbol{x})$ and $u(\boldsymbol{x})$ are the positions of the leftmost and rightmost 1 in $\boldsymbol{x}$, respectively.

Lemma 1. *Let C be an $[n,k,d]^{\mathrm{DCC}}$ code. For all $\boldsymbol{x} \in C \backslash \{\boldsymbol{0}\}$ we have*
(i) *$u(\boldsymbol{x}) = d_r$ for some r, and*
(ii) *$l(\boldsymbol{x}) = n + 1 - d_{r'}$ for some r'.*

Proof. Since $\mathscr{G}$ is a basis, there exist $a_1, a_2, \ldots, a_r$ for some r, $1 \leqslant r \leqslant k$ such that

$$\boldsymbol{x} = \sum_{i=1}^{r} a_i \boldsymbol{g}_i,$$

and $a_r = 1$. By the definition of the chain condition, we have

$$g_{ij} = 0 \quad \text{if } 1 \leqslant i \leqslant r \text{ and } d_r < j \leqslant n,$$

$$g_{id_r} = 0 \quad \text{if } 1 \leqslant i < r,$$

and

$$g_{rd_r} = 1.$$

Hence $u(\boldsymbol{x}) = d_r$. A similar argument, using the basis $\mathscr{H}$ gives (ii). $\square$

Corollary 1. *Let C be an $[n,k,d]^{\mathrm{DCC}}$ code. Then there exists a basis $\mathscr{G}$ and a permutation π of $\{1,2,\ldots,k\}$ such that*

$$\langle g_1, g_2, \ldots, g_r\rangle = D_r^{\mathrm{L}} \quad \textit{for } 1 \leqslant r \leqslant k,$$

and

$$\langle g_{\pi(1)}, g_{\pi(2)}, \ldots, g_{\pi(r)}\rangle = D_r^{\mathrm{R}} \quad \textit{for } 1 \leqslant r \leqslant k.$$

That is, we can choose $\mathscr{H}$ as a permutation of $\mathscr{G}$.

Proof. Let $\mathscr{G}$ be a basis for C satisfying (1). If $i<j$ are such that $l(g_i)=l(g_j)$, then we can replace g_j by g_i+g_j. This will not affect the property (1). Repeating these substitutions if necessary, we see that we may assume that $l(g_i)\neq l(g_j)$ for all $i\neq j$. From Lemma 1(ii) we see that

$$\{l(g_r) | 1 \leqslant r \leqslant k\} = \{n+1-d_{r'} | 1 \leqslant r' \leqslant k\},$$

and the corollary follows.

In a vector or matrix, a block of a consecutive zeros will sometimes be denoted by $\overbrace{0}^{a}$, and similarly for a block of ones.

2. Some basic results

Theorem 1. *If C is an $[n,k]^{\mathrm{DCC}}$ code, then $d_k = n$.*

Proof. Suppose that $d_k < n$. By the left chain condition $n \notin \chi(C)$. By the right chain condition $n \in \chi(C)$, a contradiction. □

Lemma 2. *If C is an $[n,k,d]$ code with $k>2$ which contains two codewords*

$$\boldsymbol{x} = (\overbrace{1}^{a}\ \overbrace{1}^{b}\ \overbrace{0}^{a}) \quad \textit{and} \quad \boldsymbol{y} = (\overbrace{0}^{a}\ \overbrace{1}^{b}\ \overbrace{1}^{a}),$$

where $a+b=d$, then $b=0$.

Proof. Write the codewords $\boldsymbol{c}$ of C as

$$\boldsymbol{c} = (\boldsymbol{c}_1 | \boldsymbol{c}_2 | \boldsymbol{c}_3)$$

where $\boldsymbol{c}_1$ and $\boldsymbol{c}_3$ have length a and $\boldsymbol{c}_2$ has length b. Let

$$\boldsymbol{z} = \boldsymbol{x} + \boldsymbol{y} = (\mathbf{1}|\mathbf{0}|\mathbf{1}).$$

For any codeword $\boldsymbol{c}$, we have $\boldsymbol{c}+\boldsymbol{z}\in C$. If $\boldsymbol{c}\notin\{\boldsymbol{0},\boldsymbol{z}\}$ we have

$$\begin{aligned}2a+2b=d+d &\leqslant w(\boldsymbol{c})+w(\boldsymbol{c}+\boldsymbol{z})\\ &=\Big(w(\boldsymbol{c}_1)+w(\boldsymbol{c}_1+\boldsymbol{1})\Big)+\Big(w(\boldsymbol{c}_2)+w(\boldsymbol{c}_2)\Big)+\Big(w(\boldsymbol{c}_3)+w(\boldsymbol{c}_3+\boldsymbol{1})\Big)\\ &=a+2w(\boldsymbol{c}_2)+a\leqslant 2a+2b\end{aligned}$$

since $w(\boldsymbol{c}_2)\leqslant b$. Hence $w(\boldsymbol{c}_2)=b$ (and $\boldsymbol{c}_2=\boldsymbol{1}$). Let $\tilde{\boldsymbol{c}}$ be a codeword in C, not in $\{\boldsymbol{0},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\}$, and let

$$\boldsymbol{c}=\tilde{\boldsymbol{c}}+\boldsymbol{x}=((\tilde{\boldsymbol{c}}_1+\boldsymbol{1})|\boldsymbol{0}|\tilde{\boldsymbol{c}}_3).$$

Then $b=w(\boldsymbol{c}_2)=w(\boldsymbol{0})=0$. □

Theorem 2. *If C is an $[n,k>2,d]^{\mathrm{DCC}}$ code, then $2d\leqslant n$.*

Proof. Let $D_1^{\mathrm{L}}=\{\boldsymbol{0},\boldsymbol{x}\}$ and $D_1^{\mathrm{R}}=\{\boldsymbol{0},\boldsymbol{y}\}$. By Lemma 2, $\chi(\boldsymbol{x})\cap\chi(\boldsymbol{y})=\emptyset$ and so

$$2d=|\chi(\boldsymbol{x})|+|\chi(\boldsymbol{y})|=|\chi(\boldsymbol{x})\cup\chi(\boldsymbol{y})|\leqslant n. \qquad \square$$

Example. The simplex codes have parameters $[2^m-1,m,2^{m-1}]$. By Theorem 2, the simplex codes do not satisfy the double chain condition. In contrast, Kasami et al. [8] showed that the closely related $[2^m,m+1,2^{m-1}]$ first order Reed–Muller codes do satisfy the double chain condition for all m.

Theorem 3. *If C is an $[n,k,d]^{\mathrm{DCC}}$ code, then*

$$d_{r+1}\leqslant d_r+d$$

for $1\leqslant r<k$. In particular $d_r\leqslant rd$ for all r and $n\leqslant kd$.

Proof. Let $1\leqslant r<k$ and let $D=\langle D_r^{\mathrm{L}},D_1^{\mathrm{R}}\rangle$. Since $n\notin\chi(D_r^{\mathrm{L}})$ and $n\in\chi(D_1^{\mathrm{R}})$, we have $\dim(D)=r+1$. Hence

$$d_{r+1}\leqslant w_S(D)\leqslant w_S(D_r^{\mathrm{L}})+w_S(D_1^{\mathrm{R}})=d_r+d. \qquad \square$$

In [11], Lafourcade and Vardy proved that for any $[n,k,s,d]$ code we have

$$n\geqslant\frac{k}{s}(d-1). \tag{3}$$

For codes satisfying the double chain condition we can give stronger bounds on n. We will also give bounds on d_r in general.

By Theorem 3, if $d=1$ for an $[n,k]^{\mathrm{DCC}}$ code C, then $n=k$ and so $C=GF(2)^n$. Further, the only $[kd,k,d]^{\mathrm{DCC}}$ codes are the $[kd,k,0,d]^{\mathrm{DCC}}$ codes generated by the matrices

$$\begin{pmatrix} \overbrace{1}^{d} & \overbrace{0}^{d} & \cdots & \overbrace{0}^{d} \\ 0 & 1 & \cdots & 0 \\ \cdots & & & \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Therefore, from now on we will assume that $s \geqslant 1$, $d \geqslant 2$, and $n < kd$.

Lemma 3. *For an* $[n,k,s,d \geqslant 2]^{\mathrm{DCC}}$ *code we have*

$$d_r + d_{k-r-s+1} \leqslant n + 1.$$

Proof. Let $i = d_r - 1$. By definition, $k_i = r - 1$ and

$$k_{n-i} = k - k_i - s_i \geqslant k - r + 1 - s$$

and so

$$n - i \geqslant d_{k-r+1-s} \quad \text{and} \quad n \geqslant d_r - 1 + d_{k-r+1-s}.$$

Corollary 2. *Let C be an $[n,k,s,d \geqslant 2]^{\mathrm{DCC}}$ code. If $r + t \leqslant k - s$, then*

$$\dim(\langle D_r^{\mathrm{L}}, D_t^{\mathrm{R}} \rangle) = r + t \quad \text{and} \quad w_S(\langle D_r^{\mathrm{L}}, D_t^{\mathrm{R}} \rangle) = d_r + d_t.$$

Corollary 3. *Let C be an $[n,k,s,d \geqslant 2]^{\mathrm{DCC}}$ code. If $r + t = k - s + 1$, then*

$$\dim(\langle D_r^{\mathrm{L}}, D_t^{\mathrm{R}} \rangle) = r + t$$

and

$$d_r + d_t - 1 \leqslant w_S(\langle D_r^{\mathrm{L}}, D_t^{\mathrm{R}} \rangle) \leqslant d_r + d_t.$$

Proof. If $r + t \leqslant k - s$, then, by Lemma 3,

$$d_r + d_t \leqslant d_r + d_{t+1} - 1 \leqslant d_r + d_{k-r-s+1} - 1 \leqslant n$$

and so $\chi(D_r^{\mathrm{L}}) \cap \chi(D_t^{\mathrm{R}}) = \emptyset$. Hence ,

$$\dim(\langle D_r^{\mathrm{L}}, D_t^{\mathrm{R}} \rangle) = r + t \quad \text{and} \quad w_S(\langle D_r^{\mathrm{L}}, D_t^{\mathrm{R}} \rangle) = d_r + d_t.$$

If $r + t = k - s + 1$ we get in the same way that

$$d_r + d_t - 1 \leqslant w_S(\langle D_r^{\mathrm{L}}, D_t^{\mathrm{R}} \rangle).$$

Assume that $\dim(\langle D_r^{\mathrm{L}}, D_t^{\mathrm{R}} \rangle) < r + t$. This is only possible if $g_r \in D_t^{\mathrm{R}}$ and so $l(g_r) \geqslant n + 1 - d_t \geqslant d_r$. Hence $l(g_r) = d_r = u(g_r)$ and $w_{\mathrm{H}}(g_r) = 1 < d$, a contradiction. $\square$

Theorem 4. *For an* $[n,k,s,d \geqslant 2]^{\mathrm{DCC}}$ *code we have*

$$d_{r+s+t-1} \geqslant d_r + d_t - 1$$

for $r \geqslant 1$, $t \geqslant 1$, *and* $r+s+t-1 \leqslant k$.

Proof. Let

$$D = \langle D_r^{\mathrm{L}}, D_{k-r-s+1}^{\mathrm{R}} \rangle.$$

By Corollary 2, $\dim(D)=k-s+1$. Since the vectors $\boldsymbol{g}_{r+1}, \boldsymbol{g}_{r+2}, \dots, \boldsymbol{g}_{r+s+t-1}$ are linearly independent, and

$$\dim(\langle D, \boldsymbol{g}_{r+1}, \boldsymbol{g}_{r+2}, \dots, \boldsymbol{g}_{r+s+t-1} \rangle) \leqslant k,$$

there exist $i_1, i_2, \dots, i_t$ such that

$$r+1 \leqslant i_1 < i_2 < \cdots < i_t \leqslant r+s+t-1$$

and

$$\boldsymbol{g}_{i_u} \in D \quad \text{for } 1 \leqslant u \leqslant t,$$

that is

$$\boldsymbol{g}_{i_u} = \boldsymbol{y}_u + \boldsymbol{z}_u,$$

where $\boldsymbol{y}_u \in D_r^{\mathrm{L}}$ and $\boldsymbol{z}_u \in D_{k-r-s+1}^{\mathrm{R}}$. Suppose

$$\sum_{u=1}^{t} a_u \boldsymbol{z}_u = 0$$

for some $a_u \in \mathrm{GF}(2)$. Then

$$\sum_{u=1}^{t} a_u \boldsymbol{g}_{i_u} = \sum_{u=1}^{t} a_u \boldsymbol{y}_u \in D_r^{\mathrm{L}}$$

and so $a_u = 0$ for all u; that is, the vectors $\boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_t$ are linearly independent. Let

$$D' = \langle \boldsymbol{z}_1, \boldsymbol{z}_2, \dots, \boldsymbol{z}_t \rangle.$$

Then

$$\max \chi(D') = d_{i_t} \leqslant d_{r+s+t-1}, \tag{4}$$

and

$$\min \chi(D') \geqslant \min \chi(D_{k-r-s+1}^{\mathrm{R}}) = n + 1 - d_{k-r-s+1} \geqslant d_r \tag{5}$$

by Lemma 3. Combining (4) and (5) we get

$$d_{r+s+t-1} \geqslant d_r + w_S(D') - 1 \geqslant d_r + d_t - 1. \qquad \square$$

Let

$$g(r,d) = \sum_{i=0}^{r-1} \left\lceil \frac{d}{2^i} \right\rceil$$

denote the Griesmer bound. It is well known that

$$d_r \geqslant g(r,d).$$

Theorem 5. *For an* $[n,k,s,d \geqslant 2]^{\mathrm{DCC}}$ *code* C, *for* $t \geqslant 1$, *and for* $1 \leqslant r \leqslant k$, *write*

$$r = a(s+t-1) + b$$

where $1 \leqslant b \leqslant s+t-1$. *Then*

$$d_r \geqslant a(g(t,d)-1) + g(b,d).$$

Proof. By Theorem 4 and induction we get

$$d_r \geqslant a(d_t - 1) + d_b \geqslant a(g(t,d)-1) + g(b,d). \qquad \square$$

Example. If d is even and $k = a(s+1)+2$ for some integer a, we can choose $t = 2$, $b = 2$ in the theorem and get

$$n \geqslant \frac{k-2}{s+1}\left(\frac{3}{2}d - 1\right) + \frac{3}{2}d,$$

compared to Lafourcade and Vardy general bound (3):

$$n \geqslant \frac{k}{s}(d-1).$$

E.g. for $s = 3$, $d = 4$, $k = 10 = 2(3+1)+2$ we get $n \geqslant 16$ compared to $n \geqslant 10$.

3. Codes with trellis complexity one

Theorem 6. *For an* $[n,k,1,d \geqslant 2]^{\mathrm{DCC}}$ *code we have*

(a) $d_{r+1} \geqslant d_r + d - 1$ *for* $1 \leqslant r < k$,
(b) $d_r \geqslant r(d-1) + 1$ *for* $1 \leqslant r < k$,
(c) $n \geqslant k(d-1) + 1$.

Proof. We see that (a) follows directly from Theorem 4 and that (b) follows from (a) by induction. Finally, (c) follows from (b) and Theorem 1, or alternatively, by putting $s = t = 1$ in Theorem 5. □

By Theorems 3 and 6, for an $[n,k,1,d \geqslant 2]^{\mathrm{DCC}}$ it is necessary that $dk - k + 1 \leqslant n < dk$. The main result of this section is to show that this is also sufficient, i.e. for all such n there do exist $[n,k,1,d]^{\mathrm{DCC}}$ codes. We do this by giving explicit code constructions of $[n,k,1,d \geqslant 2]^{\mathrm{DCC}}$ codes for all n, k, d for which $dk - k + 1 \leqslant n < dk$.

To give a compact description of the codes we will present, we introduce another notation. To a sequence $(b_0, a_1, b_1, a_2, b_2, \ldots, b_{k-1}, a_k, b_k)$ of non-negative integers we assosiate a generator matrix

$$\begin{pmatrix} \overbrace{1}^{b_0} & \overbrace{1}^{a_1} & \overbrace{1}^{b_1} & \overbrace{0}^{a_2} & \overbrace{0}^{b_2} & \cdots & \overbrace{0}^{b_{k-1}} & \overbrace{0}^{a_k} & \overbrace{0}^{b_k} \\ 0 & 0 & 1 & 1 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 1 & 1 \end{pmatrix}$$

of an $[n,k,d]$ code $C(b_0 a_1 b_1 a_2 b_2 \cdots b_{k-1} a_k b_k)$, where

$$n = \sum_{i=1}^{k} a_i + \sum_{i=0}^{k} b_i.$$

If

$$\begin{aligned} &a_r = a_{k+1-r} && \text{for } 1 \leqslant r \leqslant k, \\ &b_r = b_{k-r} && \text{for } 0 \leqslant r \leqslant k, \\ &b_0 + a_1 + b_1 = d, \\ &a_r + b_r \leqslant d && \text{for } 1 \leqslant r \leqslant k, \\ &b_{r-1} + a_r + b_r \geqslant d && \text{for } 1 \leqslant r \leqslant k, \end{aligned}$$

we call such a code an *ab*-code. If in addition

$$b_r \in \{0,1\} \quad \text{for } 0 \leqslant r \leqslant k,$$

we call the code a 1-*ab*-code. Note that this implies that

$$a_r \in \{d-2, d-1, d\} \quad \text{for } 1 \leqslant r \leqslant k.$$

For the sequence $b_0 a_1 b_1 a_2 b_2 \cdots b_{k-1} a_k b_k$ we will sometimes use a power notation, e.g. $(10^2)^2$ denotes 100100.

Lemma 4. *All* $[n,k,1,d]^{\mathrm{DCC}}$ *codes are* 1-*ab-codes.*

Proof. Let C be an $[n, k, 1, d]^{\mathrm{DCC}}$ code. Let

$$a_r = n - d_{r-1} - d_{k-r} \quad \text{for } 1 \leqslant r \leqslant k,$$
$$b_r = d_r + d_{k-r} - n \quad \text{for } 0 \leqslant r \leqslant k.$$

For $d = 1$ we get $C = \mathrm{GF}(2)^k$, and so $d_r = r$ for all r. Hence $a_r = 1$ and $b_r = 0$ for all r, and

$$C = C(01010 \cdots 010).$$

For $d \geqslant 2$, combining Theorem 3, Lemma 3, Corollary 2, and Theorem 4, we see that

$$C = C(b_0 a_1 b_1 a_2 b_2 \cdots b_{k-1} a_k b_k)$$

and that this is an 1-*ab*-code. □

Lemma 4 explains why we consider 1-*ab*-codes. However, not all 1-*ab*-codes are $[n, k, 1, d]^{\mathrm{DCC}}$ codes. For example, for $d \geqslant 2$, the code $C(0d0\delta 1\delta 0d0)$ where $\delta = d - 1$ is a 1-*ab*-code, but,

$$d_2 = w_S(\langle g_2, g_3 \rangle) = 2d - 1,$$

and

$$w_S(D_2^{\mathrm{L}}) = 2d > d_2.$$

Lemma 5. *Let C be an 1-ab-code. For each r, $1 \leqslant r \leqslant k$, there exist a set of r subscripts $i_1, i_2, \ldots, i_r$ such that*

$$d_r = w_S(\langle g_{i_1}, g_{i_2}, \ldots, g_{i_r} \rangle).$$

Proof. Let G denote the generator matrix of C. Any r-dimensional subspace D of C has a generator matrix AG where A is an $r \times k$ matrix of rank r. Row operations on A will not change the code D. Therefore, we may assume without loss of generality that $A = (a_{ij})$ is a reduced echelon matrix, that is, there exist numbers $j_1, j_2, \ldots, j_r$ such that

$$a_{ij_i} = 1 \quad \text{for } 1 \leqslant i \leqslant r,$$
$$a_{i'j_i} = 0 \quad \text{for } 1 \leqslant i \leqslant r,\ 1 \leqslant i' < i,$$
$$a_{ij} = 0 \quad \text{for } 1 \leqslant i \leqslant r,\ 1 \leqslant j < j_i. \quad \square$$

We say that D is a *quasi-diagonal subcode* if $a_{ij} = 0$ for $1 \leqslant i \leqslant r$ and $j \neq j_i$. The lemma states that for each r there exists an r-dimensional quasi-diagonal subcode D of C such that $d_r = w_S(D)$. Equivalently, if D is not quasi-diagonal, then there exists a quasi-diagonal subcode D' of the same dimension such that $w_S(D') \leqslant w_S(D)$. We show this by modifying the echelon matrix A to a matrix A' with only one non-zero element in each row. The modification can be done row by row. Suppose that the first $i-1$ rows of A contain a single non-zero element. Consider row i with its first non-zero element in position j_i. Let A' be the matrix which has the same elements as A outside row i, and which has a single 1 in row i in position j_i. Let D'' denote the r-dimensional code generated by the rows of D except row number i. Then $D = \langle D'', \boldsymbol{g}_{j_i} + \sum_{j=j_i+1}^{k} a_{ij}\boldsymbol{g}_j \rangle$ and $D' = \langle D'', \boldsymbol{g}_{j_i} \rangle$. Hence

$$w_S(D) = w_S(D'') + |\chi(D) \setminus \chi(D'')| = w_S(D'') + a_{j_i} + c$$

and

$$w_S(D') = w_S(D'') + |\chi(D') \setminus \chi(D'')| = w_S(D'') + a_{j_i} + c'$$

for some $c \geqslant 0$, $c' \in \{0, 1\}$. Here $c' = 0$ if $b_{j_i} = 0$. Similarly, $c' = 0$ if $b_{j_i} = 1$ and $j_{i+1} = j_i + 1$. In all other cases $c' = 1$. We have $w_S(D') \leqslant w_S(D)$ except when $c = 0$ and $c' = 1$. This can only occur if $d = 2$, $j_{i+1} > j_i + 1$, $b_r = 1$ for $j_i \leqslant r \leqslant j_{i+1} - 1$, and $a_{ij} = 1$ for $j_i + 1 \leqslant j_{i+1} - 1$. In this exeptional case we can choose $D' = \langle D'', \boldsymbol{g}_{j_{i+1}-1} \rangle$ to get $w_S(D') \leqslant w_S(D)$. This completes the induction. □

For a sequence $\bar{a} = (a_1, a_2, \ldots, a_k)$ define

$$\sigma(u, j) = \sigma(\bar{a}; u, j) = \sum_{i=u}^{u+j-1} a_i.$$

Lemma 6. *Let $(a_1, a_2, \ldots, a_m)$ be a sequence such that $a_i = a_{m+1-i}$ for all i, and $|\sigma(u, j) - \sigma(u', j)| \leqslant 1$ for all u, u', j such that $1 \leqslant j \leqslant m$ and $1 \leqslant u \leqslant u' \leqslant m - j + 1$. Then the 1-ab-codes C_t defined by*

$$C_t = C\Big(1a_1 1a_2 1 \ldots 1a_{m-1} 1 (a'_m 0 a'_1 1a_2 1 \ldots 1a_{m-1} 1)^t a_m 1\Big),$$

where $a'_1 = a_1 + 1$ and $a'_m = a_m + 1$, is a DCC code for all $t \geqslant 0$.

Proof. We first prove this for $t = 0$. Let

$$D = \langle \boldsymbol{g}_{i_1}, \boldsymbol{g}_{i_2}, \ldots, \boldsymbol{g}_{i_r} \rangle$$

be a subcode of C_0. Consider the last gap in the sequence $i_1, i_2, \ldots, i_r$: $i_{v+1} > i_v + 1$, but $i_{j+1} = i_j + 1$ for $j > v$. Let

$$D' = \langle \boldsymbol{g}_{i_1}, \boldsymbol{g}_{i_2}, \ldots, \boldsymbol{g}_{i_v}, \boldsymbol{g}_{i_v+1}, \ldots, \boldsymbol{g}_{i_v+(r-v)} \rangle.$$

Then

$$w_S(D) - w_S(D') = (1 + \sigma(v + 1, r - v) + 1) - (\sigma(v, r - v) + 1) \geqslant 0.$$

Now D' has one less gap in its sequence of subscripts, and we can repeat the process until we end up with a code D'' with no gaps, that is

$$D'' = \langle \boldsymbol{g}_u, \boldsymbol{g}_{u+1}, \ldots, \boldsymbol{g}_{u+r-1} \rangle$$

and $w_S(D'') \leqslant w_S(D)$. The same argument shows that

$$w_S(\langle \boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_r \rangle) \leqslant w_S(D'') \leqslant w_S(D).$$

By Lemma 5 we get

$$d_r = w_S(\langle g_1, g_2, \ldots, g_r \rangle).$$

We note that $\langle g_1, g_2, \ldots, g_r \rangle = D_r^{\mathrm{L}}$ and so

$$\chi(D_r^{\mathrm{L}}) = \chi(\langle g_1, g_2, \ldots, g_r \rangle) = \{1, 2, \ldots, d_r\}.$$

From the symmetry in the generator matrix we get

$$\chi(\langle g_{k+1-r}, g_{k+2-r}, \ldots, g_k \rangle) = \{n+1-d_r, n+2-d_r, \ldots, n\}.$$

Hence C_0 is a DCC code.

Now, consider C_t in general. Let

$$D = \langle g_{i_{01}}, g_{i_{02}}, \ldots, g_{i_{0j_0}}, \ldots, g_{i_{t1}}, g_{i_{t2}}, \ldots, g_{i_{tj_t}} \rangle$$

where

$$um + 1 \leqslant i_{u1} < i_{u2} < \cdots < i_{uj_u} \leqslant (u+1)m$$

for $0 \leqslant u \leqslant t$. Within each block we can perform the same operations as we did above. Thus we get $w_S(D') \leqslant w_S(D)$, where

$$D' = \langle g_1, g_2, \ldots, g_{j_0}, g_{m+1}, g_{m+2}, \ldots, g_{m+j_1}, \ldots, g_{tm+1}, g_{tm+2}, \ldots, g_{tm+j_t} \rangle.$$

Next we observe that if $j_u + j_{u+1} \leqslant m$, then

$$\begin{aligned} &w_S(\langle g_{um+1}, g_{um+2}, \ldots, g_{um+j_u}, g_{(u+1)m+1}, g_{(u+1)m+2}, \ldots, g_{(u+1)m+j_{u+1}} \rangle) \\ &= w_S(\langle g_{um+1}, g_{um+2}, \ldots, g_{um+j_u}, g_{um+m+1-j_{u+1}}, g_{(u+1)m+2}, \ldots, g_{um+m} \rangle) \end{aligned}$$

since $a_i = a_{m+1-i}$ and $a'_1 = a'_m$. Similarly, if $j_u + j_{u+1} > m$, then

$$\begin{aligned} &w_S(\langle g_{um+1}, g_{um+2}, \ldots, g_{um+j_u}, g_{(u+1)m+1}, g_{(u+1)m+2}, \ldots, g_{(u+1)m+j_{u+1}} \rangle) \\ &= w_S(\langle g_{um+1}, \ldots, g_{um+m}, g_{(u+1)m+m-j_u+1}, \ldots, g_{(u+1)m+j_{u+1}} \rangle). \end{aligned}$$

Hence we can move elements from one block to the preceding block without increasing the support weight. By repeatedly moving elements and removing gaps we get $w_S(D_r^{\mathrm{L}}) \leqslant w_S(D)$, where r is the dimension of D. Hence we get $d_r = w_S(D_r^{\mathrm{L}})$. By symmetry, we get as above that C_t is a DCC code.

Theorem 7. $[n, k, 1, d \geqslant 2]^{\mathrm{DCC}}$ *codes exist if and only if* $dk - k + 1 \leqslant n < dk$.

Proof. By Theorems 3 and Corollary 6, for an $[n, k, 1, d \geqslant 2]^{\mathrm{DCC}}$ it is necessary that $dk - k + 1 \leqslant n < dk$. It remains to show the if part, and we do this by giving explicit constructions of the forms described in Lemma 6. We use the notation $\{x\}$ for the integer closest to x, with the special case $\{n + 0.5\} = n$ for all integers n.

Case I: $n = kd - 2p - 1$ where $0 \leqslant p \leqslant (k-2)/2$. Use

$$a_r = d-2 \quad \text{for } r = \left\{ \frac{k-1}{2p+1} i + 1 \right\},\ 0 \leqslant i \leqslant 2p+1,$$
$$a_r = d-1 \quad \text{otherwise},$$

and $t = 0$ in Lemma 6.

Case II: k is odd and $n = kd - 2p$, where $0 < p \leqslant \frac{1}{2}(k-1)$. Use

$$a_r = a_{k+1-r} = d-2 \quad \text{for } r = \left\{ \frac{k-1}{2p} i + 1 \right\},\ 0 \leqslant i \leqslant p,$$
$$a_r = d-1 \quad \text{otherwise},$$

and $t = 0$ in Lemma 6.

Case III: k is even and $n = kd - 2p$, where $0 < p \leqslant m-1$. Let $k = \alpha m$, where α is even and m is odd. Use the construction in cases I and II (with m substituted for k) and $t = \alpha - 1$ in Lemma 6.

We have to show that the conditions in Lemma 6 are satisfied for the sequences in cases I and II. Consider case I. First we note that $[(k-1)/(2p+1)]i + 1$ is not of the form $n + 0.5$. Hence

$$k + 1 - \left\{ \frac{k-1}{2p+1} i + 1 \right\} = \left\{ k + 1 - \frac{k-1}{2p+1} i + 1 \right\}$$
$$= \left\{ \frac{k-1}{2p+1}(2p+1-i) + 1 \right\}.$$

This implies that $a_r = a_{k+1-r}$ for all r. Next, if

$$1 \leqslant u \leqslant u + j - 1 \leqslant k,$$

then

$$\sigma(u,j) = j(d-1) - \Delta(u,j)$$

where

$$\Delta(u,j) = |\mathscr{D}(u,j)|,$$

and where

$$\mathscr{D}(u,j) = \{ r \mid u \leqslant r \leqslant u + j - 1 \text{ and } a_r = d-2 \}.$$

Since $a_r = d-2$ if and only if $r = \{[(k-1)/(2p+1)]i + 1\}$ where $0 \leqslant i \leqslant 2p+1$, we get

$$\mathscr{D}(u,j) = \left\{ i | u \leqslant \frac{k-1}{2p+1} i + 1 \leqslant u + j - 1 \right\}.$$

Let $i_{\min}$ and $i_{\max}$ be the smallest and largest element of $\mathscr{D}(u,j)$. Then

$$\frac{(k-1)i_{\min} - p}{2p+1} \leqslant u \leqslant \frac{(k-1)i_{\min} + p}{2p+1}$$

and so

$$\frac{(2p+1)(u-1)-p}{k-1} \leqslant i_{\min} \leqslant \frac{(2p+1)(u-1)+p}{k-1}.$$

Similarly,

$$\frac{(2p+1)(u+j-2)-p}{k-1} \leqslant i_{\max} \leqslant \frac{(2p+1)(u+j-2)+p}{k-1}.$$

Since $\Delta(u,j) = i_{\max} - i_{\min} + 1$, we get

$$\frac{(2p+1)(j-1)-2p}{k-1} + 1 \leqslant \Delta(u,j) \leqslant \frac{(2p+1)(j-1)+2p}{k-1} + 1.$$

Therefore

$$\max_u\{\Delta(u,j)\} - \min_u\{\Delta(u,j)\} \leqslant \frac{4p}{k-1} < 2,$$

and so

$$|\sigma(u,j) - \sigma(u',j)| = |\Delta(u,j) - \Delta(u',j)| \leqslant 1$$

for all u, u' and j.

The proof of case II is similar for $p < (k-1)/2$. For $k=(p-1)/2$ we get $a_r = d-2$ for all r, $1 \leqslant r \leqslant k$ and so $\sigma(u,j) = j(d-2)$ for all u and j. □

4. Codes with trellis complexity two

We now consider the parameters n, k, d for which $[n,k,2,d \geqslant 2]^{\mathrm{DCC}}$ codes exist. Since $[n,k,1,d \geqslant 2]^{\mathrm{DCC}}$ codes exist for $n > kd - k$, we restrict our attention to $n \leqslant kd - k$. We will show that for even d, $[n,k,2,d \geqslant 2]^{\mathrm{DCC}}$ codes exist if and only if $n \geqslant \frac{1}{2}(k+1)d$. For odd d we show that $[n,k,2,d \geqslant 2]^{\mathrm{DCC}}$ codes exist for $n \geqslant \frac{1}{2}(k+1)(d-1)+k$. We believe that no $[n,k,2,d \geqslant 2]^{\mathrm{DCC}}$ codes exist for $n < \frac{1}{2}(k+1)(d-1)+k$, but we can only show a slightly weaker result.

Putting $s=t=2$ in Theorem 5 we get a lower bound on n for an $[n,k,2,d \geqslant 2]^{\mathrm{DCC}}$ code. However, we will show that this bound can be improved in most cases.

Lemma 7. *Let C be an $[n,k,2,d \geqslant 2]^{\mathrm{DCC}}$ code. If $\boldsymbol{x} \in \langle D^{\mathrm{L}}_{r-1}, D^{\mathrm{R}}_{k-r}\rangle \backslash D^{\mathrm{L}}_{r-1}$, then*

$$\boldsymbol{x} = \boldsymbol{y} + \boldsymbol{z}$$

where

$$\boldsymbol{y} \in D^{\mathrm{L}}_{r-1}, \qquad \boldsymbol{z} \in D^{\mathrm{R}}_{k-r},$$

$$u(\boldsymbol{x}) = u(\boldsymbol{z}), \qquad l(\boldsymbol{z}) \geqslant d_{r-1}.$$

Proof. Since $x \in \langle D^{\mathrm{L}}_{r-1}, D^{\mathrm{R}}_{k-r}\rangle$, by definition, there exist $y \in D^{\mathrm{L}}_{r-1}$ and $z \in D^{\mathrm{R}}_{k-r}$ such that $x = y + z$. Since $x \notin D^{\mathrm{L}}_{r-1}$ we have $z \neq \mathbf{0}$. By Lemma 3 we have

$$l(z) \geqslant n + 1 - d_{k-r} \geqslant d_{r-1}.$$

This also implies that

$$u(z) > d_{r-1} \geqslant u(y),$$

and so $u(x) = u(z)$. $\square$

If $g_r \in \langle D^{\mathrm{L}}_{r-1}, D^{\mathrm{R}}_{k-r}\rangle$, we say that g_r is of type I, otherwise it is of type II. From the proof of Theorem 4 we get the following lemma.

Lemma 8. *If g_r is of type* II, *then $g_{r+1} \in \langle D^{\mathrm{L}}_{r-1}, D^{\mathrm{R}}_{k-r}\rangle$.*

Lemma 9. *Let C be an $[n,k,2,d \geqslant 2]^{\mathrm{DCC}}$ code and $1 \leqslant r \leqslant k-1$.*

(i) *If g_r is of type* I, *then $g_r = y + z$ where $y \in D^{\mathrm{L}}_{r-1}$, $z \in D^{\mathrm{R}}_{k-r}$, and $d_r \geqslant d_{r-1} + d - 1$.*

(ii) *If g_r is of type* II, *then $g_{r+1} = y + z$ where $y \in D^{\mathrm{L}}_{r-1}$, $z \in D^{\mathrm{R}}_{k-r}$, and $d_{r+1} \geqslant d_r + \frac{1}{2}(d-1)$.*

Proof. *Case* I: g_r is of type I. By Lemma 7, g_r has the given representation and

$$d_r = u(g_r) = u(z) \geqslant l(z) + d - 1 \geqslant d_{r-1} + d - 1.$$

Case II: g_r is of type II. By Lemmas 7 and 8, g_{r+1} has the given representation with $u(z) = d_{r+1}$, and $l(z) \geqslant d_{r-1}$. Define

$$a = |\chi(z) \cap \chi(g_{r-1})|,$$
$$b = |\{i \mid i > d_{r-1}, g_{ri} = 1, \text{ and } z_i = 1\}|,$$
$$c = |\{i \mid i > d_{r-1}, g_{ri} = 1, \text{ and } z_i = 0\}|,$$
$$e = |\{i \mid i > d_{r-1}, g_{ri} = 0, \text{ and } z_i = 1\}|.$$

Then

$$d_r = d_{r-1} + b + c, \tag{6}$$

$$d_{r+1} = d_r + e, \tag{7}$$

and

$$d_r \leqslant w_S(\langle D^{\mathrm{L}}_{r-1}, g_r + z\rangle) = d_{r-1} + c + e. \tag{8}$$

Combining (6) and (8) we get

$$e \geqslant b. \tag{9}$$

By definition, if $a > 0$, then

$$a \leqslant d_{r-1} - l(z) + 1 \leqslant 1$$

and so $a = 1$. Hence

$$a + b + e = w(z) \geqslant d. \tag{10}$$

Combining this with (9) we get

$$d \leqslant 1 + 2e,$$

and so

$$d_{r+1} = d_r + e \geqslant d_r + \tfrac{1}{2}(d-1). \qquad \square$$

Theorem 8. *For an* $[n,k,2,d \geqslant 2]^{\mathrm{DCC}}$ *code, where* d *is even, we have*

$$d_r \geqslant \tfrac{1}{2}(r+1)d \quad \textit{for} \quad 1 \leqslant r \leqslant k.$$

Proof. The proof is by induction on r. We first observe that the result is true by the Griesmer bound for $r = 1$ and $r = 2$. Let $r \geqslant 2$, and suppose that the result is true up to r. By Lemma 9 we have either $d_r \geqslant d_{r-1} + d - 1$ and so

$$d_{r+1} \geqslant d_r + 1 \geqslant d_{r-1} + d \geqslant \tfrac{1}{2}(r-1)d + d = \tfrac{1}{2}(r+1)d$$

or $d_{r+1} \geqslant d_r + \frac{1}{2}d$ (since d is even) and so

$$d_{r+1} \geqslant d_r + \tfrac{1}{2}d \geqslant \tfrac{1}{2}(r-1)d + \tfrac{1}{2}d = \tfrac{1}{2}(r+1)d. \qquad \square$$

For odd d, let $\delta = (d-1)/2$. We have $d_2 \geqslant 3\delta + 2$ by the Griesmer bound, and the same argument as in the proof of Theorem 8 gives

$$d_r \geqslant (r+1)\delta + 2 \quad \text{for } r \geqslant 2.$$

However, in most cases this is weaker than the bound we obtain if we choose $t = s = 2$ in Theorem 5. The underlying results for this bound from Lemma 2 are

$$d_{r+2} \geqslant d_r + g(1,d) - 1 = d_r + 2\delta, \tag{11}$$

and

$$d_{r+3} \geqslant d_r + g(2,d) - 1 = d_r + 3\delta + 1. \tag{12}$$

Using these results we get a lower bound on d_r as in Theorem 5.

Theorem 9. *For an* $[n,k,2,d \geqslant 3]^{\mathrm{DCC}}$ *code, where* d *is odd, we have*

$$d_r \geqslant \tfrac{1}{2}(r+1)d - \tfrac{1}{6}(r - \alpha_r) \quad \textit{for } 1 \leqslant r \leqslant k,$$

where

$$\alpha_r = \begin{cases} 3 & \text{for } r \equiv 0 \pmod 3, \\ 1 & \text{for } r \equiv 1 \pmod 3, \\ 5 & \text{for } r \equiv 2 \pmod 3. \end{cases}$$

Proof. We have $d_1 = 2\delta + 1$ and we get $d_2 \geqslant 3\delta + 2$ by the Griesmer bound. Next

$$d_3 \geqslant d_1 + 2\delta = 4\delta + 1,$$

and we can show by an argument similar to the one in the appendix (but simpler) that $d_3 = 4\delta + 1$ is not possible. Hence

$$d_3 \geqslant 4\delta + 2. \tag{13}$$

This proves the theorem for $r \leqslant 3$, and the general result follows by induction using (12).

It is possible to show that

$$d_{r+5} \geqslant d_r + 5\delta + 2, \tag{14}$$

and this will give a better bound on d_r in most cases. The proof of (14) is a little technical and is given in an appendix. Using (14) we get the following bound on d_r; the proof is similar to the proof of Theorem 9.

Theorem 10. *For an $[n, k, 2, d \geqslant 5]^{\mathrm{DCC}}$ code, where d is odd, we have*

$$d_r \geqslant \tfrac{1}{2}(r+1)d - \tfrac{1}{10}(r - \beta_r) \quad \text{for } 1 \leqslant r \leqslant k,$$

where

$$\beta_r = \begin{cases} 5 & \text{for } r \equiv 0 \pmod 5, \\ 1 & \text{for } r \equiv 1 \pmod 5, \\ -3 & \text{for } r \equiv 2 \pmod 5, \\ 3 & \text{for } r \equiv 3 \pmod 5, \\ -1 & \text{for } r \equiv 4 \pmod 5. \end{cases}$$

Lemma 10. *If the ab-code $C = C(b_0 a_1 b_1 a_2 \cdots b_{k-1} a_k b_k)$ is an $[n, k, s, d]^{\mathrm{DCC}}$ code, then*

$$C' = C(b'_0 a_1 b'_1 a_2 \cdots b'_{k-1} a_k b'_k),$$

where $b'_i = b_i + 1$, is an $[n + k + 1, k, 2, d + 2]^{\mathrm{DCC}}$ code.

Proof. If g'_i is the ith row in the generator matrix for C', the correspondence $g_i \leftrightarrow g'_i$ extends to a natural 1–1 correspondence between the subspaces of D of C and the subspaces of D' of C'. For any subspace D of C of dimension r, $\chi(D)$ contains $v(D) \geqslant r + 1$ of the groups of a_i 1s (where $1 \leqslant i \leqslant k$), and $v(D_r^{\mathrm{L}}) = r + 1$. Hence,

$$w_S(D') = w_S(D) + v(D) \geqslant w_S(D_r^{\mathrm{L}}) + r + 1 = w_S((D_r^{\mathrm{L}})').$$

Hence $(D')_r^{\mathrm{L}} = (D_r^{\mathrm{L}})'$. Similarly, we have $(D')_r^{\mathrm{R}} = (D_r^{\mathrm{R}})'$. Hence C' is a DCC code. Clearly, the length has increased by $k+1$ and the minimum distance by 2. Further $s' = 2$ (except when $b_i = 0$ for all i). □

Let $\delta + 1 \leqslant u \leqslant 2\delta$. Starting from an $[n_0, k, 1, 2u - 2\delta + 1]^{\mathrm{DCC}}$ 1-*ab*-code where

$$k(2u - 2\delta + 1) - k + 1 \leqslant n_0 \leqslant k(2u - 2\delta + 1),$$

and repeating the construction in Lemma 10 a total of $2\delta - u$ times, we get an $[n, k, 2, 2\delta + 1]^{\mathrm{DCC}}$ code, where

$$(k+1)u - (2u - 2\delta) + 1 \leqslant n \leqslant (k+1)u - (2u - 2\delta) + k.$$

For $u = \delta + 1$ we get

$$(k+1)\delta + k \leqslant n \leqslant (k+1)\delta + 2k - 1,$$

for $u = \delta + 2$ we get

$$(k+1)\delta + 2k - 1 \leqslant n \leqslant (k+1)\delta + 3k - 2,$$

etc.

Similarly, starting from $[n_0, k, 1, 2u - 2\delta]^{\mathrm{DCC}}$ 1-*ab*-codes, we get $[n, k, 2, 2\delta]^{\mathrm{DCC}}$ codes for all n,

$$(k+1)\delta \leqslant n \leqslant 2k\delta - k.$$

Summarizing, we get the following result.

Theorem 11. (i) *If d is even, then there exist $[n, k, 2, d]^{\mathrm{DCC}}$ codes for all n in the range*

$$\tfrac{1}{2}(k+1)d \leqslant n \leqslant kd - k.$$

(ii) *If d is odd, then there exist $[n, k, 2, d]^{\mathrm{DCC}}$ codes for all n in the range*

$$\tfrac{1}{2}(k+1)(d-1) + k \leqslant n \leqslant kd - k.$$

Theorem 11(i) shows that the lower bound in Theorem 8 is best possible. For odd d there is a gap of approximately $\frac{1}{2}k + \frac{1}{10}k = \frac{3}{5}k$ between the lower bound in Theorem 10 and the smallest n given by Theorem 11(ii). The structure of possible $[n, k, 2, d]^{\mathrm{DCC}}$ codes with d odd and $n < (k+1)\delta + k$ is described by the next theorem (except for small δ).

Theorem 12. *If C is an $[n, k, 2, 2\delta + 1]^{\mathrm{DCC}}$, where*

$$\delta > \tfrac{1}{3}(k+5)$$

and

$$n < (k+1)\delta + k,$$

then all g_r *are of type* II, *and*

$$l(g_r) = n + 1 - d_{k+1-r}$$

for all r.

Proof. From Theorem 9 we get

$$d_r \geqslant (r+1)\delta + \tfrac{1}{3}(2r+2),$$

and similarly we can get

$$d_{r+j} \geqslant d_r + j\delta + \tfrac{1}{3}(2j-8) \qquad (15)$$

for $j \geqslant 2$ and all r.

Suppose g_r is of type I for some r, $2 \leqslant r \leqslant k-2$. Then $d_r \geqslant d_{r-1} + 2\delta$ and so

$$\begin{aligned} n = d_k &\geqslant d_r + (k-r)\delta + \tfrac{1}{3}(2k-2r-8) \\ &\geqslant d_{r-1} + (k-r+2)\delta + \tfrac{1}{3}(2k-2r-8) \\ &\geqslant r\delta + \tfrac{2}{3}r + (k-r+2)\delta + \tfrac{1}{3}(2k-2r-8) \\ &= (k+2)\delta + \tfrac{1}{3}(2k-8) > (k+1)\delta + k - 1 \end{aligned}$$

for $\delta > \frac{1}{3}(k+5)$, a contradiction. Assume that g_{k-1} is of type I. Then

$$l(g_k) = d_k - 2\delta \geqslant d_{k-1} + 1 - 2\delta \geqslant d_{k-2} + 1.$$

Hence

$$d_{k-1} \leqslant w_S(\langle D^{\mathrm{L}}_{r-2}, g_{r-1} + g_r \rangle) = d_{k-2} + d_k - d_{k-1}$$

and so

$$d_k - d_{k-1} \geqslant d_{k-1} - d_{k-2} \geqslant 2\delta.$$

Hence

$$d_k \geqslant d_{k-2} + 4\delta \geqslant (k+3)\delta + \tfrac{1}{3}(2k-2) > (k+1)\delta + k - 1$$

for $\delta > \frac{1}{6}(k-1)$.

Since all g_r are of type II, we have $d_{r+1} \geqslant d_r + \delta$ for all r (in particular, (15) is true also for $j = 1$), and $l(g_r) \geqslant d_{r-2}$ for all r. Suppose that $l(g_{r+1}) \leqslant l(g_r)$ for some r. Since $l(g_r) = n + 1 - d_{r'}$ and $l(g_{r+1}) = n + 1 - d_{r''}$ for some $r' \neq r''$ we have

$$l(g_{r+1}) \leqslant l(g_r) - \delta.$$

Hence

$$d_r \geqslant l(g_r) + 2\delta \geqslant l(g_{r+1}) + 3\delta \geqslant d_{r-1} + 3\delta \geqslant (r+3)\delta + \tfrac{2}{3}r,$$

and we get

$$\begin{aligned} n = d_k &\geqslant d_r + (k-r)\delta + \tfrac{1}{3}(2k - 2r - 8) \\ &\geqslant (k+3)\delta + \tfrac{1}{3}(2k-8) > (k+1)\delta + k - 1 \end{aligned}$$

for $\delta > \frac{1}{6}(k+5)$, again a contradiction. Therefore $l(g_{r+1}) > l(g_r)$ for all r. By Lemma 1, this implies that $l(g_r) = n + 1 - d_{k+1-r}$ for all r. □

A computer search showed that there are no $[n,k,2,2\delta+1]^{\mathrm{DCC}}$ codes with $\delta > \frac{1}{3}(k+5)$ and $n < (k+1)\delta + k$ for $k \leqslant 12$. If there exist any $[n,k,2,2\delta+1]^{\mathrm{DCC}}$ codes with $n < (k+1)\delta + k$ at all is an open question.

Appendix A

In this appendix we prove the following lemma.

Lemma A.1. *For an $[n,k,2,d \geqslant 5]^{\mathrm{DCC}}$ code, where d is odd, we have*

$$d_{r+5} \geqslant d_r + 5\delta + 2. \tag{A.1}$$

Proof. By (11) and (12) we have

$$d_{r+5} \geqslant d_{r+3} + 2\delta \geqslant d_r + 5\delta + 1. \tag{A.2}$$

We will show that $d_{r+5} = d_r + 5\delta + 1$ is not possible. Suppose

$$d_{r+5} = d_r + 5\delta + 1 \tag{A.3}$$

for some r. By (9) we have

$$d_{r+3} = d_r + 3\delta + 1. \tag{A.4}$$

Similarly, since

$$d_{r+5} \geqslant d_{r+2} + 3\delta + 1 \geqslant d_r + 5\delta + 1,$$

we get

$$d_{r+2} = d_r + 2\delta. \tag{A.5}$$

Since

$$d_{r+5} - d_{r+4} < d_{r+5} - d_{r+3} = 2\delta = d - 1,$$

we conclude from Lemma 9(i) that $\boldsymbol{g}_{r+5}$ is of type II. Similar arguments show that $\boldsymbol{g}_j$ is of type II for all

$$j \in \{r+1, r+2, r+3, r+4\}.$$

Hence, by Lemma 7, for $j \in \{r+2, r+3, r+4, r+5\}$ there exist vectors $\boldsymbol{y}_j \in D^{\mathrm{L}}_{j-2}$ and $z_j \in D^{\mathrm{R}}_{k-j}$ such that

$$\boldsymbol{g}_j = \boldsymbol{y}_j + z_j, \tag{A.6}$$

$$u(z_j) = d_j, \tag{A.7}$$

$$l(z) \geqslant d_{j-2}. \tag{A.8}$$

We have $u(\boldsymbol{x}) - l(\boldsymbol{x}) \geqslant d - 1 = 2\delta$ for all codewords $\boldsymbol{x}$. In particular,

$$d_r + 2\delta = d_{r+2} = u(z_{r+2}) \geqslant l(z_{r+2}) + 2\delta \geqslant d_r + 2\delta$$

and so

$$l(z_{r+2}) = d_r.$$

Similarly, we get

$$l(z_{r+5}) = d_{r+3} = d_r + 3\delta + 1.$$

By the Griesmer bound

$$d_{r+5} - l(z_{r+4}) = w_S(\langle z_{r+4}, z_{r+5}\rangle) \geqslant d_2 \geqslant 3\delta + 1$$

and so $l(z_{r+4}) \leqslant d_r + 2\delta$. On the other hand,

$$l(z_{r+4}) \geqslant d_{r+2} = d_r + 2\delta,$$

and so

$$l(z_{r+4}) = d_r + 2\delta.$$

To determine $l(z_{r+3})$ requires a little more effort. First

$$d_{r+4} - l(z_{r+3}) = w_S(\langle z_{r+3}, z_{r+4}\rangle) \geqslant d_2 \geqslant 3\delta + 1,$$

and so

$$l(z_{r+3}) \leqslant d_r + \delta = l(z_{r+4}) - \delta.$$

Also

$$l(z_{r+3}) \geqslant d_{r+1} \geqslant d_r + 1 = l(z_{r+2}) + 1.$$

Hence $l(z_j) = n + 1 - d_{k+1-j}$ for $j \in \{r+2, r+3, r+4\}$, and so

$$\begin{aligned} d_{k-r-2} - d_{k-r-3} &= l(z_{r+4}) - l(z_{r+3}) \\ &\leqslant l(z_{r+4}) - l(z_{r+2}) - 1 = 2\delta - 1 < d - 1. \end{aligned}$$

Therefore, g_{k-r-2} is of type II, and so

$$l(z_{r+3}) - l(z_{r+2}) = d_{k-r-1} - d_{k-r-2} \geqslant \delta,$$

and $l(z_{r+3}) \geqslant d_r + \delta$. Therefore,

$$l(z_{r+3}) = d_r + \delta.$$

Hence we have the following situation:

$$\begin{array}{l} \\ z_{r+2} = (\\ z_{r+3} = (\\ z_{r+4} = (\\ z_{r+5} = (\end{array} \begin{array}{ccccccccc} \overbrace{}^{d_r-1} & \overbrace{}^{\delta} & \overbrace{}^{\delta} & \overbrace{}^{1} & \overbrace{}^{\delta} & \overbrace{}^{1} & \overbrace{}^{\delta} & \overbrace{}^{\delta} & \overbrace{}^{n-d_{r+5}} \\ & 1 & 1 & 1 & & & & & \\ & & 1 & a & 1 & 1 & & & \\ & & & 1 & 1 & b & 1 & & \\ & & & & & 1 & 1 & 1 & \end{array} \begin{array}{l} \\ ) \\ ) \\ ) \\ ) \end{array}$$

where $a, b \in \{0, 1\}$, and all the elements which are left out are zero. We have

$$d_{r+2} + \delta + 1 = d_{r+3} \leqslant w_S(\langle D^{\mathrm{L}}_{r+2}, z_{r+3} + z_{r+4} \rangle) = d_{r+2} + \delta + (1 - b)$$

and so $b = 0$. Similarly, $a = 0$. However, this implies that

$$w(z_{r+2} + z_{r+3} + z_{r+4} + z_{r+5}) = 2\delta < d,$$

a contradiction since $z_{r+2} + z_{r+3} + z_{r+4} + z_{r+5} \in C$. $\square$

References

[1] S. Encheva and T. Kløve, Codes satisfying the chain condition, IEEE Trans. Inform. Theory 40 (1994) 175–180.

[2] G.D. Forney, Dimension/length profiles and trellis complexity of linear block codes, IEEE Trans. Inform. Theory 40 (1994) 1741–1752.

[3] G.D. Forney, Dimension/length profiles and trellis complexity of lattices, IEEE Trans. Inform. Theory 40 (1994) 1753–1772.

[4] T. Helleseth, T. Kløve and J. Mykkeltveit, The weight distribution of irreducible cyclic codes, Discrete Math. 18 (1977) 179–211.

[5] T. Helleseth, T. Kløve and Ø. Ytrehus, Generalized Hamming weights of linear codes, IEEE Trans. Inform. Theory 38 (1992) 1133–1140.

[6] T. Helleseth, T. Kløve and Ø. Ytrehus, Codes and the chain condition, Proc. Internat. Workshop on Algebraic and Combinatorial Coding Theory, Voneshta Voda, Bulgaria (1992) 88–91.

[7] T. Helleseth, T. Kløve and Ø. Ytrehus, Codes, weight hierarchies, and chains, Proc. ICCS/ISITA '92, Singapore (1992) 608–612.

[8] T. Kasami, T. Takata, T. Fujiwara and S. Lin, On the optimum bit orders with respect to the state complexity of trellis diagrams for binary linear codes, IEEE Trans. Inform. Theory 39 (1993) 242–245.

[9] T. Kløve, Support weight distribution of linear codes, Discrete Math. 106/107 (1992) 311–316.
[10] T. Kløve, Minimum support weights of binary codes, IEEE Trans. Inform. Theory 39 (1993) 648–654.
[11] A. Lafourcade and A. Vardy, Asymptotically good codes have infinite trellis complexity, IEEE Trans. Inform. Theory 41 (1995) 555–559.
[12] A. Vardy and Y. Be'ery, Maximum-likelihood soft decision decoding of BCH codes, IEEE Trans. Inform. Theory 40 (1994) 546–554.
[13] V.K. Wei, Generalized Hamming weights for linear codes, IEEE Trans. Inform. Theory 37 (1991) 1412–1418.