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Fixed points and exponential stability for stochastic Volterra–Levin equations ☆

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ABSTRACT

In this paper we study a stochastic Volterra–Levin equation. By using fixed point theory, we give some conditions for ensuring that this equation is exponentially stable in mean square and is also almost surely exponentially stable. Our result generalizes and improves on the results in [14,1,30].

1. Introduction

For more than one hundred years Lyapunov's direct method has been the primary technique for dealing with stability problems in deterministic/stochastic differential equations and functional differential equations. Yet numerous difficulties with the theory and application to specific problems persist and it does seem that new methods are needed to address those difficulties. Recently, Burton and other authors have applied fixed point theory to investigate the stability for deterministic systems, and it has been shown that some of these difficulties vanish on applying fixed point theory; for example, see the monograph in [1] and papers [2–13].

To the best of the author's knowledge, up to now, there have been a few papers in which the fixed point theory is used to deal with the stability for stochastic (delay) differential equations; see [14–18]. More precisely, Appleby in [14] (also see [1, pp. 315–328]) studied the almost sure stability for some classical equations by splitting the stochastic differential equation into two equations, one being a fixed stochastic problem and the other a deterministic stability problem with forcing function. In [15], Luo used a method different to that in [14] to investigate the mean square asymptotic stability by means of fixed point theory for neutral stochastic differential equations. In [16–18], Luo used the fixed point theory to study the exponential stability of mild solutions of stochastic partial differential equations with bounded delays and with infinite delays.

The stability theory of stochastic differential equations with/without delay has been considered by many authors over the last few years; see the monographs [19–29] among others.

In the present paper, we focus on the exponential stability for the classical stochastic Volterra–Levin equations, which has been discussed in [14], [1, pp. 315–328] only for stability, not for exponential stability. Our method is based on the contraction fixed point principle, and is different from the usual method. Some conditions of an averaging nature are

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obtained to ensure that the equation is exponentially stable in mean square and is also almost surely exponentially stable, while all the known conditions are pointwise conditions. Moreover, our result generalizes and improves on the results in [14,1,30].

2. Main result

Let $\{\Omega, \mathcal{F}, P\}$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions, i.e., the filtration is right continuous and $\mathcal{F}_0$ contains all P-null sets. Let $\{B(t),\ t \geq 0\}$ denote a standard Brownian motion defined on $\{\Omega, \mathcal{F}, P\}$. Let $L > 0$. We denote by $C([-L, 0]; \mathbb{R})$ the family of all continuous functions $\varphi : [-L, 0] \to \mathbb{R}$ with $\|\varphi\| = \sup_{-L\leq\theta\leq 0} |\varphi(\theta)|$. For I an interval of $\mathbb{R}$, we use the notation $C(I; \mathbb{R})$ to denote the family of continuous functions $\varphi : I \to \mathbb{R}$. We have the mappings $\sigma \in C([0, \infty); \mathbb{R}),\ p \in C([-L, 0]; \mathbb{R})$ and $g \in C(\mathbb{R}; \mathbb{R})$.

Consider the following Volterra–Levin equation perturbed by additive noise of the form [14,1]

$$\mathrm{d}x(t) = -\left(\int_{t-L}^{t} p(s-t)g(x(s))\mathrm{d}s\right)\mathrm{d}t + \sigma(t)\mathrm{d}B(t), \quad t \geq 0 \tag{2.1}$$

with the initial condition

$$x(s) = \psi(s) \in C([-L, 0]; \mathbb{R}), \quad -L \leq s \leq 0. \tag{2.2}$$

As in [14] or [1, pp. 315–328], throughout this paper, we always assume that the following conditions on g and p hold:

$$xg(x) \geq 0, \qquad g(0) = 0, \quad \text{and} \quad \gamma := \lim_{x\to 0} \frac{g(x)}{x} \text{ exists.} \tag{2.3}$$

$$\text{There exists a } \alpha > 0 \text{ such that } \frac{g(x)}{x} \geq 2\alpha. \tag{2.4}$$

$$\text{There exists a } K > 0 \text{ such that for all } x, y \in \mathbb{R}, \quad |g(x) - g(y)| \leq K|x - y|; \tag{2.5}$$

and

$$2K \int_{-L}^{0} |p(s)s|\mathrm{d}s < 1, \qquad \int_{-L}^{0} p(s)\mathrm{d}s = 1. \tag{2.6}$$

By Theorem 7.3.1 in [1], under the condition (2.5), Eq. (2.1) has a unique continuous solution. We also mention here that the unperturbed equation of Eq. (2.1), i.e.,

$$x'(t) = -\int_{t-L}^{t} p(s-t)g(x(s))\mathrm{d}s,$$

is first used in [31] and later in [32] to model a certain biological problem.

Definition 2.1. Eq. (2.1) with the initial condition (2.2) is said to be exponentially stable in mean square if there exists a pair of positive constants λ and C such that

$$E|x(t)|^2 \leq CE\|\psi\|^2 \mathrm{e}^{-\lambda t}, \quad t \geq 0. \tag{2.7}$$

Definition 2.2. Eq. (2.1) is said to be almost surely exponentially stable if there exists a $\lambda > 0$ such that there is a finite random variable β such that

$$|x(t)| \leq \beta \mathrm{e}^{-\lambda t} \quad \text{a.s. for all } t \geq 0, \tag{2.8}$$

or equivalently if

$$\limsup_{t\to\infty} \frac{1}{t} \ln |x(t)| \leq -\lambda \quad \text{a.s.}$$

Theorem 2.1. *Suppose that the conditions (2.3)–(2.6) hold. Moreover, if one of the following two conditions holds:*

(i) $\int_0^t \mathrm{e}^{4\alpha s}\sigma^2(s)\mathrm{d}s$ *is bounded for all* $t \geq 0$,
(ii) $\int_0^\infty \mathrm{e}^{4\alpha s}\sigma^2(s)\mathrm{d}s = \infty$ *and* $\mathrm{e}^{\alpha t}\sigma^2(t) \to 0$ *as* $t \to \infty$,

then Eq. (2.1) is exponentially stable in mean square, that is, $\mathrm{e}^{\alpha t}E|x(t)|^2 \to 0$ *as* $t \to \infty$.

Proof. Define a continuous function $a(t) : [0, \infty) \to [0, \infty)$ by

$$a(t) := \begin{cases} \dfrac{g(x(t))}{x(t)}, & \text{if } x(t) \neq 0, \\ \gamma, & \text{if } x(t) = 0. \end{cases}$$

Thus our equation is

$$\mathrm{d}x(t) = -a(t)x(t)\mathrm{d}t + \mathrm{d}\left(\int_{-L}^{0} p(s) \int_{t+s}^{t} g(x(u))\mathrm{d}u\mathrm{d}s\right) + \sigma(t)\mathrm{d}B(t), \quad t \geq 0. \tag{2.9}$$

Let $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ be the Banach space of all bounded and continuous in mean square $\mathcal{F}_0$-adapted processes $\phi(t, \omega) : [-L, \infty] \times \Omega \to R$ with the supremum norm

$$\|\phi\|_{\mathcal{B}} := \sup_{t \geq 0} E\,|\phi(t)|^2 \quad \text{for } \phi \in \mathcal{B}.$$

Denote by S the complete metric space with the supremum metric consisting of functions $\phi \in \mathcal{B}$ such that $\phi(s) = \psi(s)$ on $s \in [-L, 0]$ and $E\,|\phi(t, \omega)|^2 \to 0$ as $t \to \infty$.

Define an operator $\Phi : S \to S$ by $\Phi(x)(t) = \psi(t)$ for $t \in [-L, 0]$ and for $t \geq 0$,

$$\Phi(x)(t) := \sum_{i=1}^{4} I_i(t), \tag{2.10}$$

where

$$I_1(t) := \mathrm{e}^{-\int_0^t a(u)\mathrm{d}u}\left(\psi(0) - \int_{-L}^{0} p(s) \int_{s}^{0} g(\psi(u))\mathrm{d}u\mathrm{d}s\right),$$
$$I_2(t) := \int_{-L}^{0} p(s) \int_{t+s}^{t} g(x(u))\mathrm{d}u\mathrm{d}s,$$
$$I_3(t) := -\int_0^t \mathrm{e}^{-\int_v^t a(s)\mathrm{d}s} a(v) \int_{-L}^{0} p(s) \int_{v+s}^{v} g(x(u))\mathrm{d}u\mathrm{d}s\mathrm{d}v,$$
$$I_4(t) := \int_0^t \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\sigma(s)\mathrm{d}B(s).$$

We first verify the mean square continuity of Φ on $[0, \infty)$. Let $x \in S$, $t_1 \geq 0$, and $|r|$ be sufficiently small; then

$$E|\Phi(x)(t_1 + r) - \Phi(x)(t_1)|^2 \leq 4\sum_{i=1}^{4} E|I_i(t_1 + r) - I_i(t_1)|^2.$$

It can easily be shown that

$$E|I_i(t_1 + r) - I_i(t_1)|^2 \to 0, \quad i = 1, 2, 3$$

as $r \to 0$. Further, by using Burkhölder–Davis–Gundy inequality [24], we get

$$\begin{aligned} E|I_4(t_1 + r) - I_4(t_1)|^2 &\leq 2E\left|\int_0^{t_1}\left(\mathrm{e}^{-\int_{t_1}^{t_1+r} a(u)\mathrm{d}u} - 1\right)\mathrm{e}^{-\int_0^{t_1} a(u)\mathrm{d}u}\sigma(s)\mathrm{d}B(s)\right|^2 + 2E\left|\int_{t_1}^{t_1+r}\mathrm{e}^{-\int_s^{t_1+r} a(u)\mathrm{d}u}\sigma(s)\mathrm{d}B(s)\right|^2 \\ &\leq 2E\int_0^{t_1}\left(\mathrm{e}^{-\int_{t_1}^{t_1+r} a(u)\mathrm{d}u} - 1\right)^2 \mathrm{e}^{-2\int_0^{t_1} a(u)\mathrm{d}u}\sigma^2(s)\mathrm{d}s + 2E\int_{t_1}^{t_1+r}\mathrm{e}^{-2\int_s^{t_1+r} a(u)\mathrm{d}u}\sigma^2(s)\mathrm{d}s \\ &\to 0 \end{aligned} \tag{2.11}$$

as $r \to 0$. Thus, Φ is indeed mean square continuous on $[0, \infty)$.

Next, we show that $\Phi(S) \subset S$. It is easy to get $\mathrm{e}^{\alpha t}E|I_i(t)|^2 \to 0$ as $t \to \infty$, $i = 1, 2$. We only need to prove $\mathrm{e}^{\alpha t}E|I_3(t)|^2 \to 0$ and $\mathrm{e}^{\alpha t}E|I_4(t)|^2 \to 0$ as $t \to \infty$. In fact,

$$\begin{aligned} \mathrm{e}^{\alpha t}E|I_3(t)|^2 &= \mathrm{e}^{\alpha t}E\left(\int_0^t \mathrm{e}^{-\int_v^t a(s)\mathrm{d}s}a(v)\int_{-L}^{0} p(s)\int_{v+s}^{v} g(x(u))\mathrm{d}u\mathrm{d}s\mathrm{d}v\right)^2 \\ &\leq \mathrm{e}^{\alpha t}E\left(\int_0^t \mathrm{e}^{-\int_v^t a(u)\mathrm{d}u}a(v)\mathrm{d}v\right)\left(\int_0^t \mathrm{e}^{-\int_v^t a(u)\mathrm{d}u}a(v)\left(\int_{-L}^{0} p(s)\int_{v+s}^{v} g(x(u))\mathrm{d}u\mathrm{d}s\right)^2\mathrm{d}v\right) \end{aligned}$$

$$
\begin{aligned}
&\le \mathrm{e}^{\alpha t} E \int_0^t \mathrm{e}^{-\int_v^t a(u)\mathrm{d}u} a(v) \left(\int_{-L}^0 p^2(s)\mathrm{d}s\right)\left(\int_{-L}^0 \left(\int_{v+s}^v g(x(u))\mathrm{d}u\right)^2 \mathrm{d}s\right)\mathrm{d}v \\
&\le \mathrm{e}^{\alpha t}\left(\int_{-L}^0 p^2(s)\mathrm{d}s\right) E \int_0^t \mathrm{e}^{-\int_v^t a(u)\mathrm{d}u} a(v)\left(\int_{-L}^0 (-s)\int_{v+s}^v g^2(x(u))\mathrm{d}u\mathrm{d}s\right)\mathrm{d}v \\
&\le \left(K^2L^2\int_{-L}^0 p^2(s)\mathrm{d}s\right)\mathrm{e}^{\alpha t} E\int_0^t \mathrm{e}^{-\int_v^t a(u)\mathrm{d}u} a(v)\int_{v-L}^v x^2(u)\mathrm{d}u\mathrm{d}v \\
&:= C_1 \mathrm{e}^{\alpha t} E\int_0^t \mathrm{e}^{-\int_v^t a(u)\mathrm{d}u} a(v)\int_{v-L}^v x^2(u)\mathrm{d}u\mathrm{d}v, \qquad (2.12)
\end{aligned}
$$

where $C_1 := K^2L^2\int_{-L}^0 p^2(s)\mathrm{d}s$.

For any $\epsilon > 0$, there exists $T_1 > 0$ such that $s \ge T_1 - L$ implies $\mathrm{e}^{\alpha s}E|x(s)|^2 < \epsilon$. Hence, we have

$$
\begin{aligned}
\mathrm{e}^{\alpha t}E|I_3(t)|^2 &\le C_1\mathrm{e}^{\alpha t}E\int_0^{T_1}\mathrm{e}^{-\int_v^t a(u)\mathrm{d}u}a(v)\int_{v-L}^v x^2(u)\mathrm{d}u\mathrm{d}v + C_1\mathrm{e}^{\alpha t}E\int_{T_1}^t \mathrm{e}^{-\int_v^t a(u)\mathrm{d}u}a(v)\int_{v-L}^v x^2(u)\mathrm{d}u\mathrm{d}v \\
&\le C_1 LE\left(\sup_{-L\le s\le T_1}|x(s)|^2\right)\int_0^{T_1}\mathrm{e}^{-\alpha(t-2v)}a(v)\mathrm{d}v \\
&\quad + C_1\mathrm{e}^{\alpha t}\int_{T_1}^t \mathrm{e}^{-\alpha(t-v)}\mathrm{e}^{-\frac{1}{2}\int_v^t a(u)\mathrm{d}u}a(v)\int_{v-L}^v \mathrm{e}^{-\alpha u}\mathrm{e}^{\alpha u}x^2(u)\mathrm{d}u\mathrm{d}v \\
&\le \mathrm{e}^{-\alpha t}C_1 LE\left(\sup_{-L\le s\le T_1}|x(s)|^2\right)\int_0^{T_1}\mathrm{e}^{2\alpha v}a(v)\mathrm{d}v + \frac{2(\mathrm{e}^{\alpha L}-1)}{\alpha}\epsilon. \qquad (2.13)
\end{aligned}
$$

Thus, we have $\mathrm{e}^{\alpha t}E|I_3(t)|^2 \to 0$ as $t \to \infty$.

On the other hand, from our conditions (i) and (ii), we have

$$
\mathrm{e}^{\alpha t}E|I_4(t)|^2 \le \mathrm{e}^{\alpha t}\int_0^t \mathrm{e}^{-2\int_v^t a(u)\mathrm{d}u}\sigma^2(v)\mathrm{d}v \le \frac{\int_0^t \mathrm{e}^{4\alpha v}\sigma^2(v)\mathrm{d}v}{\mathrm{e}^{3\alpha t}} \to 0 \quad \text{as } t\to\infty. \qquad (2.14)
$$

So we conclude that $\Phi(S) \subset S$.

Thirdly, we will show that Φ is contractive. For $x, y \in S$, we have

$$
\begin{aligned}
E\sup_{s\in[0,t]}|\Phi(x)(s)-\Phi(y)(s)|^2 &\le E\sup_{s\in[0,t]}\left(\int_{-L}^0 |p(v)|\int_{s+v}^s |g(x(u))-g(y(u))|\mathrm{d}u\mathrm{d}v\right. \\
&\quad \left. + \int_0^s \mathrm{e}^{-\int_v^s a(v)\mathrm{d}s}a(v)\int_{-L}^0|p(\tau)|\int_{v+\tau}^v |g(x(u))-g(y(u))|\mathrm{d}u\mathrm{d}\tau\mathrm{d}v\right)^2 \\
&\le E\sup_{s\in[0,t]}|x(s)-y(s)|^2\left(2K\int_{-L}^0|p(s)s|\mathrm{d}s\right)^2. \qquad (2.15)
\end{aligned}
$$

Thus by (2.6) we know that Φ is a contraction mapping.

Hence by the Contraction Mapping Principle, Φ has a unique fixed point $x(t)$ in S, which is a solution of (2.9) with $x(s) = \psi(s)$ on $[-L, 0]$ and $\mathrm{e}^{\alpha t}E|x(t)|^2 \to 0$ as $t \to \infty$. This completes the proof. □

Theorem 2.2. *Suppose that all the conditions of Theorem 2.1 hold. Then Eq. (2.1) is almost surely exponentially stable.*

Proof. Let N be a sufficiently large positive integer. Let $N \le t \le N+1$; then

$$
\begin{aligned}
x(t) &= \mathrm{e}^{-\int_N^t a(u)\mathrm{d}u}\left(x(N) - \int_{-L}^0 p(s)\int_{N+s}^N g(x(u))\mathrm{d}u\mathrm{d}s\right) + \int_{-L}^0 p(s)\int_{t+s}^t g(x(u))\mathrm{d}u\mathrm{d}s \\
&\quad - \int_N^t \mathrm{e}^{-\int_v^t a(s)\mathrm{d}s}a(v)\int_{-L}^0 p(s)\int_{v+s}^v g(x(u))\mathrm{d}u\mathrm{d}s\mathrm{d}v + \int_N^t \mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\sigma(s)\mathrm{d}B(s). \qquad (2.16)
\end{aligned}
$$

Thus, for any fixed $\varepsilon_N > 0$, we obtain

$$
P\left\{\sup_{N\le t\le N+1}|x(t)| > \varepsilon_N\right\} \le P\left\{\sup_{N\le t\le N+1}\mathrm{e}^{-\int_N^t a(u)\mathrm{d}u}\left|x(N) - \int_{-L}^0 p(s)\int_{N+s}^N g(x(u))\mathrm{d}u\mathrm{d}s\right| > \varepsilon_N/4\right\}
$$

$$
\begin{aligned}
&+P\left\{\sup_{N\le t\le N+1}\left|\int_{-L}^{0}p(s)\int_{t+s}^{t}g(x(u))\mathrm{d}u\mathrm{d}s\right|>\varepsilon_N/4\right\}\\
&+P\left\{\sup_{N\le t\le N+1}\left|\int_{N}^{t}\mathrm{e}^{-\int_v^t a(s)\mathrm{d}s}a(v)\int_{-L}^{0}p(s)\int_{v+s}^{v}g(x(u))\mathrm{d}u\mathrm{d}s\mathrm{d}v\right|>\varepsilon_N/4\right\}\\
&+P\left\{\sup_{N\le t\le N+1}\left|\int_{N}^{t}\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\sigma(s)\mathrm{d}B(s)\right|>\varepsilon_N/4\right\}\\
=&\;J_1+J_2+J_3+J_4,\quad \text{say.}
\end{aligned}
\tag{2.17}
$$

In view of Theorem 2.1, there is a $C>0$ such that $E|x(t)|^2\le C\mathrm{e}^{-\alpha t},\ t\ge 0$. So we have that

$$
\begin{aligned}
J_1 &\le (4/\varepsilon_N)^2E\sup_{N\le t\le N+1}\mathrm{e}^{-2\int_N^t a(u)\mathrm{d}u}\left|x(N)-\int_{-L}^{0}p(s)\int_{N+s}^{N}g(x(u))\mathrm{d}u\mathrm{d}s\right|^2\\
&\le (4/\varepsilon_N)^2\times 2\left(E|x(N)|^2+E\left|\int_{-L}^{0}p(s)\int_{N+s}^{N}g(x(u))\mathrm{d}u\mathrm{d}s\right|^2\right)\\
&\le (4/\varepsilon_N)^2\times 2\left(C\mathrm{e}^{-\alpha N}+LE\left|\int_{-L}^{0}p^2(s)\left(\int_{N+s}^{N}|g(x(u))|\mathrm{d}u\right)^2\mathrm{d}s\right)\\
&\le (4/\varepsilon_N)^2\times 2\left(C\mathrm{e}^{-\alpha N}+LE\left|\int_{-L}^{0}p^2(s)|s|\int_{N+s}^{N}|g(x(u))|^2\mathrm{d}u\mathrm{d}s\right)\\
&\le (4/\varepsilon_N)^2\times 2\left(C\mathrm{e}^{-\alpha N}+K^2L\left|\int_{-L}^{0}p^2(s)|s|\int_{N+s}^{N}E|x(u)|^2\mathrm{d}u\mathrm{d}s\right)\\
&\le (4/\varepsilon_N)^2\times 2C\left(1+\frac{LK^2(\mathrm{e}^{\alpha L}-1)}{\alpha}\int_{-L}^{0}p^2(s)|s|\mathrm{d}s\right)\mathrm{e}^{-\alpha N},
\end{aligned}
\tag{2.18}
$$

$$
\begin{aligned}
J_2 &\le (4/\varepsilon_N)^2E\sup_{N\le t\le N+1}\left|\int_{-L}^{0}p(s)\int_{t+s}^{t}g(x(u))\mathrm{d}u\mathrm{d}s\right|^2\\
&\le (4/\varepsilon_N)^2E\sup_{N\le t\le N+1}L\left|\int_{-L}^{0}p^2(s)\left(\int_{t+s}^{t}g(x(u))\mathrm{d}u\right)^2\mathrm{d}s\right.\\
&\le (4/\varepsilon_N)^2\sup_{N\le t\le N+1}L\left|\int_{-L}^{0}p^2(s)|s|\int_{t+s}^{t}K^2E|x(u)|^2\mathrm{d}u\mathrm{d}s\right.\\
&\le (4/\varepsilon_N)^2\frac{CLK^2(\mathrm{e}^{\alpha L}-1)}{\alpha}\left(\int_{-L}^{0}p^2(s)|s|\mathrm{d}s\right)\mathrm{e}^{-\alpha N},
\end{aligned}
\tag{2.19}
$$

$$
\begin{aligned}
J_3 &\le (4/\varepsilon_N)^2E\sup_{N\le t\le N+1}\left|\int_{N}^{t}\mathrm{e}^{-\int_v^t a(s)\mathrm{d}s}a(v)\int_{-L}^{0}p(s)\int_{v+s}^{v}g(x(u))\mathrm{d}u\mathrm{d}s\mathrm{d}v\right|^2\\
&\le (4/\varepsilon_N)^2E\sup_{N\le t\le N+1}\left\{\int_{N}^{t}\mathrm{e}^{-\int_v^t a(s)\mathrm{d}s}a(v)\mathrm{d}v\int_{N}^{t}\mathrm{e}^{-\int_v^t a(s)\mathrm{d}s}a(v)\left(\int_{-L}^{0}p(s)\int_{v+s}^{v}g(x(u))\mathrm{d}u\mathrm{d}s\right)^2\mathrm{d}v\right\}\\
&\le (4/\varepsilon_N)^2E\sup_{N\le t\le N+1}\int_{N}^{t}\mathrm{e}^{-\int_v^t a(s)\mathrm{d}s}a(v)\left(\int_{-L}^{0}p(s)\int_{v+s}^{v}g(x(u))\mathrm{d}u\mathrm{d}s\right)^2\mathrm{d}v\\
&\le (4/\varepsilon_N)^2E\sup_{N\le t\le N+1}\int_{N}^{t}\mathrm{e}^{-\int_v^t a(s)\mathrm{d}s}a(v)L\int_{-L}^{0}p^2(s)\left(\int_{v+s}^{v}g(x(u))\mathrm{d}u\right)^2\mathrm{d}s\mathrm{d}v\\
&\le (4/\varepsilon_N)^2LE\sup_{N\le t\le N+1}\int_{N}^{t}\mathrm{e}^{-\int_v^t a(s)\mathrm{d}s}a(v)\int_{-L}^{0}p^2(s)|s|\int_{v+s}^{v}g^2(x(u))\mathrm{d}u\mathrm{d}s\mathrm{d}v\\
&\le (4/\varepsilon_N)^2L\sup_{N\le t\le N+1}\int_{N}^{t}\mathrm{e}^{-\int_v^t a(s)\mathrm{d}s}a(v)\int_{-L}^{0}p^2(s)|s|\int_{v+s}^{v}K^2E|x(u)|^2\mathrm{d}u\mathrm{d}s\mathrm{d}v\\
&\le (4/\varepsilon_N)^2\frac{CLK^2(\mathrm{e}^{\alpha L}-1)}{\alpha}\left(\int_{-L}^{0}p^2(s)|s|\mathrm{d}s\right)\mathrm{e}^{-\alpha N},
\end{aligned}
\tag{2.20}
$$

and

$$
J_4 \le (4/\varepsilon_N)^2E\sup_{N\le t\le N+1}\left|\int_{N}^{t}\mathrm{e}^{-\int_s^t a(u)\mathrm{d}u}\sigma(s)\mathrm{d}B(s)\right|^2
$$

$$
\begin{aligned}
&\leq (4/\varepsilon_N)^2 E \sup_{N\leq t\leq N+1} \int_N^t \mathrm{e}^{-2\int_s^t a(u)\mathrm{d}u}\sigma^2(s)\mathrm{d}s \\
&\leq (4/\varepsilon_N)^2 E \sup_{N\leq t\leq N+1} \int_N^t \mathrm{e}^{-4\alpha(t-s)}\sigma^2(s)\mathrm{d}s \\
&\leq (4/\varepsilon_N)^2 \frac{\int_N^{N+1} \mathrm{e}^{4\alpha s}\sigma^2(s)\mathrm{d}s}{\mathrm{e}^{3\alpha N}} \mathrm{e}^{-\alpha N}.
\end{aligned} \tag{2.21}
$$

From the conditions (i) and (ii), we know that for sufficiently large N, there exists a positive constant L_1 such that

$$
\frac{\int_N^{N+1} \mathrm{e}^{4\alpha s}\sigma^2(s)\mathrm{d}s}{\mathrm{e}^{3\alpha N}} < L_1.
$$

Thus we have $J_4 < (4/\varepsilon_N)^2 L_1 \mathrm{e}^{-\alpha N}$.

Using these estimates, it follows that

$$
P\left\{\sup_{N\leq t\leq N+1} |x(t)| > \varepsilon_N\right\} \leq \left(D/\varepsilon_N^2\right)\mathrm{e}^{-\alpha N},
$$

where

$$
D := 16\left\{2C + L_1 + \frac{4CLK^2(\mathrm{e}^{\alpha L}-1)}{\alpha}\int_{-L}^0 |p^2(s)s|\mathrm{d}s\right\}.
$$

Hence, if we set $\varepsilon_N = D^{1/2}\mathrm{e}^{-\alpha N/4}$, then

$$
P\left\{\sup_{N\leq t\leq N+1} |x(t)| > D^{1/2}\mathrm{e}^{-\alpha N/4}\right\} \leq \mathrm{e}^{-\alpha N/2}.
$$

Therefore, by the Borel–Cantelli lemma we conclude that there exists a random time $0 < T(\omega)$ such that

$$
|x(t)| \leq D^{1/2}\mathrm{e}^{\alpha/4}\mathrm{e}^{-\alpha t/4} \quad \text{a.s. for } t > T(\omega).
$$

The proof is completed. □

Remark 2.1. In [14,1], Eq. (2.1) is almost surely stable if the conditions (i) and (ii) in Theorem 2.1 are replaced by

$$
\sigma^2(t)\ln t \to 0 \quad \text{as } t \to \infty. \tag{2.22}
$$

However, if we let $\alpha \to 0$, then from our proof we know that the corresponding condition is

$$
\int_0^t \sigma^2(s)\mathrm{d}s \quad \text{is bounded for all } t > 0, \tag{2.23}
$$

or

$$
\int_0^\infty \sigma^2(s)\mathrm{d}s = \infty \quad \text{and} \quad \sigma^2(t) \to 0 \quad \text{as } t \to \infty. \tag{2.24}
$$

Obviously, our conditions (2.23) and (2.24) are better than Appleby's condition (2.22). For example, let $\sigma^2(t) = \frac{1}{\ln(t+1)}$, $t > 0$; then the condition (2.24) is satisfied, but the condition (2.22) fails. Hence, our result generalizes and improves on the results in [14,1].

Remark 2.2. Consider an ordinary equation with damped stochastic perturbation of the form

$$
\mathrm{d}x(t) = f(x(t))\mathrm{d}t + \sigma(t)\mathrm{d}B(t) \quad \text{on } t \geq 0 \tag{2.25}
$$

with initial data $x(0) = \psi \in R$, where $f : R \to R$, $\sigma : R \to R$, B is a one-dimensional Brownian motion. Assume that there is a positive constant α such that $xf(x) \leq -2\alpha x^2$ for all $x \in R$. Then by Theorem 2.1 in [30] Eq. (2.25) is almost surely exponentially stable if there exists a positive constant C such that

$$
\mathrm{e}^{4\alpha t}\sigma^2(t) \leq C \quad \text{for all } t \geq 0. \tag{2.26}
$$

The above condition (2.26) is a pointwise condition. However, by Theorem 2.2 in the present paper, if

$$
\int_0^t \mathrm{e}^{4\alpha s}\sigma^2(s)\mathrm{d}s \quad \text{is bounded for all } t \geq 0, \tag{2.27}
$$

or

$$\int_0^\infty e^{4\alpha s}\sigma^2(s)ds = \infty \quad \text{and} \quad e^{\alpha t}\sigma^2(t) \to 0 \quad \text{as } t \to \infty, \tag{2.28}$$

then Eq. (2.25) is almost surely exponentially stable. Obviously, (2.27) is a condition of an averaging nature. In addition, it is easily seen that conditions (2.27) and (2.28) are better than the condition (2.26). In this sense, the result in this paper improves on that in the paper [30].

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