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Over-Determined Boundary Value Problems

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A problem of some physical interest is finding that integral curve of a differential equation which best fits some experimental data. For example we might measure the displacement $Y(t_k)$ of a pendulum at a number of times t_k. Supposing the motion of the pendulum is described by

$$y''(t) + \sin y(t) = 0, \tag{1}$$

we then want to find a solution $y(t)$ such that $y(t_k)$ is close to $Y(t_k)$ for all k. Bellman and Kalaba in [1] and elsewhere have used least squares as a criterion for closeness of fit. We shall study the Chebyshev criterion of minimizing the maximum error. After making some reasonable assumptions about the differential equation and the data, we shall prove the existence of a best fit and characterize it. The characterization is computationally useful. We shall also study the question of uniqueness of the best fit. The results are obtained quite easily because of an interesting connection with the theory of approximation by unisolvent functions.

We shall fit the integral curves of

$$y^{(n)}(t) + f(t, y(t), y'(t),..., y^{(n-1)}(t)) = 0 \tag{2}$$

to $\{Y(t_1),..., Y(t_m)\}$. (Other possibilities such as a system of first order equations could be handled in the same way.) The data points $t_1,..., t_m$ are to lie in $[a, b]$. We are interested in an integral curve of (2) which minimizes

$$\max_{1 \leqslant k \leqslant m} |y(t_k) - Y(t_k)|.$$

We shall encounter difficulties unless we restrict $[a, b]$. The example (1) shows this clearly for it has a nontrivial solution $y(t)$ which vanishes at suitable t_k, $k = 0, 1,...$. Since $y(t) \equiv 0$ is a solution of (1) which also fits $Y(t_k) = 0$, $k = 0, 1,...$, with zero error, we see there are two best fits to this data and they need not be close to each other. We shall suppose $b - a$ is small enough that this sort of interpolation problem has at most one solution. Specifically we assume

(U) For any $x_1, \ldots, x_n$ such that $a \leqslant x_1 < x_2 < \cdots < x_{n-1} < x_n \leqslant b$ and any $z_1, \ldots, z_n$, (2) has a unique solution $y(t)$ such that

$$y(x_k) = z_k \qquad k = 1, \ldots, n.$$

We shall also suppose the problem is over-determined, i.e., $m \geqslant n$. The problem is then a generalization of the idea of a boundary value problem for (2).

THEOREM. *If* (U) *holds for* (2) *on* $[a, b]$ *and if initial value problems for* (2) *have unique solutions on* $[a, b]$, *then there exists a solution* $y(t)$ *of* (2) *which minimizes* (3). *The minimal value* d *of* (3) *is attained for* $y(t)$ $n + 1$ *times, at the points* $t_0 < t_1 < \cdots < t_n$ *say, with alternating sign*

$$y(t_j) - Y(t_j) = (-1)^j (y(t_0) - Y(t_0)) \qquad j = 0, 1, \ldots, n.$$

If $w(t)$ *is any solution of* (2) *for which there are* $\tau_0 < \tau_1 < \cdots < \tau_n$ *with*

$$\operatorname{sign}(w(\tau_j) - Y(\tau_j)) = (-1)^j \operatorname{sign}(w(\tau_0) - Y(\tau_0)) \qquad j = 0, 1, \ldots, n$$

then

$$\min_j | w(\tau_j) - Y(\tau_j) | \leqslant d \leqslant \max_j | w(\tau_j) - Y(\tau_j) | .$$

Because of the uniqueness assumption, the integral curves are uniquely specified by n parameters such as

$$y(a) = a_1, \qquad y'(a) = a_2, \ldots, y^{(n-1)}(a) = a_n .$$

Let us denote the curve specified by $A = (a_1, \ldots, a_n)$ as $y(t, A)$. The assumption (U) about the family of approximating functions $y(t, A)$ is precisely Tornheim's definition of an n-parameter family [2]. Such families are more commonly called unisolvent families today (Motzkin [3], Rice [4, p. 70 ff.]). The theorem is a translation to this context of the results of [2, 3]. There are few interesting examples of nonlinear unisolvent families and this source of such families apparently has not been explicitly noted before.

If we fit $y(t, A)$ to a *continuous* curve $Y(t)$ on $[a, b]$, the best approximation is unique; however, this continuity is necessary [2]. To obtain uniqueness in our case we need *strong* unisolvence, meaning that $y(t, A_1) - y(t, A_2)$ has at most $n - 1$ zeros in $[a, b]$ if $A_1 \neq A_2$ where a zero without change of sign in its neighborhood is counted twice. Our assumptions imply strong unisolvence when $n = 2$ so by [3] we have the

COROLLARY. *When* $n = 2$ *the best approximation of the Theorem is unique.*

There has been considerable interest in criteria which imply the validity of (U) for (2), especially when $n = 2$. Lasota and Opial [5] have given a suitable condition of general n and the papers Schrader [6], Shampine [7] give conditions for $n = 2$. The last paper asserts that (U) holds for the equation (1) if only $b - a < \pi$. The equation is Lipschitzian so that uniqueness is true of initial value problems. We see the corollary is applicable provided all the t_k lie in an interval $[a, b]$ with $b - a < \pi$.

When $n = 2$ the use of the characterization to compute the best approximation is qualitatively the same as fitting a straight line to this data, c.f., Rice [4, p. 53]. After numerically obtaining an integral curve we compute the errors at the points t_k and their signs. If, say, the largest positive error is greater in magnitude than the largest negative error, we wish to raise the integral curve to decrease (3). This can be done conveniently by using various ways of specifying the integral curves. If it is convenient to solve boundary value problems for (2), let us specify $y(t, A)$ by $y(a) = a_1$, $y(b) = a_2$. Because of (U) if we change only one parameter, the change in $y(t, A)$ is monotone. For example if $A' = (a_1, a_2')$ with $a_2' > a_2$, then $y(t, A') > y(t, A)$ on $(a, b]$. A similar procedure can be used when it is convenient to solve initial value problems for (2). We use $y(a) = a_1$, $y'(a) = a_2$ and vary a_2 to obtain the best fit with a_1 fixed. We then switch to $y(b) = y(b, A)$, $y'(b) = b_2$ and vary b_2, etc. In either case as soon as we have produced a good enough approximation that the error alternates in sign, we have a bound from the theorem on the optimal error and how close we are to it.

With the hypotheses of the corollary the alternating behavior of the best approximation also holds for best weighted pth power approximation, $0 < p \leqslant \infty$; however, uniqueness of the best approximation is not necessarily present [8, 9].

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