Lipschitz Continuity of the Absolute Value and Riesz Projections in Symmetric Operator Spaces

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A principal result of the paper is that if \( E \) is a symmetric Banach function space on the positive half-line with the Fatou property then, for all semifinite von Neumann algebras \((\mathcal{M}, \tau)\), the absolute value mapping is Lipschitz continuous on the associated symmetric operator space \( E(\mathcal{M}, \tau) \) with Lipschitz constant depending only on \( E \) if and only if \( E \) has non-trivial Boyd indices. It follows that if \( \mathcal{M} \) is any von Neumann algebra, then the absolute value map is Lipschitz continuous on the corresponding Haagerup \( L^p \)-space, provided \( 1 < p < \infty \).

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INTRODUCTION

The purpose of the present paper is to characterise those symmetric spaces of measurable operators affiliated with a semifinite von Neumann algebra for which the absolute value mapping is Lipschitz continuous. Norm continuity of this mapping has been established by Kosaki [Ko1, 4]

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for the $L^p$-spaces associated with an arbitrary von Neumann algebra and by Chilin and Sukochev [CS] in the setting of symmetric operator spaces associated with separable symmetric Banach function spaces. It was shown by Kato [Ka] for the operator norm on the space $L^p(\mathcal{H})$, and in [DD] for a much wider class of symmetric operator spaces, that the absolute value map is “almost” Lipschitz continuous, in a certain sense. However, it was shown by Kato [Ka] that if $\mathcal{H}$ is infinite-dimensional, then the absolute value mapping is not Lipschitz continuous. Kato’s method is based on an earlier example of McIntosh [McI] showing the failure of certain commutator estimates for the operator norm. Subsequently, it was shown by Davies [Da1] that if $1 \leq p \leq \infty$, then there exists a constant $c(p) > 0$ such that the Lipschitz estimate
\[
\| |x| - |y| \|_p \leq c(p) \| x - y \|_p
\]
holds for all $x, y$ in the Schatten class $\mathcal{C}_p$, if and only if $1 < p < \infty$. Equivalent to such Lipschitz estimates are commutator estimates of the type
\[
\|[|x|, y]\|_p \leq c(p) \|[x, y]\|_p,
\]
where, as usual, $[u, v]$ denotes the commutator $uv - vu$. The results of [Da1] (see also [Ab, AD]) are based on a non-trivial application of a theorem of V. I. Macaev [GK2] concerning triangular truncation to obtain matrix estimates in the Schatten $p$-norm for certain Schur–Hadamard multipliers. The results of Davies were placed in the more general framework of trace ideals by H. Kosaki [Ko2] who pointed out, via some earlier results of Arazy [Ar], that the existence of Lipschitz-type estimates for the absolute value mapping is in fact equivalent to existence of the corresponding commutator estimates and that each in turn is equivalent to the non-triviality of the Boyd indices of the underlying symmetric sequence space. It was this paper of Kosaki that provided much of the stimulus for the investigations in the present paper; in fact, we will show that the results of [Da1], as formulated in [Ko2], continue to hold in the general setting of symmetric spaces of measurable operators affiliated with a semifinite von Neumann algebra. While the extension to the non-commutative $L^p$-spaces associated with the hyperfinite $II_1$ and $II_f$ factors may well be deduced from the matrix case, as noted by Davies [Da1], there are new technical and conceptual difficulties which arise in the more general setting, even in the case of more general symmetric operator spaces associated with these same factors, as well as in the case of the $L^p$-spaces associated with more general semifinite von Neumann algebras; in particular, the approximation techniques of [Da1, Ko2] simply fail and our approach here is to use methods based on the theory of decreasing rearrangements of measurable...
operators as developed, for example, in [FK, DDP3]. At the heart of our analysis is a general version of the Riesz projection theorem for the non-commutative $L^p$-spaces associated with a general semifinite von Neumann algebra. The existence of such projections in the non-commutative setting was first established by Zsidó [Zs] via methods which go back to Arveson [Arv]. It will be convenient here to adopt the more general approach of [BGM4] (see also [FS]) as our basic starting point. The essential idea which underlies the present paper is that the existence of a generalised Riesz projection yields an approach to the classical theorem of Macaev which does not depend on the theory of triangular truncation as given in [GK2]. This permits the implementation of the ideas of [Da1] in the present more general setting. The circle of ideas which relate Macaev's theorem to the classical Riesz projection theorem has been explored by Berkson, et al. [BGM1, 3, 4], Asmar, et al. [ABG], and Ferleger and Sukochev [FS]. The approach of these authors highlights the fact, implicit in [Zs], that the non-commutative $L^p$-spaces are UMD-spaces, provided $1 < p < \infty$.

In the first section below, we gather some of the basic facts concerning symmetric spaces of measurable operators that will be needed throughout. In the second section, we show the equivalence of Lipschitz estimates for the absolute value mapping and certain commutator estimates (Theorem 2.2). The approximation steps necessary to reduce the main results of the paper to the setting of certain finite operator-valued matrices are given in Lemma 2.3 and Proposition 2.4 and these results are of independent interest. The third section of the paper contains our principal results. In particular, we show (Theorem 3.3) that existence of Lipschitz estimates for the absolute value in any non-commutative symmetric operator space is equivalent to non-triviality of the Boyd indices of the underlying (commutative) symmetric function space. In particular, it follows that the absolute value mapping is Lipschitz continuous in the non-commutative $L^p$-space associated with an arbitrary semifinite von Neumann algebra provided $1 < p < \infty$. Moreover, we show (Theorem 3.6) that this result continues to hold in the Haagerup $L^p$-spaces associated with an arbitrary von Neumann algebra. In the fourth section, we take the opportunity to place both Zsidó's theorem and Macaev's theorem in a more general perspective (Theorems 4.1, 4.4) by giving characterisations for the validity of each of these theorems for symmetric operator spaces in terms of non-triviality of the Boyd indices of the underlying symmetric Banach function space. These results give complete extensions of earlier results of Arazy [Ar] and Kosaki [Ko2] in the trace ideal setting. In the final section, we give an application of our methods to the characterisation of those symmetric operator spaces in which every shell decomposition is unconditional, extending earlier of Arazy [Ar] for trace ideals.
The authors wish to thank Hideki Kosaki for much stimulating correspondence concerning the theme of this paper and in particular for pointing out that Theorem 3.6 is essentially a direct consequence of Theorem 3.3. The authors also thank László Zsido for bringing to their attention the reference [Zs].

1. PRELIMINARIES

In this section we collect some of the basic facts, notation, and tools that will be used in this paper. Unless stated otherwise, we denote by \( \mathcal{H} \) a semi-finite von Neumann algebra on the Hilbert space \( H \) with a fixed faithful and normal semifinite trace \( \tau \). The identity in \( \mathcal{H} \) is denoted by 1. A linear operator \( x: \text{dom}(x) \to \mathcal{H} \), with domain \( \text{dom}(x) \subseteq H \), is called affiliated with \( \mathcal{H} \) if \( ux = xu \) for all unitary \( u \) in the commutant \( \mathcal{H}' \) of \( \mathcal{H} \). The closed and densely defined operator \( x \), affiliated with \( \mathcal{H} \), is called \( \tau \)-measurable if for every \( \epsilon > 0 \) there exists an orthogonal projection \( p \in \mathcal{H} \) such that \( p(\mathcal{H}) \subseteq \text{dom}(x) \) and \( \tau(1 - p) < \epsilon \). The collection of all \( \tau \)-measurable operators is denoted by \( \mathcal{M} \). With the sum and product defined as the respective closures of the algebraic sum and product, \( \mathcal{M} \) is a \(*\)-algebra. For \( \epsilon, \delta > 0 \) we denote by \( N(\epsilon, \delta) \) the set of all \( x \in \mathcal{H} \) for which there exists an orthogonal projection \( p \in \mathcal{H} \) such that \( p(\mathcal{H}) \subseteq \text{dom}(x) \), \( \|xp\| \leq \epsilon \) and \( \tau(1 - p) \leq \delta \). The sets \( \{ N(\epsilon, \delta) : \epsilon, \delta > 0 \} \) form a base at 0 for a metrizable Hausdorff topology in \( \mathcal{M} \), which is called the measure topology. Equipped with this measure topology, \( \mathcal{M} \) is a complete topological \(*\)-algebra. These facts and their proofs can be found in the papers [Ne, Te].

Next we recall the notion of generalized singular value function. Given a self-adjoint operator \( x \) in \( \mathcal{H} \) we denote by \( e^x(\cdot) \) the spectral measure of \( x \). Now assume that \( x \in \mathcal{H} \). Then \( e^{is}(B) \in \mathcal{H} \) for all Borel sets \( B \subseteq \mathbb{R} \), and there exists \( s > 0 \) such that \( \tau(e^{is}(s, \infty)) < \infty \). For \( x \in \mathcal{H} \) and \( t \geq 0 \) we define

\[
\mu_t(x) = \inf\{ s \geq 0 : \tau(e^{is}(s, \infty)) \leq t \}.
\]

The function \( \mu_t(x): [0, \infty) \to [0, \infty) \) is called the generalized singular value function (or decreasing rearrangement) of \( x \); note that \( \mu_t(x) \leq \infty \) for all \( t > 0 \). For the basic properties of this singular value function we refer the reader to [FK]; some additional properties can be found in [DDP1, DDP2]. We note that a sequence \( \{ x_n \} \subseteq \mathcal{H} \) converges to 0 for the measure topology if and only if \( \mu_t(x) \to 0 \) for all \( t > 0 \).

If we consider \( \mathcal{H} = L^\infty(\mathbb{R}^+, m) \), where \( m \) denotes Lebesgue measure on \( \mathbb{R}^+ \), as an abelian von Neumann algebra acting via multiplication on the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^+, m) \), with the trace given by integration with
respect to \( m \), it is easy to see that \( \mathcal{M} \) consists of all measurable functions on \( \mathbb{R}^+ \) which are bounded except on a set of finite measure, and that for \( f \in \mathcal{M} \), the generalized singular value function \( \mu(f) \) is precisely the decreasing rearrangement of the function \( |f| \) (and in this setting, \( \mu(f) \) is frequently denoted by \( f^* \)).

Using the generalized singular value function, it is possible to construct certain Banach spaces of measurable operators. In particular, the non-commutative \( L^p \)-spaces \((1 \leq p \leq \infty)\) associated with \((\mathcal{M}, \tau)\) can be defined by

\[
L^p(\mathcal{M}, \tau) = \{ x \in \mathcal{M} : \mu(x) \in L^p(\mathbb{R}^+, m) \},
\]

equipped with the norm \(|x|_p := ||\mu(x)||_p, x \in L^p(\mathcal{M}, \tau)\). It is not difficult to see that this definition coincides with the definition of non-commutative \( L^p \)-spaces as in [Ne, Te]. If \( \mathcal{M} = L(\mathcal{H}) \) with standard trace, then these non-commutative \( L^p \)-spaces are precisely the Schatten classes \( \mathcal{C}_p \), \( 1 \leq p < \infty \).

We will consider non-commutative spaces of more general form and now briefly describe their construction. By \( L^0(\mathbb{R}^+, m) \) we denote the space of all \( \mathcal{C} \)-valued Lebesgue measurable functions on \( \mathbb{R}^+ \) (with identification \( m \)-a.e.). A Banach space \((E, || \cdot ||_E)\), where \( E \subseteq L^0(\mathbb{R}^+, m) \), is called a rearrangement-invariant Banach function space if it follows from \( f \in E, g \in L^0(\mathbb{R}^+, m) \) and \( \mu(g) \leq \mu(f) \) that \( g \in E \) and \( ||g||_E \leq ||f||_E \). Furthermore, \( (E, || \cdot ||_E) \) is called a symmetric Banach function space if it has the additional property that \( f, g \in E \) and \( g \ll f \) imply that \( ||g||_E \leq ||f||_E \). Here \( g \ll f \) denotes submajorization in the sense of Hardy, Littlewood, and Polya:

\[
\int_0^t \mu(g) \, ds \leq \int_0^t \mu(f) \, ds, \quad \text{for all } t > 0.
\]

For the general theory of rearrangement-invariant Banach function spaces, we refer the reader to [KPS, BS, LT], although in the latter two references the class of function spaces considered is more restrictive. We shall need the following notion. If \( E \) is a rearrangement-invariant Banach function space on \( \mathbb{R}^+ \), then \( E \) is said to have the Fatou property if it follows from \( \{ f_n \}_{n \geq 1} \subseteq E, f_n \in L^0(\mathbb{R}^+, m), f_n \to f \) a.e. on \( \mathbb{R}^+ \), and \( \sup_n ||f_n||_E < \infty \) that \( f \in E \) and \( ||f||_E \leq \lim \inf_{n \to \infty} ||f_n||_E \). We note that the Fatou property implies that \( E \) is fully symmetric in the sense that \( f \in E, g \in L^0(\mathbb{R}^+, m) \) and \( g \ll f \) implies that \( g \in E \) and \( ||g||_E \leq ||f||_E \). This latter property is equivalent to \( E \) being an exact interpolation space for the pair \( (L^1(\mathbb{R}^+, m), L^\infty(\mathbb{R}^+, m)) \). If \( E \) has the Fatou property or if \( E \) is separable, then \( E \) is fully symmetric. Somewhat weaker than the notion of Fatou property of a rearrangement invariant space \( E \) is the notion of a Fatou norm. If \( E \) is a

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rearrangement-invariant Banach function space on $\mathbb{R}^+$, then the norm $\| \cdot \|_E$ on $E$ is said to be a Fatou norm if the unit ball of $E$ is closed with respect to almost everywhere convergence. Any rearrangement-invariant Banach function space with a Fatou norm is a symmetric Banach function space. The norm on the rearrangement invariant Banach function space $E$ is a Fatou norm if and only if the natural embedding of $E$ into its second associate space is an isometry. On the other hand, the natural embedding of $E$ into its second associate space is a surjective isometry if and only if $E$ has the Fatou property. See, for example, [KPS, Chap. II].

Given a semifinite von Neumann algebra $(\mathcal{M}, \tau)$ and a symmetric Banach function space $(E, \| \cdot \|_E)$ on $(\mathbb{R}^+, m)$ we define the corresponding non-commutative space $E(\mathcal{M}, \tau)$ by setting $E(\mathcal{M}, \tau) = \{ x \in \mathcal{M} : \| x \# \|_E \}$. Equipped with the norm $\| x \# \|_{E(\mathcal{M}, \tau)} := \| \mu(x) \|_E$, the space $(E(\mathcal{M}, \tau), \| \cdot \|_{E(\mathcal{M}, \tau)})$ is a Banach space and is called the (non-commutative) symmetric operator space associated with $(E, \| \cdot \|_E)$. An extensive discussion of the various properties of such spaces can be found in [DDP1, 2, 3]. We note that if $E$ is a symmetric Banach function space on $\mathbb{R}^+$ with the Fatou property, then the non-commutative space $E(\mathcal{M}, \tau)$ has the Fatou property in the following sense: if $\{ x_n \} \subseteq E(\mathcal{M}, \tau)$, $x_n \to x$ for the measure topology and $\sup \| x_n \# \|_{E(\mathcal{M}, \tau)} < \infty$, then $x \in E(\mathcal{M}, \tau)$ and $\| x \# \|_{E(\mathcal{M}, \tau)} \leq \liminf_{n \to \infty} \| x_n \# \|_{E(\mathcal{M}, \tau)}$. If $E$ is a symmetric Banach function space on $\mathbb{R}^+$, then $E$ is separable if and only if the norm on $E$ is order continuous in the sense that $0 \leq f, 0$ in $E$ implies that $\| f \# \|_E \downarrow 0$, and in this case, the norm is order continuous on the non-commutative space $E(\mathcal{M}, \tau)$. It follows in particular that if $E$ is separable, then $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$ is $\| \cdot \# \|_{E(\mathcal{M}, \tau)}$-dense in $E(\mathcal{M}, \tau)$.

We shall need the following result, due to O. Tychonov [Ty], concerning the continuity of the absolute value for the measure topology. Since the proof of this result is somewhat inaccessible, we outline a proof for the sake of convenience and completeness.

**Theorem 1.1 (Tychonov).** If $x \in \mathcal{M}$, $\{ x_n \} \subseteq \mathcal{M}$ and if $x_n \to x$ for the measure topology, then also $|x_n| \to |x|$ for the measure topology.

**Proof.** Since $\mathcal{M}$ is a topological *-algebra, it suffices to show that the function $z \to \sqrt{z}$ is continuous on the positive cone of $\mathcal{M}$. Suppose then that $0 \leq x_n, x \in \mathcal{M}$, $n \geq 1$, and that $\mu(x_n - x) \to 0$ for all $t > 0$. Since

$$\sqrt{z + x_1} - \sqrt{z} \leq x_1, \quad \forall 0 \leq z \in \mathcal{M}, \quad \forall x \geq 0,$$
it follows via [FK, Lemma 2.5(v)] that
\[ \mu_f(x^{1/2} - x_n^{1/2}) \leq 2\delta + \mu_{\nu_3}(x + x1)^{1/2} - (x_n + x1)^{1/2} \]
for all \( t > 0 \). Changing notation, we may assume that \( x \geq x1 \) and \( x_n \geq x1 \) for all \( n \geq 1 \) and some \( x > 0 \). In particular, \( x, x_n \) are invertible in \( \mathcal{A} \) and \( \|x^{-1}\|_\infty, \|x_n^{-1}\|_\infty \leq 1/x \) for all \( n \geq 1 \). From the equality
\[ x^{-1} - x_n^{-1} = x^{-1}(x - x_n)x^{-1}, \quad n \geq 1 \]
it follows that
\[ \mu_f(x^{-1} - x_n^{-1}) \leq \frac{1}{x^2}\mu_f(x_n - x), \quad n \geq 1, \quad t > 0, \]
so that also \( x_n^{-1} \to x^{-1} \) for the measure topology. Since \( \mu_f(x_n) \to \mu_f(x) \) almost everywhere, it follows that
\[ \sup_{n \geq 1} \mu_f(x_n^{1/2}) = \sup_{n \geq 1} \mu_f(x_n)^{1/2} < \infty \]
for all \( t > 0 \). From the equality
\[ x_n^{1/2} - x_n^{-1/2} = x_n^{1/2}(x^{-1/2} - x_n^{-1/2}) x^{1/2} = x_n^{1/2}(x^{-1/2} - (x_n^{-1})^{1/2}) x^{1/2}, \quad n \geq 1, \]
it follows via [FK, Lemma 2.5 (vii)] that
\[ \mu_f(x_n^{1/2} - x_n^{-1/2}) \leq \mu_{\nu_3}(x^{-1/2} - (x_n^{-1})^{1/2}) \cdot \mu_{\nu_3}(x^{1/2}) \cdot \sup_{n \geq 1} \mu_{\nu_3}(x_n^{1/2}), \quad n \geq 1, \]
for all \( t > 0 \). By replacing \( x, x_n \) by \( x^{-1}, x_n^{-1} \) respectively, it follows that we may assume that \( x, x_n \in \mathcal{A} \) and that \( \|x\|_\infty, \|x_n\|_\infty \leq 1/x = K \) for all \( n \geq 1 \). In particular, it follows that the spectrum of \( x, x_n \) lie in the interval \([0, K]\) for all \( n \geq 1 \). If \( \varepsilon > 0 \) is given, let \( p \) be any polynomial such that
\[ \sup_{\lambda \in [0, K]} |\sqrt{\lambda} - p(\lambda)| < \varepsilon. \]
It follows that
\[ \mu_f(\sqrt{x} - p(x)) \leq \varepsilon, \quad \mu_f(\sqrt{x_n} - p(x_n)) \leq \varepsilon \]
for all \( t > 0 \) and so
\[ \mu_f(\sqrt{x} - \sqrt{x_n}) \leq 2\varepsilon + \mu_{\nu_3}(p(x_n) - p(x)) \]
for all \( t > 0 \). The assertion of the theorem now follows since \( p(x_n) \to p(x) \) for the measure topology. \( \blacksquare \)
It will be convenient to introduce the following notion of convergence for \( \| \cdot \|_\infty \)-bounded nets in \( \mathcal{M} \). If \( \{ x_\alpha \}_{\alpha \in \mathcal{A}} \subseteq \mathcal{M} \) is a net and if \( x \in \mathcal{M} \), then we will write \( x_\alpha \to x \) if and only if (i) there exists \( M > 0 \) such that \( \| x_\alpha \|_\infty \leq M \) for all \( \alpha \), (ii) for all \( u \in L^1(\mathcal{M}, \tau) \cap \mathcal{M} \),

\[
\| u(x_\alpha - x) \|_1 \to 0 \quad \text{and} \quad \| (x_\alpha - x) u \|_1 \to 0.
\]

In the special case \( \mathcal{M} = L^1(\mathcal{H}) \), it is not difficult to see that if \( \{ x_\alpha \}_{\alpha \in \mathcal{A}} \subseteq \mathcal{M} \) is a bounded net, then \( x_\alpha \to x \) if and only if \( x_\alpha \to x \) and \( x^*_\alpha \to x^* \) for the strong operator topology.

We remark immediately that if \( \{ e_\alpha \}_{\alpha \in \mathcal{A}} \) is an upwards directed family of orthogonal projections in \( \mathcal{M} \) such that \( e_\alpha \uparrow 1 \), then it follows also that \( e_\alpha \to 1 \) since \( \| (1 - e_\alpha) z \|_1 \to 0 \) and \( \| z(1 - e_\alpha) \|_1 \to 0 \) for all \( z \in L^1(\mathcal{M}, \tau) \).

The proof of the following result is straightforward, and we omit the details.

**Lemma 1.2.** If \( \{ x_\alpha \}_{\alpha \in \mathcal{A}}, \{ y_\beta \}_{\beta \in \mathcal{B}} \subseteq \mathcal{M} \) are nets in \( \mathcal{M} \) such that \( x_\alpha \to x \in \mathcal{M} \) and \( y_\beta \to y \in \mathcal{M} \), then \( x_\alpha y_\beta \to x y \).

It now follows that if \( \{ x_\alpha \}_{\alpha \in \mathcal{A}} \subseteq \mathcal{M} \), if \( x_\alpha \to x \in \mathcal{M} \), and if \( p \) is any polynomial, then also \( p(x_\alpha) \to p(x) \). Consequently, if \( \{ e_\alpha \}_{\alpha \in \mathcal{A}} \) is any upwards directed family of orthogonal projections in \( \mathcal{M} \) such that \( e_\alpha \uparrow 1 \), then \( e_\alpha x e_\alpha \to x \) and \( p(e_\alpha x e_\alpha) \to p(x) \) for all \( x \in \mathcal{M} \) and all polynomials \( p \).

**Lemma 1.3.** Let \( \{ x_\alpha \}_{\alpha \in \mathcal{A}} \) be a net of self-adjoint elements of \( \mathcal{M} \) such that \( x_\alpha \to x \in \mathcal{M} \), and suppose that the spectrum of \( x \), \( x_\alpha \), is contained in the interval \( [-M, M] \) for some \( M > 0 \) and every index \( \alpha \). If \( f \) is any continuous real function on \( [-M, M] \), then \( f(x_\alpha) \to f(x) \).

**Proof.** Let \( \{ p_n \} \) be a sequence of real polynomials such that \( p_n \to f \) uniformly on \( [-M, M] \). If \( u \in L^1(\mathcal{M}, \tau) \cap \mathcal{M} \), then

\[
\| u f(x_\alpha) - u f(x) \|_1 \leq \| u f(x_\alpha) - u p_n(x_\alpha) \|_1 + \| u p_n(x_\alpha) - u p_n(x) \|_1 + \| u p_n(x) - u f(x) \|_1
\]

\[
\leq 2 \| u \|_1 \max_{[-M, M]} | f - p_n | + \| u p_n(x_\alpha) - u p_n(x) \|_1.
\]

From this, it follows that \( \| u f(x_\alpha) - u f(x) \|_1 \to 0 \) and similarly \( \| f(x_\alpha) u - f(x) u \|_1 \to 0 \).

**Corollary 1.4.** If \( \{ x_\alpha \}_{\alpha \in \mathcal{A}} \) is a net of self-adjoint elements of \( \mathcal{M} \) such that \( x_\alpha \to x \in \mathcal{M} \), then \( \| x_\alpha \|_1 \to \| x \|_1 \).
Corollary 1.5. If \( \{ e_n \} \subseteq \mathcal{M} \) is any upwards directed family of orthogonal projections in \( \mathcal{M} \) such that \( e_n^*, e_n \rightarrow 1 \) and if \( x = x^* \in \mathcal{M} \), then 
\[
|e_n x e_n| \rightarrow_\ast \|x\|.
\]

Proposition 1.6. Let \( E \) be a symmetric Banach function space on \( \mathbb{R}^+ \) with the Fatou property. If \( \{ x_n \} \subseteq \mathcal{M} \), if \( x_n \rightarrow_x x \in \mathcal{M} \) and if \( \|x_n\|_{E(\mathcal{M}, \tau)} \leq C \) for every index \( n \), then \( x \in E(\mathcal{M}, \tau) \) and \( \|x\|_{E(\mathcal{M}, \tau)} \leq C \).

Proof. Let \( u \in L^1(\mathcal{M}, \tau) \cap \mathcal{M} \) and suppose that \( \|u\|_{\mathcal{M}} \leq 1 \). Since \( x_n \rightarrow_x x \), it follows that \( \|ux_n - ux\|_1 \rightarrow 0 \) and so \( ux_n \rightarrow ux \) for the measure topology; moreover \( \|ux_n\|_{E(\mathcal{M}, \tau)} \leq \|x_n\|_{E(\mathcal{M}, \tau)} \leq C \) for every index \( n \). Via the Fatou property of \( E \), this implies that \( ux \in E(\mathcal{M}, \tau) \) and \( \|ux\|_{E(\mathcal{M}, \tau)} \leq C \). It follows that \( u \|x\| \in E(\mathcal{M}, \tau) \) and \( \|ux\|_{E(\mathcal{M}, \tau)} \leq C \) for every \( u \in L^1(\mathcal{M}, \tau) \cap \mathcal{M} \) for which \( \|u\|_{\mathcal{M}} \leq 1 \). Via [DDP3, Lemma 1.5], it now follows again from the Fatou property of \( E \) that \( x \in E(\mathcal{M}, \tau) \) and \( \|x\|_{E(\mathcal{M}, \tau)} \leq C \), and this completes the proof of the proposition.

2. Lipschitz Continuity and Commutator Estimates

In this section, we will show that Lipschitz continuity of the absolute value in symmetric operator spaces is equivalent to certain commutator estimates for a certain class of operator-valued matrices. In the trace ideal setting, this equivalence is given in Kosaki [Ko]. While our argument follows the same directions as that for trace ideals, there are important differences in the approximation techniques required to reduce the general inequalities to the matrix setting. The details follow.

We denote by \( \mathcal{M} := \mathcal{M} \otimes M_2(\mathbb{C}) \) the von Neumann algebra of all \( 2 \times 2 \) matrices

\[
[x_{ij}] = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},
\]

with \( x_{ij} \in \mathcal{M}, i, j = 1, 2 \), acting on the Hilbert space \( \mathcal{H} \otimes \mathcal{H} \). If \( 0 \leq [x_{ij}] \in \mathcal{M} \), the trace \( \tau_1 \) is defined by setting

\[
\tau_1([x_{ij}]) = \tau(x_{11}) + \tau(x_{22});
\]

thus \( (\mathcal{M}_1, \tau_1) \) is a semifinite von Neumann algebra. Now suppose that \( x_{ij} \in \mathcal{M}_1, (i, j = 1, 2) \) and define \( X : \text{dom}(X) \rightarrow \mathcal{H} \otimes \mathcal{H} \) by

\[
X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix},
\]
where \( \text{dom}(X) = [\text{dom}(x_{11}) \cap \text{dom}(x_{21})] \oplus [\text{dom}(x_{12}) \cap \text{dom}(x_{22})] \). It is easy to see that \( X \) is affiliated with \( \mathcal{M} \) and is \( \tau_1 \)-premeasurable (in the sense of [Te]). Consequently, \( X \) is closable and the closure of \( X \) is \( \tau_1 \)-measurable (see [Te, Proposition 19]). With a slight abuse of notation, we continue to denote this closure by

\[
X = \begin{pmatrix}
  x_{11} & x_{12} \\
  x_{21} & x_{22}
\end{pmatrix}
\]

and so \( [x_{ij}] \in \mathcal{M} \).

The proof of the following result is essentially the same as in the trace ideal setting and we omit the details. See, for example, [Ko2] Lemma 1.

**Lemma 2.1.** (i) If \( x \in \mathcal{M} \), then

\[
\mu\left(\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right) = \mu\left(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = \mu\left(\begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}\right) = \mu(x).
\]

(ii) If \( x, y, z, w \in \mathcal{M} \) then

\[
\mu(x), \mu(y), \mu(z), \mu(w) \leq \mu\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}\right) \leq \mu(x) + \mu(y) + \mu(z) + \mu(w).
\]

We note that it follows immediately from the preceding Lemma 2.1 that if \( E \) is a symmetric Banach function space on \( \mathbb{R}^+ \), then the operator matrix \( [x_{ij}] \), \( (i, j = 1, 2) \) belongs to \( E(\mathcal{M}, \tau) \) if and only if \( x_{ij} \in E(\mathcal{M}, \tau) \) \( (i, j = 1, 2) \) and in this case

\[
\|x_{kl}\|_{E(\mathcal{M}, \tau)} \leq \|[x_{ij}]\|_{E(\mathcal{M}, \tau)} \leq \sum_{i,j=1}^{2} \|x_{ij}\|_{E(\mathcal{M}, \tau)}, \quad (k, l = 1, 2).
\]

**Theorem 2.2.** Let \( E \) be a symmetric Banach function space on \( \mathbb{R}^+ \) with the Fatou property. The following statements are equivalent.

(i) There exists a constant \( c(E) > 0 \) depending only on \( E \) such that for every semifinite von Neumann algebra \( (\mathcal{M}, \tau) \), \( x, y \in \mathcal{M} \), \( x - y \in E(\mathcal{M}, \tau) \), imply that \( |x| - |y| \in E(\mathcal{M}, \tau) \) and

\[
\| |x| - |y| \|_{E(\mathcal{M}, \tau)} \leq c(E) \|x - y\|_{E(\mathcal{M}, \tau)}.
\]

(ii) There exists a constant \( c(E) > 0 \) depending only on \( E \) such that for every semifinite von Neumann algebra \( (\mathcal{M}, \tau) \), \( x = x^* \in \mathcal{M} \), \( y \in \mathcal{M} \), \( [x, y] \in E(\mathcal{M}, \tau) \) imply that \( [x, y] \in E(\mathcal{M}, \tau) \) and

\[
\|[x, y]\|_{E(\mathcal{M}, \tau)} \leq c(E) \|[x, y]\|_{E(\mathcal{M}, \tau)}.
\]
(iii) There exists a constant $c(E) > 0$ depending only on $E$ such that for every semifinite von Neumann algebra $(\mathcal{M}, \tau)$, $x = x^*, \ z = z^*, \ y \in \mathcal{M}$, $xy - yz \in E(\mathcal{M}, \tau)$ imply that $|x| \ y - y \ |z| \in E(\mathcal{M}, \tau)$ and
\[
\|x\ y - y \ |z| \|_{E(\mathcal{M}, \tau)} \leq c(E) \|xy - yz\|_{E(\mathcal{M}, \tau)}.
\]

(iv) There exists a constant $c(E) > 0$ depending only on $E$ such that for every semifinite von Neumann algebra $(\mathcal{M}, \tau)$, $0 \leq x, \ z \in \mathcal{M}$, $y \in \mathcal{M}$, $xy + yz \in E(\mathcal{M}, \tau)$ imply that $xy - yz \in E(\mathcal{M}, \tau)$ and
\[
\|xy - yz\|_{E(\mathcal{M}, \tau)} \leq c(E) \|xy + yz\|_{E(\mathcal{M}, \tau)}.
\]

Proof. (i) $\Rightarrow$ (ii). We assume first that $x = x^* \in \mathcal{M}$, that $y = y^* \in \mathcal{M}$, and that $[x, y] \in E(\mathcal{M}, \tau)$. We follow first the argument of [Ab, Theorem 1].

The simple identity
\[
y^n x - x^n = y^{n-1}(yx - xy) + (y^{n-1}x - xy^{n-1}) y
\]
\[
y^n x - x^n = y^{n-1}[y, x] + y^{n-2}[y, x] y + y^{n-3}[y, x] y^2 + \cdots + [y, x] y^{n-1},
\]
$n = 1, 2, ...$

shows that
\[
\|y^n x - x^n\|_{E(\mathcal{M}, \tau)} \leq n \|[x, y]\|_{E(\mathcal{M}, \tau)} \|y\|_{\mathcal{M}}^{-1}, \quad n \geq 1.
\]

For each $\epsilon > 0$, it follows that
\[
e^{\epsilon \tau}xe^{-\epsilon \tau} - x = -i\epsilon [x, y]e^{-\epsilon \tau} + R(\epsilon),
\]
where
\[
\|R(\epsilon)\|_{E(\mathcal{M}, \tau)} \leq \epsilon \|e^{\epsilon \tau} - 1\| \|[x, y]\|_{E(\mathcal{M}, \tau)}.
\]

Similarly, for each $\epsilon > 0$,
\[
e^{\epsilon \tau} |x| e^{-\epsilon \tau} - |x| = -i\epsilon [|x|, y] e^{-\epsilon \tau} + S(\epsilon)
\]
where
\[
\|S(\epsilon)\| \leq \epsilon \|e^{\epsilon \tau} - 1\| \|[|x|, y]\|_{\mathcal{M}}.
\]

The assumption (i) now implies that
\[
|e^{\epsilon \tau}xe^{-\epsilon \tau}| - |x| = e^{\epsilon \tau} |x| e^{-\epsilon \tau} - |x| \in E(\mathcal{M}, \tau)
\]
and that
\[ \| e^{\epsilon y} x e^{-\epsilon y} - x \|_{E(\mathcal{A}, \tau)} \leq c(E) \| e^{\epsilon y} x e^{-\epsilon y} - x \|_{E(\mathcal{A}, \tau)} \]
\[ \leq c(E) + \epsilon \| e^{\epsilon y} x e^{-\epsilon y} - x \|_{E(\mathcal{A}, \tau)} \]
\[ + \epsilon \| R(\epsilon) \|_{E(\mathcal{A}, \tau)} \]
for all \( \epsilon > 0 \). We observe that
\[ \frac{e^{\epsilon y} x e^{-\epsilon y} - x}{\epsilon} \to [x, y] \]
for the measure topology as \( \epsilon \to 0 \). The Fatou property of \( E \) now implies that \([x, y] \in E(\mathcal{A}, \tau)\) and
\[ \|([x, y])_{E(\mathcal{A}, \tau)} \leq c(E) \|([x, y])_{E(\mathcal{A}, \tau)} \]
To remove the assumption that \( y \in \mathcal{A} \), suppose that \( y = y^* \in \mathcal{A} \), \( x = x^* \in \mathcal{A} \) and that \([x, y] \in E(\mathcal{A}, \tau)\). Let \( \{e_n\} \subseteq \mathcal{A} \) be a sequence of spectral projections of \( y \) for which \( e_n \uparrow 1, \tau(1 - e_n) \to 0 \) and \( e_n y = ye_n \), \( n \geq 1 \). Since
\[ e_n[x, y] e_n = [e_n xe_n, ye_n], \quad n \geq 1, \]
it follows that \([e_n xe_n, ye_n] \in E(\mathcal{A}, \tau), n \geq 1, \) and
\[ \|([e_n xe_n], ye_n)]_{E(\mathcal{A}, \tau)} \leq c(E) \|([x, y])_{E(\mathcal{A}, \tau)} \]
\[ n \geq 1. \]
Since \( e_n xe_n \to x, ye_n \to y \) for the measure topology, it follows from Tychonov’s theorem (Theorem 1.1) that \( [e_n xe_n, ye_n] \to [x, y] \) for the measure topology. By the Fatou property, it follows that \([x, y] \in E(\mathcal{A}, \tau)\) and
\[ \|([x, y])_{E(\mathcal{A}, \tau)} \leq c(E) \|([x, y])_{E(\mathcal{A}, \tau)} \]
A similar argument now removes the restriction that \( x \in \mathcal{A} \). The restriction that \( y = y^* \) may be removed by observing that the equalities
\[ R(z, y) = i[z, 3y], \quad J(z, y) = -i[z, 3y] \]
hold for all \( z = z^* \in \mathcal{A}, y \in \mathcal{A} \). We omit the simple details and this suffices to complete the proof of the implication (i) \( \Rightarrow \) (ii).

(ii) \( \Rightarrow \) (iii). If \( x = x^*, z = z^* \in \mathcal{A}, y \in \mathcal{A} \) and \( xy - yz \in E(\mathcal{A}, \tau) \), set
\[ X = \begin{pmatrix} x & 0 \\ 0 & z \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & y \\ -y^* & 0 \end{pmatrix}, \]
and observe that

\[
\begin{pmatrix}
0 & xy - yz \\
-(xy - yz)^* & 0
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & |x| |y - |z||^* \\
-(|x| |y - |z||)^* & 0
\end{pmatrix}
\]

We now apply (ii) in \((\vec{\mathcal{M}}, \tau_1)\) and appeal to Lemma 2.1.

(iii) \(\Rightarrow\) (iv) is trivial.

(iv) \(\Rightarrow\) (ii). Suppose that \(x = x^*, y \in \vec{\mathcal{M}}\). Let \(p, q\) be orthogonal projections in \(\mathcal{M}\) such that \(pq = 0\), \(p + q = 1\) and \(px = x^+, qx = q\).

Observe that

\[
[|x|, y] = p[x, y] q - q[x, y] p + (x^- qy - qy p x^+) + (x^+ q y - q y q x^-).
\]

As is easily checked,

\[x^+ q y + q y q x^- = p[x, y] q, \quad x^- q y + q y q x^+ = -q[x, y] p.\]

Consequently, if \([x, y] \in E(\mathcal{M}, \tau)\), it follows from (iv) that \([|x|, y] \in E(\mathcal{M}, \tau)\) and

\[
\|\left[|x|, y\right]_{E(\mathcal{M}, \tau)}\| \leq 2 \left\| [x, y]_{E(\mathcal{M}, \tau)} \right\| + x^+ q y - q y q x^+ \| E(\mathcal{M}, \tau)\|
\]

\[+ c(E) \left\| x^+ q y + q y q x^+ \right\| E(\mathcal{M}, \tau)\]

\[\leq 2 \left\| [x, y]_{E(\mathcal{M}, \tau)} \right\| + c(E) \left\| x^+ q y + q y q x^+ \right\| E(\mathcal{M}, \tau)\]

\[\leq (2 + 2c(E)) \left\| [x, y]_{E(\mathcal{M}, \tau)} \right\|.
\]

(ii) \(\Rightarrow\) (i). Assume first that \(x = x^*, y = y^* \in \vec{\mathcal{M}}\) and set

\[
X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

so that

\[
[ X, Y ] = \begin{pmatrix} 0 & x - y \\ 0 & 0 \end{pmatrix}, \quad [|X|, Y] = \begin{pmatrix} 0 & |x| - |y| \\ 0 & 0 \end{pmatrix}.
\]

It follows from (ii) and Lemma 2.1 that if \(x - y \in E(\mathcal{M}, \tau)\) then \(|x| - |y| \in E(\mathcal{M}, \tau)\) and

\[
\| |x| - |y| \|_{E(\mathcal{M}, \tau)} \leq c(E) \| x - y \|_{E(\mathcal{M}, \tau)}.
\]
To remove the self-adjointness assumption, suppose that \( x, y \in \mathcal{M} \) and that \( x - y \in \mathcal{A} \). If

\[
X = \begin{pmatrix} 0 & x^* \\ x & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & y^* \\ y & 0 \end{pmatrix},
\]

then \( X = X^*, \ Y = Y^* \in \mathcal{M}_1 \) and \( X - Y \in \mathcal{E}(\mathcal{A}, \tau_1) \). Since

\[
|X| = \begin{pmatrix} |x| & 0 \\ 0 & |x^*| \end{pmatrix}, \quad |Y| = \begin{pmatrix} |y| & 0 \\ 0 & |y^*| \end{pmatrix}, \quad |X - Y| = \begin{pmatrix} |x - y| & 0 \\ 0 & |(x - y)^*| \end{pmatrix},
\]

An appeal to Lemma 2.1 shows that \( |x| - |y| \in \mathcal{E}(\mathcal{A}, \tau) \) and

\[
\| |x| - |y| \|_{\mathcal{E}(\mathcal{A}, \tau)} \leq 2c(E) \| x - y \|_{\mathcal{E}(\mathcal{A}, \tau)}.
\]

With this the proof of the theorem is complete.

**Lemma 2.3.** Let \( E \) be a symmetric Banach function space on \( \mathbb{R}^+ \) with the Fatou property. The following statements are equivalent:

(i) There exists a constant \( c(E) > 0 \) depending only on \( E \) such that for every semifinite von Neumann algebra \( (\mathcal{A}, \tau) \), \( x, y \in \mathcal{M} \), \( x - y \in \mathcal{E}(\mathcal{A}, \tau) \) imply \( |x| - |y| \in \mathcal{E}(\mathcal{A}, \tau) \) and

\[
\| |x| - |y| \|_{\mathcal{E}(\mathcal{A}, \tau)} \leq c(E) \| x - y \|_{\mathcal{E}(\mathcal{A}, \tau)}.
\]

(ii) There exists a constant \( c(E) > 0 \) depending only on \( E \) such that for every semifinite von Neumann algebra \( (\mathcal{A}, \tau) \), \( x = x^* \), \( y = y^* \in L^1(\mathcal{A}, \tau) \cap \mathcal{M} \) implies

\[
\| |x| - |y| \|_{\mathcal{E}(\mathcal{A}, \tau)} \leq c(E) \| x - y \|_{\mathcal{E}(\mathcal{A}, \tau)}.
\]

**Proof.** It is clear that only the implication (ii) \( \Rightarrow \) (i) requires proof. We assume first that \( x = x^* \), \( y = y^* \in \mathcal{A} \) and that \( x - y \in \mathcal{E}(\mathcal{A}, \tau) \). Let \( \{ e_\alpha \} \subseteq \mathcal{A} \) be any family of orthogonal projections such that \( e_\alpha \uparrow 1 \) and \( \tau(e_\alpha) < \infty \) for every index \( \alpha \). It follows from (ii) that

\[
\| e_\alpha x e_\alpha - e_\alpha y e_\alpha \|_{\mathcal{E}(\mathcal{A}, \tau)} \leq c(E) \| x - y \|_{\mathcal{E}(\mathcal{A}, \tau)},
\]

for every index \( \alpha \). From Corollary 1.5, it follows that

\[
|e_\alpha x e_\alpha| \rightarrow^* \| x \|, \quad |e_\alpha y e_\alpha| \rightarrow^* \| y \|,
\]

and so Proposition 1.6 implies that \( |x| - |y| \in \mathcal{E}(\mathcal{A}, \tau) \) and

\[
\| |x| - |y| \|_{\mathcal{E}(\mathcal{A}, \tau)} \leq c(E) \| x - y \|_{\mathcal{E}(\mathcal{A}, \tau)}.
\]
Now suppose that \( x = x^* \), \( y = y^* \in \mathcal{A} \) and that \( x - y \in E(\mathcal{A}, \tau) \). There exist orthogonal projections \( \{ e_n \} \subseteq \mathcal{A} \) such that \( e_n \to 1 \), \( \tau(1 - e_n) \to 0 \) and such that \( xe_n, ye_n \in \mathcal{A}, n \geq 1 \). It follows that \( e_nxe_n \to x, e_nye_n \to y \) for the measure topology. Applying the first part of the proof to \( e_nxe_n, e_nye_n \in \mathcal{A}, n \geq 1 \), we obtain that

\[
\| e_nxe_n - e_nye_n \|_{E(\mathcal{A}, \tau)} \leq c(E) \| e_nxe_n - e_nye_n \|_{E(\mathcal{A}, \tau)},
\]

By Tychonov's theorem, \( |e_nxe_n - e_nye_n| \to |x - y| \) for the measure topology. Since \( E \) has the Fatou property, it follows that \( |x - y| \in E(\mathcal{A}, \tau) \) and

\[
\| x - y \|_{E(\mathcal{A}, \tau)} \leq c(E) \| x - y \|_{E(\mathcal{A}, \tau)}.
\]

The assumption that \( x, y \) are self-adjoint is removed by the same argument as in the last step of the proof of the implication (ii) \( \Rightarrow \) (i) of Theorem 2.2.

The proposition which follows shows that the estimates of Theorem 2.2 can be reduced to showing a multiplier estimate for a very special class of operator-valued matrices. For ease of reference, we continue with the numbering of Theorem 2.2.

**Proposition 2.4.** Let \( E \) be a symmetric Banach function space on \( \mathbb{R}^+ \) with the Fatou property. The following statements are equivalent:

(ii) There exists a constant \( c(E) > 0 \) depending only on \( E \) such that for all semifinite von Neumann algebras \( (\mathcal{A}, \tau) \), \( x = x^* \in \mathcal{A} \), \( y \in \mathcal{A} \), \( [x, y] \in E(\mathcal{A}, \tau) \) imply \( |x| - |y| \in E(\mathcal{A}, \tau) \) and

\[
\| [x, y] \|_{E(\mathcal{A}, \tau)} \leq c(E) \| x - y \|_{E(\mathcal{A}, \tau)}.
\]

(vi) There exists a constant \( c(E) > 0 \) depending only on \( E \) such that for all semifinite von Neumann algebras \( (\mathcal{A}, \tau) \), for all pairs of finite sequences \( \{ p_1, p_2, ..., p_n \} \), \( \{ q_1, q_2, ..., q_m \} \) of mutually orthogonal projections on \( \mathcal{A} \), for all choices \( 0 \leq \lambda_1, \lambda_2, ..., \lambda_n \), \( 0 \leq \mu_1, \mu_2, ..., \mu_m \in \mathbb{R} \), and all \( y \in E(\mathcal{A}, \tau) \) it follows that

\[
\left\| \sum_{i=1}^n \sum_{j=1}^m (\lambda_i - \mu_j) p_i q_j \right\|_{E(\mathcal{A}, \tau)} \leq c(E) \left\| \sum_{i=1}^n \sum_{j=1}^m (\lambda_i + \mu_j) p_i q_j \right\|_{E(\mathcal{A}, \tau)}.
\]
Lipschitz Continuity

0 ≤ \lambda_1, \lambda_2, ..., \lambda_n; 0 ≤ \mu_1, \mu_2, ..., \mu_n ∈ \mathbb{R}_+ \text{ and for all } y ∈ E(\mathcal{M}, τ), \text{ it follows that }

\left\| \sum_{i,j=1}^{n} (\lambda_i - \mu_j) p_i y q_j \right\|_{E(\mathcal{M}, τ)} \leq c(E) \left\| \sum_{i,j=1}^{n} (\lambda_i + \mu_j) p_i y q_j \right\|_{E(\mathcal{M}, τ)}.

(vii) Same as (v) with the additional restriction that \( p, q_j = 0 \) for all \( 1 ≤ i ≤ n, 1 ≤ j ≤ m. \)

(viii) There exists a constant \( c(E) > 0 \) depending only on \( E \) such that for all semifinite von Neumann algebras \( (\mathcal{M}, τ) \), whenever \( x ∈ E(\mathcal{M}, τ) \) is of the form

\[ x = (\lambda_1 p_1 + \lambda_2 p_2 + ... + \lambda_n p_n) - (\mu_1 q_1 + \mu_2 q_2 + ... + \mu_m q_m), \]

where \( p_1, p_2, ..., p_n, q_1, q_2, ..., q_m \) are mutually orthogonal projections in \( \mathcal{M} \)
and \( 0 ≤ \lambda_1, \lambda_2, ..., \lambda_n, \mu_1, \mu_2, ..., \mu_m ∈ \mathbb{R}_+ \), then

\[ \|[x,y]\|_{E(\mathcal{M}, τ)} ≤ c(E) \|[x,y]\|_{E(\mathcal{M}, τ)}, \]

for all \( y ∈ L^1(\mathcal{M}, τ) \cap \mathcal{M}. \)

Proof. The implication (ii) ⇒ (v) follows by observing that

\[
\left( \sum_{k=1}^{n} \lambda_k p_k \right) \left( \sum_{i=1}^{n} \sum_{j=1}^{m} p_i y q_j \right) - \left( \sum_{i=1}^{n} \sum_{j=1}^{m} p_i y q_j \right) \left( \sum_{k=1}^{m} \mu_k q_k \right)
= \sum_{i,j=1}^{n,m} (\lambda_i - \mu_j) p_i y q_j,
\]

and

\[
\left( \sum_{k=1}^{n} \lambda_k p_k \right) \left( \sum_{i=1}^{n} \sum_{j=1}^{m} p_i y q_j \right) + \left( \sum_{i=1}^{n} \sum_{j=1}^{m} p_i y q_j \right) \left( \sum_{k=1}^{m} \mu_k q_k \right)
= \sum_{i,j=1}^{n,m} (\lambda_i + \mu_j) p_i y q_j,
\]

and by applying the equivalence of (ii) and (iv) of Theorem 2.2.

The implication (v) ⇒ (vi) is immediate.

(vi) ⇒ (vii). Define orthogonal projections \( e_k, 1 ≤ k ≤ m + n \) by setting \( e_k = p_k \) if \( 1 ≤ k ≤ n \) and \( e_n + k = q_k \) if \( 1 ≤ k ≤ m \). From (vi), we obtain that

\[
\left\| \sum_{i,j=1}^{n+m} (\lambda_i - \mu_j) e_i z e_j \right\|_{E(\mathcal{M}, τ)} ≤ c(E) \left\| \sum_{i,j=1}^{n+m} (\lambda_i + \mu_j) e_i z e_j \right\|_{E(\mathcal{M}, τ)}.
\]
for all \( z \in E(\mathcal{H}, \tau) \). The assertion of (vii) now follows by taking
\[
z = \sum_{i=1}^{n} \sum_{j=1}^{m} p_i y_j, \quad y \in E(\mathcal{H}, \tau).
\]

(vii) \(\Rightarrow\) (viii). If \( p = \sum_{i=1}^{n} p_i, \quad q = \sum_{j=1}^{m} q_j \), then \( pq = 0 \). It is clear also that we may assume that \( p + q = 1 \). If we observe that, for each \( y \in L^1(\mathcal{H}, \tau) \cap \mathcal{H} \),
\[
p([x], y) = \sum_{i=1}^{n} \sum_{j=1}^{m} (\lambda_i - \lambda_j) p_i y_j = p[x, y] p,
\]
\[
p([x], y) = \sum_{i=1}^{n} \sum_{j=1}^{m} (\lambda_i - \mu_j) p_i y_j,
\]
\[
p(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} (\mu_j + \lambda_i) p_i y_j,
\]
\[
q([x], y) = \sum_{i=1}^{n} \sum_{j=1}^{m} (\mu_j - \lambda_i) q_j y_i,
\]
\[
q(x, y) = -\sum_{i=1}^{n} \sum_{j=1}^{m} (\mu_j + \lambda_i) q_j y_i,
\]
\[
q(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{m} (\mu_j - \lambda_j) q_i y_j = -q[x, y] q.
\]

then it follows from (vii) that
\[
\|p([x], y) p\|_{E(\mathcal{H}, \tau)} = \|p[x, y] p\|_{E(\mathcal{H}, \tau)},
\]
\[
\|p([x], y) q\|_{E(\mathcal{H}, \tau)} \leq c(E) \|p[x, y] q\|_{E(\mathcal{H}, \tau)},
\]
\[
\|q([x], y) p\|_{E(\mathcal{H}, \tau)} \leq c(E) \|q[x, y] p\|_{E(\mathcal{H}, \tau)},
\]
\[
\|q([x], y) q\|_{E(\mathcal{H}, \tau)} = \|q[x, y] q\|_{E(\mathcal{H}, \tau)}.
\]

It follows that
\[
\|[x], y]\|_{E(\mathcal{H}, \tau)} = \|(p + q([x], y)(p + q))\|_{E(\mathcal{H}, \tau)}
\leq 2(1 + c(E)) \|[x, y]\|_{E(\mathcal{H}, \tau)},
\]
for each \( y \in L^1(\mathcal{H}, \tau) \cap \mathcal{H} \), and this suffices to complete the proof of the implication.

(viii) \(\Rightarrow\) (ii). We suppose that \( x = x^*, \quad y = y^* \in L^1(\mathcal{H}, \tau) \cap \mathcal{H} \). There exists a sequence \( \{x_n\} \in L^1(\mathcal{H}, \tau) \cap \mathcal{H} \) such that each \( x_n, n \geq 1 \) is a linear combination of spectral projections of \( x \) and such that \( x_n \to x, \ |x_n| \to |x| \) in
L^1(\mathcal{M}, \tau) \cap \mathcal{M}. It follows also that \([x_n, y] \to [x, y], [|x_n|, y] \to [|x|, y]\)

in \(L^1(\mathcal{M}, \tau) \cap \mathcal{M}\) and hence also in \(E(\mathcal{M}, \tau)\) by the continuity of the embedding of \(L^1(\mathcal{M}, \tau) \cap \mathcal{M}\) into \(E(\mathcal{M}, \tau)\). It follows from the assumption (vii) that

\[
\|[x_n, y]\|_{E(\mathcal{M}, \tau)} \leq c(E) \|[x_n, y]\|_{E(\mathcal{M}, \tau)},
\]

for all \(n \geq 1\), and so also

\[
\|[x, y]\|_{E(\mathcal{M}, \tau)} \leq c(E) \|[x, y]\|_{E(\mathcal{M}, \tau)}.
\]

The matrix argument in the proof of the implication (ii) \(\Rightarrow\) (i) of Theorem 2.2 now shows that if \(x = x^*, y = y^* \in L^1(\mathcal{M}, \tau) \cap \mathcal{M}\), then

\[
\|[x] - |y|\|_{E(\mathcal{M}, \tau)} \leq c(E) \|[x] - y\|_{E(\mathcal{M}, \tau)}.
\]

The proof of the implication (viii) \(\Rightarrow\) (ii) is now completed by an appeal to Lemma 2.3 preceding and the equivalence (i) \(\Leftrightarrow\) (ii) of Theorem 2.2.

This suffices to conclude the proof of the proposition.

3. LIPSCHITZ ESTIMATES

We now gather some of the tools concerning representations of locally compact abelian groups that will be used in the sequel. For details and proofs, we refer the reader to [Arv, Zs, St]. It will be convenient to adopt the notation and setting of [Zs], and refer to this source for terminology not explained in the text. In the present section, we need the concepts introduced below only in the case of compact groups and strongly continuous representations; the more general setting will be needed for the following section.

Let \(G\) be a locally compact abelian group with any (fixed) choice of Haar measure \(dg\), normalised if \(G\) is compact, with dual group \(\hat{G}\). The pairing between \(G\) and \(\hat{G}\) is denoted by \(\langle g, \gamma \rangle\) with \(g \in G\) and \(\gamma \in \hat{G}\). Let \((X, \mathcal{F})\) be a dual pair of (complex) Banach spaces in the sense of [Zs], that is, \((X, \mathcal{F})\) is a pair of (complex) Banach spaces together with a bilinear functional

\[
(x, \phi) \mapsto \langle x, \phi \rangle, \quad (x, \phi) \in X \times \mathcal{F}
\]

such that

(i) \(\|x\| = \sup_{\phi \in X, ||\phi|| \leq 1} |\langle x, \phi \rangle|\) for any \(x \in X\);
(ii) \(\|\phi\| = \sup_{x \in X, ||x|| \leq 1} |\langle x, \phi \rangle|\) for any \(\phi \in \mathcal{F}\).
(iii) the convex hull of every relatively $\mathcal{F}$-compact subset of $X$ is relatively $\mathcal{F}$-compact;

(iv) the convex hull of every relatively $X$-compact subset of $\mathcal{F}$ is relatively $X$-compact.

We denote by $B_\mathcal{F}(X)$ the Banach space of all $\mathcal{F}$-continuous linear operators on $X$. Let $U = \{U_\gamma\}_{\gamma \in G} \subseteq B_\mathcal{F}(X)$ be an $\mathcal{F}$-continuous representation of $G$. The representation $\{U_\gamma\}_{\gamma \in G} \subseteq B_\mathcal{F}(X)$ is said to be bounded if $\sup \{||U_\gamma||: g \in G\} < \infty$. If $G$ is compact, then any $\mathcal{F}$-continuous representation is bounded. If $\{U_\gamma\}_{\gamma \in G}$ is a bounded representation of $G$, and $x \in X$, $f \in L^1(G)$, we define the Arveson convolution $f \ast_U x \in X$ by setting

$$f \ast_U x = \int_G f(g) U_\gamma(x) \, dg,$$

where the integral is taken with respect to the weak topology on $X$ induced by $\mathcal{F}$. For $x \in X$, we define the Arveson spectrum by

$$\text{sp}_U(x) = \cap \{Z(\hat{f}) : f \in L^1(G) \text{ and } f \ast_U x = 0\},$$

where $\hat{f}$ denotes the Fourier transform of $f$, and $Z(\hat{f}) = \{\gamma \in \hat{G} : \hat{f}(\gamma) = 0\}$. For any closed subset $F \subseteq \hat{G}$, we define the corresponding spectral subspace $X_U^F$ by setting

$$X_U^F = \{x \in X : \text{sp}_U(x) \subseteq F\},$$

which is a closed subspace of $X$. We remark that

$$X_U^F = \mathcal{F}\text{-closure of } \left(\bigcup X_U^K : K \subseteq F, K \text{ compact}\right).$$

Further, if $F_1, F_2 \subseteq \hat{G}$ are closed and if $F_1 \cap F_2 = \emptyset$, then $X_U^{F_1} \cap X_U^{F_2} = \{0\}$. If $\gamma \in \hat{G}$, then we denote the spectral subspace $X_U^{\gamma}$ by $X_U^\gamma$. We shall refer to this spectral subspace as the eigenspace corresponding to $\gamma$. It is not difficult to show that

$$X_U^\gamma = \{x \in X : U_\gamma(x) = \langle g, -\gamma \rangle x, \forall g \in G\}.$$

We note that the eigenspace $X_U^\gamma$ may be $\{0\}$.

For the remainder of this section, we will assume that $G$ is compact. If $F \subseteq \hat{G}$, then $X_U^F$ is the $\mathcal{F}$-closure of the linear hull of $\bigcup X_U^{\gamma}$ and $\{U_\gamma\}_{\gamma \in G}$ is said to have the projection property on $F$ if and only if there exists an $\mathcal{F}$-continuous linear projection $P_U^F$ with range the spectral subspace $X_U^F$ and kernel $X_U^{\bar{G}\setminus F}$. Since $G$ is compact, it follows that $\{U_\gamma\}_{\gamma \in G}$
has the projection property on each singleton subset of $\mathcal{G}$. In fact, writing $\mathcal{P}_\gamma^U$ instead of $\mathcal{P}_{\{\gamma\}}^U$ for each $\gamma \in \mathcal{G}$, it is not difficult to see that

$$\mathcal{P}_\gamma^U(x) = \gamma \ast_U x, \quad x \in X, \gamma \in \mathcal{G}$$

yields the desired projection.

Now suppose in addition that $\mathcal{G}$ is connected. In this case, $\mathcal{G}$ can be (totally) ordered: there exists a positive cone $\Sigma \subseteq \mathcal{G}$ satisfying $\Sigma + \Sigma \subseteq \Sigma$, $\Sigma \cap (-\Sigma) = \{0\}$ and $\Sigma \cup (-\Sigma) = \mathcal{G}$. Such an ordering is, in general, not unique. If $\{\mathcal{U}_\gamma\}_{\gamma \in \mathcal{G}}$ has the projection property on the set $\mathcal{G}$, then the corresponding projection $\mathcal{P}_\gamma^U$ is called the generalized Riesz projection (associated with the representation $\{\mathcal{U}_\gamma\}_{\gamma \in \mathcal{G}}$ and positive cone $\Sigma$). In general, this projection need not exist.

Before proceeding, we need some additional terminology. It is convenient to recall the following definition.

A Banach space $X$ is said to have the UMD-property if for some $p \in (1, \infty)$, there exists a constant $C_p(X)$, which depends only on $p$ and $X$ such that for all $n \in \mathbb{N}$

$$\left\| \sum_{j=1}^n \varepsilon_j d_j \right\|_{\mathcal{L}^p(X)} \leq C_p(X) \left\| \sum_{j=1}^n d_j \right\|_{\mathcal{L}^p(X)}$$

for every $X$-valued martingale difference sequence $\{d_j\}$ and for every $\{\varepsilon_j\} \in \{-1, 1\}^\infty$. Here $\mathcal{L}^p(X) = \mathcal{L}^p(\Omega, \mathcal{F}, \mu)$ denotes the set of all strongly measurable functions $f$ on the probability space $(\Omega, \mathcal{F}, \mu)$ with values in the Banach space $X$ such that $\|f\|_{\mathcal{L}^p(X)} := (\int_\Omega \|f(\omega)\|^p \, d\mu(\omega))^{1/p} < \infty$. We shall be concerned here with the characterisation of the UMD-property due to Burkholder [Bur1, 2] and Bourgain [Bou] in terms of the $X$-valued Hilbert transform, which we formulate as follows. The Banach space $X$ has the UMD-property if and only if there exists $1 < p < \infty$ and a constant $C_p(X)$, which depends only on $p$ and $X$ such that for all finite sequences $\{x_n\}_{n \in \mathbb{N}} \subseteq X$,

$$\int_\mathbb{T} \left\| \sum_{k=1}^n e^{ik\theta} x_k \right\|_X^p \, d\mu(\theta) \leq C_p(X) \int_\mathbb{T} \left\| \sum_{k=-n}^n e^{ik\theta} x_k \right\|_X^p \, d\mu(\theta),$$

where $d\mu(\theta)$ denotes normalised Lebesgue measure on $\mathbb{T}$. We note that this characterisation is equivalent to those more frequently given in terms of various Hilbert kernels. If $X$ has the UMD-property for some $1 < p < \infty$, then $X$ has the UMD-property for all $1 < p < \infty$. We remark that the classical Riesz projection theorem for $L^p(\mathbb{T})$, $1 < p < \infty$, is simply equivalent to the assertion that the Banach space $\mathbb{C}$ has the UMD-property. A discussion of equivalent formulations of the UMD-property which stress
the role played by the Hilbert kernel may be found in [BGM1, 2, 3]. More information concerning UMD-spaces may be found in [RdF, Bou, Bur1, 2]. We refer also to the survey [Buk] and the references therein.

If we consider the strongly continuous representation \( t \to S_t \) of the group \( \mathbb{T} \) on the Banach space \( L^p(\mathbb{T}; X) \) \( (1 < p < \infty) \) given by \( S_t F(\theta) = F(\theta - t) \) for all \( F \in L^p(\mathbb{T}; X) \) and all \( \theta, t \in \mathbb{T} \), and if we consider in \( \mathbb{T} = \mathbb{Z} \) the standard positive cone \( \mathbb{Z}^+ \), then the existence of the corresponding Riesz projection is equivalent to the UMD-property of \( X \). Indeed, this follows by simply observing that in this special case, for each \( \gamma \in \mathbb{T} = \mathbb{Z} \), the corresponding eigenspace \( X_\gamma \) is of the form \( \{ x e^{i \gamma t} : x \in X \} \).

The result concerning UMD-spaces which we will use in this paper is the following which is based on a transference argument.

**Proposition 3.1** [BGM4, Theorem 4.1]. Let \( U = \{ U_g \}_{g \in G} \) be a strongly continuous representation of the compact connected abelian group \( G \) on the UMD-space \( X \) and let \( \Sigma \) be the positive cone of a linear ordering of \( G \). Then the corresponding generalized Riesz projection \( P^U_\Sigma \) exists in \( X \) and satisfies

\[
\| P^U_\Sigma \| \leq C_X M^2,
\]

where \( M = \sup \{ \| U_g \| : g \in G \} \) and \( C_X \leq C_p(X) \) for all \( 1 < p < \infty \).

We will apply this result in the situation where \( X \) is a non-commutative \( L^p \)-space, \( X = L^p(\mathbb{A}, \tau) \) with \( 1 < p < \infty \). It is of course well known that if \( (\mathbb{A}, \tau) \) is a semifinite von Neumann algebra, then each of the spaces \( L^p(\mathbb{A}, \tau) \), \( 1 < p < \infty \), has the UMD-property, and that this may be proved as in the commutative case via the Cotlar bootstrap method. We note explicitly that the UMD-constant depends only on \( p \) and not on the semifinite von Neumann algebra \( (\mathbb{A}, \tau) \). See, for example, [BGM1, 2, 3] and the references contained therein.

We are now in a position to present the main results of this paper. In the matrix setting, the lemma which follows is due to Davies [Da1, Corollary 5]. The essential core of the argument of [Da1] is the theorem of Macaev in the trace ideal setting. It seems that the present approach via spectral subspaces, even in the matrix setting, yields some additional insight into the method of [Da1].

**Lemma 3.2.** If \( 1 < p < \infty \), then there exists a constant \( K_p > 0 \), which depends only on \( p \), such that

\[
\left\| \sum_{m, n=1}^N \frac{\lambda_m - \mu_n}{\lambda_m + \mu_n} p_m x p_n \right\| \leq K_p \| x \|_p
\]
for all semifinite von Neumann algebras (\mathcal{A}, \tau), for all finite sequences
p_1, p_2, ..., p_N of mutually orthogonal projections in \mathcal{A}, for all \( x \in L^p(\mathcal{A}, \tau) \), and
for all choices 0 \leq \lambda_1, \lambda_2, ..., \lambda_N; \mu_1, \mu_2, ..., \mu_N \in \mathbb{R}
with \( \lambda_m + \mu_n > 0 \) for all \( m, n = 1, 2, ..., N \).

**Proof.** Let the projections \( p_1, p_2, ..., p_N \in \mathcal{A} \) be fixed. Since the left
hand side of the stated inequality depends continuously on \( \lambda_1, \lambda_2, ..., \lambda_N; \mu_1, \mu_2, ..., \mu_N \),
we may assume without loss of generality that each of \( \lambda_1, \lambda_2, ..., \lambda_N; \mu_1, \mu_2, ..., \mu_N \) are positive, mutually different rational numbers.

By multiplication by an appropriate common denominator, it suffices to
show that there exists a constant \( K_p > 0 \) such that
\[
\left\| \sum_{m,n=1}^{N} \frac{k_m-l_n}{k_m+l_n} p_m x p_n \right\|_p \leq K_p \| x \|_p,
\]
for all \( x \in L^p(\mathcal{A}, \tau) \) and for all natural numbers \( k_1, k_2, ..., k_N, l_1, l_2, ..., l_N \)
such that \((k_m, l_n) \neq (k_r, l_s)\) whenever \((m,n) \neq (r,s)\). Furthermore, we may
assume that \( p_1 + p_2 + \cdots + p_N = 1 \). For \( t = (t_1, t_2) \in \mathbb{T}^2 \), define unitary
operators \( u_t, v_t \in \mathcal{A} \) by setting
\[
u_{t_m} = \sum_{m=1}^{N} e^{it_m \tau_1} p_m, \quad \nu_{t_n} = \sum_{n=1}^{N} e^{it_n \tau_2} p_n,
\]
and isometries
\[
R_t : L^p(\mathcal{A}, \tau) \to L^p(\mathcal{A}, \tau)
\]
by setting
\[
R_t(x) = u_t x v_t, \quad x \in L^p(\mathcal{A}, \tau).
\]
It is clear that the mapping \( t \to R_t \) is a strongly continuous (even uniformly
continuous) representation of \( \mathbb{T}^2 \) on \( L^p(\mathcal{A}, \tau) \). The dual group of \( \mathbb{T}^2 \) may
be identified with \( \mathbb{Z}^2 \) via the pairing
\[
\langle t, \gamma \rangle = e^{it_1 \gamma_1 + it_2 \gamma_2}
\]
for all \( t = (t_1, t_2) \in \mathbb{T}^2 \) and all \( \gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}^2 \). For \( \gamma \in \mathbb{Z}^2 \), the corresponding
eigenspace
\[
L^p(\mathcal{A}, \tau)_\gamma = \{ x \in L^p(\mathcal{A}, \tau) : R_t(x) = \langle t, -\gamma \rangle x, \forall t \in \mathbb{T}^2 \}
\]
\[
is given by
\[
L^p(\mathcal{A}, \tau)_\gamma = p_m L^p(\mathcal{A}, \tau) p_n
\]
if \( \gamma = (k_m, l_m) \) for some \( m, n = 1, \ldots, N \) and \{0\} otherwise. For \( 0 \leq \theta < \pi/2 \), we define a (linear) group ordering on \( \mathbb{Z}^2 \) via the positive cone

\[
(\mathbb{Z}^2)^+ \,:= \{ (\gamma_1, \gamma_2) \in \mathbb{Z}^2 : \gamma_2 < (\tan \theta) \gamma_1, \text{ or } \gamma_2 = (\tan \theta) \gamma_1 \text{ and } \gamma_1 \geq 0 \}.
\]

The corresponding spectral subspaces of \( L^p(\mathcal{M}, \tau) \) are given by

\[
L^p(\mathcal{M}, \tau)^+ = \text{span}\{ L^p(\mathcal{M}, \tau) : \gamma \in (\mathbb{Z}^2)^+ \}
\]

and

\[
L^p(\mathcal{M}, \tau)^- = \text{span}\{ L^p(\mathcal{M}, \tau) : \gamma \in \mathbb{Z}^2 \}.
\]

Let \( P_{\varphi} \) be the projection onto \( L^p(\mathcal{M}, \tau)^+ \) along \( L^p(\mathcal{M}, \tau)^- \); that is,

\[
P_{\varphi} x = \sum_{\{(m, n) : \gamma \in (\mathbb{Z}^2)^+\}} p_m \gamma^m, \quad x \in L^p(\mathcal{M}, \tau).
\]

It follows from Proposition 3.1 that \( \|P_{\varphi}\| \leq C_p \), where \( C_p \) is a constant depending only on \( p \). Note that \( P_0 = 0 \). Let \( 0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_M \leq \pi/2 \) be such that

\[
\{ \frac{1}{k_m} : m, n = 1, \ldots, N \} = \{ \tan \theta_j : j = 1, \ldots, M \}.
\]

If \( F_j = \{(m, n) : l_m = \tan \theta_j \} \), \( j = 1, \ldots, M \), then

\[
(P_{\theta_j} - P_{\theta_{j-1}}) x = \sum_{(m, n) \in F_j} p_m \gamma^m, \quad j = 1, 2, \ldots, M,
\]

and hence

\[
\sum_{m, n = 1}^N \frac{k_m - l_m}{k_m + l_m} p_m \gamma^m = \sum_{j = 1}^M \sum_{(m, n) \in F_j} \frac{1 - l_m/k_m}{1 + l_m/k_m} p_m \gamma^m = \sum_{j = 1}^M \frac{1 - \tan \theta_j}{1 + \tan \theta_j} (P_{\theta_j} - P_{\theta_{j-1}}) x
\]

for all \( x \in L^p(\mathcal{M}, \tau) \). We define the function \( g \) on \([0, \pi/2]\) by setting

\[
g(\theta) = \frac{1 - \tan \theta}{1 + \tan \theta}, \quad 0 \leq \theta < \pi/2
\]
and \( g(\pi/2) = -1 \). It follows that
\[
\sum_{m,n=1}^{N} \frac{k_m - l_n}{k_m + l_n} P_m x P_n = \sum_{j=1}^{M} g(\theta_j)(P_{\theta_j} - P_{\theta_{j-1}}) x \\
= \sum_{j=1}^{M-1} (g(\theta_j) - g(\theta_{j+1})) P_{\theta_j} x + g(\theta_M) P_{\theta_M} x.
\]

Since \( g \) is continuously differentiable on \([0, \pi/2]\) with \( \|g\|_{\infty} = 1 \), \( \|g'\|_{\infty} = 2 \), it follows that
\[
\left\| \sum_{m,n=1}^{N} \frac{k_m - l_n}{k_m + l_n} P_m x P_n \right\|_p \\
\leq \sum_{j=1}^{M-1} \|g(\theta_j) - g(\theta_{j+1})\| \|P_{\theta_j}\| \|x\|_p + \|g(\theta_M)\| \|P_{\theta_M}\| \|x\|_p \\
\leq (\pi + 1) C_p \|x\|_p,
\]

for all \( x \in L^p(\mathcal{M}, \tau) \). This completes the proof of the lemma. 

Before proceeding, we need to recall the notions of upper and lower indices in rearrangement invariant spaces. Let \( E \) be a symmetric Banach function space on \( \mathbb{R}^+ \). For \( s > 0 \), the dilation operator \( D_s : E \to E \) is defined by setting
\[
D_s f(t) = f(ts), \quad t > 0, \quad f \in E.
\]

The lower and upper Boyd indices (or dilation exponents) of \( E \) are defined by
\[
\underline{\sigma}_E = \lim_{s \to 0} \ln \left( \frac{\|D_s\|}{s} \right), \quad \bar{\sigma}_E = \lim_{s \to \infty} \ln \left( \frac{\|D_s\|}{s} \right),
\]

and satisfy \( 0 \leq \underline{\sigma}_E \leq \bar{\sigma}_E \leq 1 \) (see [KPS]). As is easily checked, if \( E = L^p, 1 \leq p < \infty \), then \( \sigma_E = \bar{\sigma}_E = 1/p \). It is by now a classical result of D. W. Boyd [Bo1] that if \( 1 \leq p < q \leq \infty \), then \( E \) is an interpolation space for the pair \( (L^p, L^q) \) if and only if \( 1/q \leq \underline{\sigma}_E \leq \bar{\sigma}_E < 1/p \). We remark that in [Bo1, BS, LT], this result is proved under the additional hypothesis (at least) that \( E \) is an interpolation space for the pair \( (L^1, L^\infty) \). However, it is not difficult to see, using in particular [KPS, Lemma II 4.7], that this additional assumption on \( E \) can be omitted. In particular, it follows that the condition \( 0 \leq \underline{\sigma}_E \leq \bar{\sigma}_E < 1/p \) already implies that \( E \) is an interpolation space for the pair
$(L^1, L^∞)$. If $0 < \lambda_E \leq \lambda_E < 1$, we shall say simply that $E$ has non-trivial Boyd indices.

**Theorem 3.3.** Let $E$ be a symmetric Banach function space on $\mathbb{R}^+$ with Fatou norm. The following statements are equivalent.

(i) $E$ has non-trivial Boyd indices.

(ii) There exists a constant $c(E)$ which depends only on $E$ such that

$$\left\| \sum_{m,n=1}^N \frac{\lambda_m - \mu_n}{\lambda_m + \mu_n} p_m x p_n \right\|_{E(\mathcal{A}, \tau)} \leq c(E) \| x \|_{E(\mathcal{A}, \tau)},$$

for all semifinite von Neumann algebras $(\mathcal{A}, \tau)$, for all finite sequences $p_1, p_2, ..., p_N$ of mutually orthogonal projections in $\mathcal{A}$, for all $x \in E(\mathcal{A}, \tau)$ and all choices $0 \leq \lambda_1, \lambda_2, ..., \lambda_N, \mu_1, \mu_2, ..., \mu_N \in \mathbb{R}$ with $\lambda_n + \mu_m > 0$ for all $m, n = 1, 2, ..., N$.

(iii) There exists a positive constant $c(E) > 0$ which depends only on $E$ such that

$$\left\| \sum_{1 \leq n < m \leq N} p_m x p_n \right\|_{E(\mathcal{A}, \tau)} \leq c(E) \| x \|_{E(\mathcal{A}, \tau)},$$

for all $x \in E(\mathcal{A}, \tau)$, for every semifinite von Neumann algebra $(\mathcal{A}, \tau)$, for all finite sequences $p_1, p_2, ..., p_N$ of mutually orthogonal projections in $\mathcal{A}$.

**Proof.** (i) $\Rightarrow$ (ii). Without loss of generality, we may assume that $p_1 + p_2 + \cdots + p_N = 1$. Choose $1 < p < q < \infty$ such that $1/q < \lambda_E \leq \lambda_E < 1/p$. Boyd’s theorem implies that $E$ is an interpolation space for the pair $(L^p, L^q)$. It follows from [DDP2, Theorem 3.4] that $E(\mathcal{A}, \tau)$ is an interpolation space for the pair $(L^p(\mathcal{A}, \tau), L^q(\mathcal{A}, \tau))$. The implication (i) $\Rightarrow$ (ii) is now an immediate consequence of Lemma 3.2.

(ii) $\Rightarrow$ (iii). In assertion (ii), we set $\lambda_m = k^m$, $\mu_n = k^n$, $1 \leq m, n \leq N$, $k \in \mathbb{N}$. Letting $k \to \infty$, we obtain that

$$\left\| \sum_{1 \leq n < m \leq N} p_m x p_n - \sum_{1 \leq m < n \leq N} p_n x p_m \right\|_{E(\mathcal{A}, \tau)} \leq c(E) \| x \|_{E(\mathcal{A}, \tau)},$$

From [CKS, Corollary 3.4], it follows that

$$\left\| \sum_{n=1}^N p_n x p_n \right\|_{E(\mathcal{A}, \tau)} \leq \| x \|_{E(\mathcal{A}, \tau)}.$$
Consequently,
\[ \left\| \sum_{1 \leq n < m < N} p_m x p_n \right\|_{E(\mathcal{A}, \tau)} \]
\[ \leq \frac{1}{2} \left\| \sum_{1 \leq n < m < N} p_m x p_n - \sum_{1 \leq m < n < N} p_m x p_n \right\|_{E(\mathcal{A}, \tau)} \]
\[ + \frac{1}{2} \left\| \sum_{n, m = 1}^{N} p_m x p_n \right\|_{E(\mathcal{A}, \tau)} + \frac{1}{2} \left\| \sum_{n = 1}^{N} p_n x p_n \right\|_{E(\mathcal{A}, \tau)} \]
\[ \leq (1 + \frac{1}{2} c(E)) \| x \|_{E(\mathcal{A}, \tau)} \]
and this suffices to complete the proof of the implication (ii) \( \Rightarrow \) (iii).

(iii) \( \Rightarrow \) (i). We suppose first that \( \alpha_x = 1 \). Since the operator of triangular truncation is unbounded on the Schatten class \( \mathcal{C}_1 \), there exist a natural number \( N \) and an \( N \times N \) matrix \( A \) such that if \( B = \left[ b_{ij} \right]_{i,j=1}^{N} \) is the \( N \times N \) matrix defined by setting
\[ b_{ij} = \begin{cases} a_{ij}, & \text{if } 1 \leq i < j \leq N; \\ 0, & \text{otherwise}, \end{cases} \]
then
\[ \| A \|_1 = 1, \quad \| B \|_1 \geq 2 c(E). \]
Here \( \| \cdot \|_1 \) denotes the usual trace norm. Since \( \alpha_x = 1 \), it follows from [LT, 2.b.6], that there exist non-negative, disjointly supported, equidistributed functions \( \{ f_i \}_{i=1}^{N} \) with \( \| f_i \|_E = 1, 1 \leq i \leq N \), and
\[ \frac{1}{2} \sum_{i=1}^{N} |a_i| \leq \sum_{i=1}^{N} a_i f_i \]
for every choice of scalars \( \{ a_i \}_{i=1}^{N} \subseteq \mathbb{C} \). Let \( \mathcal{A} \) be \( L^\infty(\mathbb{R}^+) \otimes M_N(\mathbb{C}) \) equipped with the trace \( \tau \) given by \( \lambda \otimes \text{tr} \) where \( \lambda \) denotes the trace on \( L^\infty(\mathbb{R}^+) \) induced by Lebesgue measure, and \( \text{tr} \) denotes the canonical trace on \( M_N(\mathbb{C}) \). Observe that \( E(\mathcal{A}, \tau) \) may be identified with the space \( E \otimes M_N(\mathbb{C}) \) of all \( N \times N \) matrices \( \left[ g_{ij} \right]_{i,j=1}^{N} \) with \( g_{ij} \in E \). We now observe that
\[ \| f_i \otimes C \|_{E(\mathcal{A}, \tau)} = \left\| \sum_{j=1}^{N} s_j(C) f_i \right\|_E \]
for any \( C \in M_N(\mathbb{C}) \), where \( s_1(C), s_2(C), \ldots, s_N(C) \) denote the singular values of \( C \) arranged in non-increasing order. In fact, let \( C \) be any \( N \times N \) matrix,
and denote by $D$ the diagonal matrix with entries $s_1(C), s_2(C), \ldots, s_N(C)$. If $U, V \in M_N(\mathbb{C})$ are unitary matrices for which $C = UDV$, then

$$\mu(f_1 \otimes C) = \mu(1 \otimes U : f_1 \otimes D \cdot 1 \otimes V) = \mu(f_1 \otimes D) = \mu\left(\sum_{j=1}^{N} s_j(C) f_j\right),$$

where the last inequality follows easily from the definition of the natural trace on $\mathcal{M}$, and this proves our assertion. It now follows that

$$\|f_1 \otimes A\|_{E_1, \mathcal{M}, \tau} = \left\|\sum_{j=1}^{N} s_j(A) f_j\right\|_E \leq \left\|\sum_{j=1}^{N} s_j(A)\right\| \|f_1\|_E = 1.$$

On the other hand,

$$\|f_1 \otimes B\|_{E_1, \mathcal{M}, \tau} = \left\|\sum_{j=1}^{N} s_j(B) f_j\right\|_E \geq \frac{1}{2} \left(\sum_{j=1}^{N} s_j(B)\right) \geq \frac{1}{2} c(E).$$

Assertion (iii) clearly implies that

$$\|f_1 \otimes B\|_{E_1, \mathcal{M}, \tau} \leq c(E) \|f_1 \otimes A\|_{E_1, \mathcal{M}, \tau},$$

and this yields a contradiction. Essentially the same argument shows that the assertion of (iii) implies that $s_E > 0$, and this suffices to complete the proof of the theorem.

The principal result of this paper now follows and is an immediate consequence of Theorem 2.2, Proposition 2.4 and Theorem 3.3 via the equivalence of the statements of Theorem 3.3(ii) and Proposition 2.4(vi).

**Theorem 3.4.** If $E$ is a symmetric Banach function space on $\mathbb{R}^+$ with the Fatou property then the following statements are equivalent.

(i) $E$ has non-trivial Boyd indices.

(ii) There exists a constant $c(E) > 0$ which depends only on $E$ such that, for all semifinite von Neumann algebras $(\mathcal{M}, \tau)$, $x, y \in \mathcal{M}$, $x - y \in E(\mathcal{M}, \tau)$ imply $|x| - |y| \in E(\mathcal{M}, \tau)$ and

$$\| |x| - |y| \|_{E_1, \mathcal{M}, \tau} \leq c(E) \| x - y \|_{E_1, \mathcal{M}, \tau}.$$

(iii) There exists a constant $c(E) > 0$ which depends only on $E$ such that, for all semifinite von Neumann algebras $(\mathcal{M}, \tau)$, $x = x^* \in \mathcal{M}$, $y \in \mathcal{M}$, $[x, y] \in E(\mathcal{M}, \tau)$ imply $|[x, y]| \in E(\mathcal{M}, \tau)$ and

$$\|[x, y]\|_{E_1, \mathcal{M}, \tau} \leq c(E) \|[x, y]\|_{E_1, \mathcal{M}, \tau}.$$
There exists a constant \( c(E) > 0 \) which depends only on \( E \) such that, for all semifinite von Neumann algebras \((\mathcal{M}, \tau)\), \( xy - yz \in E(\mathcal{M}, \tau) \) imply
\[
|xy - yz|_{E(\mathcal{M}, \tau)} \leq c(E) \|xy - yz\|_{E(\mathcal{M}, \tau)}.
\]

There exists a constant \( c(E) > 0 \) which depends only on \( E \) such that, for all semifinite von Neumann algebras \((\mathcal{M}, \tau)\), \( 0 \leq x, z \in \mathcal{M}, y \in \mathcal{M} \), \( xy + yz \in E(\mathcal{M}, \tau) \) imply
\[
|xy + yz|_{E(\mathcal{M}, \tau)} \leq c(E) \|xy + yz\|_{E(\mathcal{M}, \tau)}.
\]

In the case of trace ideals, the preceding theorem is due to Kosaki [Ko2]. Let us note that the assertion of (i) in the preceding theorem is automatically satisfied in the special case that \( E = L^p(\mathbb{R}^+) \), \( 1 < p < \infty \). In this case, we obtain the following special case of Theorem 3.4 which reduces to the estimates given by Davies [Da] for the Schatten \( p \)-classes.

Corollary 3.5. If \((\mathcal{M}, \tau)\) is a semifinite von Neumann algebra, and if \( 1 < p \leq \infty \), then there exist constants \( c_1(p), c_2(p) > 0 \) which depend only on \( p \) such that

(i) \( x, y \in \mathcal{M}, x - y \in L^p(\mathcal{M}, \tau) \) imply \(|x| - |y| \in L^p(\mathcal{M}, \tau)\) and
\[
\| |x| - |y| \|_p \leq c_1(p) \| x - y \|_p.
\]

(ii) \( x = x^*, y \in \mathcal{M}, [x, y] \in L^p(\mathcal{M}, \tau) \) imply \([|x|, y] \in L^p(\mathcal{M}, \tau)\) and
\[
\|[|x|, y]| \|_p \leq c_2(p) \|[x, y]\|_p.
\]

We now assume that \( \mathcal{M} \) is a general von Neumann algebra (not necessarily semifinite). Let \( \mathcal{U} \) be the crossed product of \( \mathcal{M} \) by the modular automorphism group \( \{\sigma_t\}_{t \in \mathbb{R}} \) of a fixed weight on \( \mathcal{M} \). It is proved in [Ta2] that \( \mathcal{U} \) admits the dual action \( \{\theta_s\}_{s \in \mathbb{R}} \) and the normal faithful semifinite trace \( \tau \) satisfying \( \tau \circ \theta_s = e^{-s} \tau, s \in \mathbb{R} \). For \( 1 \leq p \leq \infty \), the Haagerup \( L^p \)-space, denoted \( L^p(\mathcal{U}) \), associated with \( \mathcal{M} \) consists of those \( \tau \)-measurable operators \( x \) affiliated with \( \mathcal{U} \) which satisfy \( \theta_s(x) = e^{-sp}x, s \in \mathbb{R} \). For full details, see [Te]. If \( p \geq 1 \), the spaces \( L^p(\mathcal{U}) \) are Banach spaces. Further, it is shown in [FK], Lemma 4.8, that if \( x \in L^p(\mathcal{U}) \), \( 1 \leq p < \infty \), then
\[
\mu_\tau(x) = t^{-1/p} \| x \|_p, \quad t > 0,
\]
where the decreasing rearrangement is taken relative to the canonical trace on \( \mathcal{U} \).
We recall (see for example [BS, Chap. 4]) that if $1 \leq p \leq \infty$, then the Lorentz space $L^{p,\infty}(\mathbb{R}^+, m)$ consists of those (Lebesgue) measurable functions on $\mathbb{R}^+$ for which
\[
\|f\|_{p,\infty} = \sup_{0 < t < \infty} \left\{ t^{1/p} \mu_{t}(f) \right\} < \infty.
\]
If $1 < p \leq \infty$, then the space $L^{p,\infty}(\mathbb{R}^+, m)$ equipped with the equivalent Calderon norm $\| \cdot \|_{(p, \infty)}$ given by
\[
\|f\|_{(p, \infty)} = \sup_{0 < t < \infty} \left\{ t^{1/p-1} \int_{0}^{t} \mu_{s}(f) \, ds \right\}, \quad f \in L^{p,\infty}(\mathbb{R}^+, m),
\]
is a symmetric Banach function space on $\mathbb{R}^+$ with the Fatou property. The proposition which follows, which is due to H. Kosaki [Ko3], is now a simple consequence of these remarks.

**Proposition 3.6.** If $1 < p < \infty$ then the Haagerup space $L^p(\mathcal{H})$ is a closed subspace of the symmetric operator space $L^{p,\infty}((\mathcal{H}, \tau))$ associated with the semifinite von Neumann algebra $(\mathcal{H}, \tau)$. Moreover, if $1/q + 1/p = 1$, then

\[
\|x\|_p = q \|x\|_{(p, \infty)}
\]
for all $x \in L^p(\mathcal{H})$.

Let us now remark that if $1 < p_1 < p < p_2 < \infty$, then it follows from [BL, Theorem 5.3.1] that the Lorentz space $L^{p,\infty}$ is an interpolation space for the pair $(L^{p_1}, L^{p_2})$. We obtain immediately the following consequence of Theorem 3.4.

**Theorem 3.7.** If $1 < p < \infty$, and if $L^p(\mathcal{H})$ is the Haagerup $L^p$-space, then there exist constants $K_1(p) > 0$, $K_2(p) > 0$ such that

\[
\| |x| - |y| \|_p \leq K_1(p) \|x-y\|_p
\]
for all $x, y \in L^p(\mathcal{H})$ and

\[
\|[x, y]\|_p \leq K_2(p) \|[x, y]\|_p
\]
for all $x = x^* \in L^p(\mathcal{H})$ and $y \in \mathcal{H}$.
4. RIESZ PROJECTIONS AND A GENERALISED MACAEOV THEOREM

Let $G$ be a locally compact abelian group with Haar measure $dg$ and dual group $\hat{G}$, let $(X, \mathcal{F})$ be a dual pair of (complex) Banach spaces in the sense of [Zs] and let $\{U_g\}_{g \in G} \subseteq B_\mathcal{F}(X)$ be an $\mathcal{F}$-continuous representation of $G$. If $S$ is an arbitrary subset of $\hat{G}$, the spectral subspace $X^S_U$ is defined by setting

$$X^S_U = \mathcal{F} - \text{closure of} \left( \bigcup_{K \subseteq S, K \text{ compact}} \{X^U_K: K \subseteq S, K \text{ compact}\} \right).$$

Following [Zs], $\{U_g\}_{g \in G}$ is said to have the weak projection property on $S \subseteq \hat{G}$ if for any closed $F \subseteq \mathcal{F}$, $F - \text{closure of} \left( \bigcup_{K \subseteq S, K \text{ compact}} X^U_K \right) = X^F_U$.

$\{U_g\}_{g \in G}$ is said to have the projection property on $S \subseteq \hat{G}$ if $\{U_g\}_{g \in G}$ has the weak projection property on $S$ and there exists an $\mathcal{F}$-continuous linear projection $P^S_U$ with range the spectral subspace $X^S_U$ and kernel $H^U_{G \backslash S}$. We remark that if $\{U_g\}_{g \in G}$ has the projection property on $S \subseteq \hat{G}$, then

$$P^S_U U_g = U_g P^S_U, \quad g \in G.$$

Further, the family of subsets $S \subseteq \hat{G}$ for which $\{U_g\}_{g \in G}$ has the projection property on $S$ is a ring of subsets of $\hat{G}$ and the mapping $S \mapsto P^S_U$ defined on this ring is finitely additive.

If $(\mathcal{M}, \tau)$ is a semifinite von Neumann algebra, a $G$-flow on $(\mathcal{M}, \tau)$ is an ultraweakly continuous representation $U = \{U_g\}_{g \in G}$ of $G$ on $\mathcal{M}$ by $*$-automorphisms of $\mathcal{M}$ which preserve the trace $\tau$. It is clear that any $G$-flow $U$ on $\mathcal{M}$ is a group of isometries on $\mathcal{M}$ and since $U$ is trace preserving, it follows easily that $U$ extends to a group of trace-preserving isometries on $L^1(\mathcal{M}, \tau)$. A simple interpolation argument now shows that any $G$-flow $U$ on $\mathcal{M}$ extends to a group of rearrangement-preserving maps on $L^1(\mathcal{M}, \tau) + \mathcal{M}$. It follows that if $E$ is any symmetric Banach function space on $\mathbb{R}^+$, then any $G$-flow $U$ on $\mathcal{M}$ extends to a group $U^E = \{U^E_g\}_{g \in G}$ of isometries on $E(\mathcal{M}, \tau)$.

Suppose now that $E$ is a separable symmetric Banach function space on $\mathbb{R}^+$. In this case, the extension $U^E$ of the $G$-flow $U$ is uniquely determined by its restriction to $L^1(\mathcal{M}, \tau) \cap \mathcal{M}$. To show that $U^E$ is a strongly continuous group of isometries on $E(\mathcal{M}, \tau)$, it suffices by [St, Lemma 13.4] to show continuity of the map $g \mapsto \tau(U^E_g(x) y)$, $g \in G$ for all $y \in E(\mathcal{M}, \tau)$.
and \( x \in E(\mathcal{A}, \tau) \). Here, \( X' \) denotes the Banach dual of the Banach space \( X \). By separability of \( E \), it follows that \( E' \) is again a symmetric Banach function space on \( \mathbb{R}^+ \), that \( E(\mathcal{A}, \tau)' = E'(\mathcal{A}, \tau) \subseteq L^1(\mathcal{A}, \tau) + \mathcal{A} \) and that \( L^1(\mathcal{A}, \tau) \cap \mathcal{A} \) is dense in \( E(\mathcal{A}, \tau) \). It suffices further to show continuity of the map \( g \to \tau(U^g(x) y) \), \( g \in G \) for all \( x \in E(\mathcal{A}, \tau) \cap \mathcal{A} \), \( y \in \mathcal{A} \). This however follows by observing that, for all \( x \in L^1(\mathcal{A}, \tau) \cap \mathcal{A} \), \( y \in \mathcal{A} \) and \( g \in G \),

\[
\tau(U^g(x) y) = \tau(U^g(x) y) = \tau(x U^g(y))
\]

and the assertion follows.

**Theorem 4.1.** If \( E \) is a separable symmetric Banach function space on \( \mathbb{R}^+ \), then the following statements are equivalent.

1. \( E \) has non-trivial Boyd indices.
2. There exists a constant \( c(E) > 0 \) which depends only on \( E \) such that for every semifinite von Neumann algebra \( \mathcal{A} \), for every \( G \)-flow \( \{U_g\}_{g \in G} \) on \( \mathcal{A} \) and for all closed semigroups \( \Sigma \subseteq G \) such that \( \Sigma \cup (-\Sigma) \) is a group, the \( G \)-flow \( \{U_g\}_{g \in G} \) has the projection property on \( \Sigma \) and

\[
\|P^G_{\Sigma} \| \leq c(E).
\]

We shall again refer to the projection whose existence is given by the preceding theorem as the Riesz projection determined by the \( G \)-flow \( U \) and the cone \( \Sigma \). In the special case that \( G \) is the circle group \( \mathbb{T} \) and the flow \( U \) is given by right translation on the abelian von Neumann algebra \( L^1(\mathbb{T}) \), with trace given by integration with respect to normalised Haar measure, the preceding theorem yields the existence of the classical Riesz projection in \( L^1(\mathbb{T}) \) by taking \( \Sigma \) to be the set of non-negative integers \( \mathbb{Z}^+ \). Moreover the theorem contains as a special case Bochner’s extension of the Riesz projection theorem to the case of compact connected groups. In the special case that \( E = L^p(\mathbb{R}^+, m) \), \( 1 < p < \infty \), the assertion (i) is trivially satisfied and in this case, the theorem is due to L. Zsidó [Zs]. In the case that \( G \) is compact and connected, the implication (i) \( \Rightarrow \) (ii) is proved in [FS] by using the non-commutative lifting of Boyd’s interpolation theorem [DDP3] applied to the \( L^p \)-specialisation. These remarks apply verbatim to the locally compact setting, and suffice to prove the implication (i) \( \Rightarrow \) (ii).

The reverse implication (ii) \( \Rightarrow \) (i) follows from Theorem 3.3 by noting that the assertion of (ii) implies condition (iii) of that theorem. See, for example, the first part of the proof of the implication (ii) \( \Rightarrow \) (i) of Theorem 4.1 below. It is of interest to note that if \( G \) is \( \mathbb{R} \), and if \( U \) is given by right translation on the corresponding \( L^p \)-space, then Theorem 4.1 essentially recovers Boyd’s characterisation of those rearrangement invariant spaces on \( \mathbb{R} \) on
which the Hilbert transform acts continuously, at least in the separable case. In particular, in the commutative setting, the implication (ii) $\Rightarrow$ (i) is essentially Boyd’s theorem [Bo2; BS, Theorem 6.10 and Corollary 6.11]. We have remarked earlier that the existence of a certain Riesz projection is equivalent to the UMD-property. It is not without interest therefore to note explicit that the special case of Theorem 4.1 due to Zsidó [Zs] implies almost immediately that the spaces $L^p(\mathcal{M}, \tau)$, $1 < p < \infty$, have the UMD-property.

**Corollary 4.2.** If $(\mathcal{M}, \tau)$ is a semifinite von Neumann algebra, then each of the spaces $L^p(\mathcal{M}, \tau)$, $1 < p < \infty$, has the UMD-property.

**Proof.** It is well known (see, for example [BGM1]) that if we set $(\mathcal{M}_1, \tau_1) = (\mathcal{M} \otimes L^\infty(\mathbb{T}), \tau \otimes dm)$, then $L^p(\mathcal{M}_1, \tau_1)$ is isometrically isomorphic to $L^p(\mathbb{T}, L^p(\mathcal{M}, \tau))$, for $1 < p < \infty$. We now define the flow \( \{U_t\}_{t \in \mathbb{T}} \) on $(\mathcal{M}_1, \tau_1)$ by setting

\[
U_t(f \otimes x) = f(t - \cdot) \otimes x, \quad t \in \mathbb{T}, \quad f \otimes x \in \mathcal{M}_1.
\]

If $n \in \mathbb{Z}$, it is a simple matter to check that the spectral subspace $L^p(\mathcal{M}_1, \tau_1)_n$ is given by

\[
L^p(\mathcal{M}_1, \tau_1)_n = e^{in} \otimes L^p(\mathcal{M}, \tau),
\]

and the assertion of the corollary now follows directly from the implication (i) $\Rightarrow$ (ii) of Theorem 4.1.

To complete this circle of ideas, we note finally that if \( \{U_g\}_{g \in G} \) is a bounded strongly continuous representation of the locally compact group $G$ on a UMD-space $X$, and if the dual group $\hat{G}$ is ordered with Haar-measurable positive cone, then the corresponding Riesz projection exists and has norm depending only on the space $X$ and the bound of the representation. In the compact case, this theorem was established in [BGM4] and the general case is given in [ABG]. The specialisation of this theorem to spaces $L^p(\mathcal{M}, \tau)$ again recovers Zsidó’s theorem, although it does yield more since it was shown also in [BGM3] that $L^p$-spaces associated with general von Neumann algebras are also UMD-spaces if $1 < p < \infty$. In the function space setting where it is possible to compare this result with that of Theorem 3.1, let us note that an effective criterion that permits ready recognition of the UMD-property in separable symmetric Banach function spaces on $\mathbb{R}^+$ does not appear to be known; on the other hand, it is well known that there exist separable symmetric Banach function spaces on $\mathbb{R}^+$ with non-trivial Boyd indices which are not reflexive, and therefore cannot have the UMD-property.
Closely related to, and in some sense equivalent to, the existence of the Riesz projection in the non-commutative setting is the well known theorem of Macaev: if \( 1 < p < \infty \), there exists a constant \( c(p) > 0 \) such that if \( x \) is a compact quasinilpotent operator in a separable Hilbert space, and if \( \exists x \) belongs to the Schatten ideal \( \mathcal{C}_p \), then so does \( 9x \) and

\[
\|9x\|_p \leq c(p) \|3x\|_p.
\]

Macaev's theorem is proved in [GK2] via the theory of triangular truncation with respect to a chain. For a further discussion of Macaev's theorem in the setting of trace ideals, the reader is referred to [Ar, Ko1].

A general version of Macaev's theorem in the setting of \( L^p \)-spaces associated with general von Neumann algebras may be found in [BGM3]. However, it seems that the first general approach to Macaev's theorem via spectral subspace ideas may be found in [Zs]. While not formulating explicitly a general Macaev-type theorem as in [BGM3], Zsidó [Zs] showed that the classical Macaev theorem followed directly from his general theorem on the Riesz projection. We wish to point out that Zsidó's method yields a general Macaev theorem for a wide class of symmetric operator spaces.

Let \( \Sigma \subseteq \hat{G} \) be a closed semigroup such that \( \Sigma \cup (-\Sigma) \) is a group, and \( \Sigma \cap (-\Sigma) = \{0\} \). Let \( U = \{ U^\lambda \}_{\lambda \in \Sigma} \) be a bounded strongly continuous representation of \( G \) on the Banach space \( X \). We need the following.

**Lemma 4.3.** (i) \( X_{\Sigma \backslash \{0\}}^U \cap X_{-\Sigma \backslash \{0\}}^U = \{0\}, X_{\Sigma \backslash \{0\}}^U + X_{-\Sigma \backslash \{0\}}^U \cap X_{\{0\}}^U = \{0\} \).

(ii) If \( X \) is a complex Banach space with involution \( * \) then

\[
(x_{\Sigma \backslash \{0\}}^U)^* = x_{-\Sigma \backslash \{0\}}^U, \quad (x_{-\Sigma \backslash \{0\}}^U)^* = x_{\Sigma \backslash \{0\}}^U.
\]

The first assertion of statement (i) of the lemma may be found in [ABG, Remarks (6.6)]. An alternative proof follows from the final part of the argument in the last paragraph of the proof of [Zs, Theorem 5.7]. The second assertion of (i) follows from [ABG, Theorem 2.1(iii)]. The statement of (ii) follows from [Zs, Corollary 2.4(ii)]. If \((\mathcal{A}, \tau)\) is a semifinite von Neumann algebra, we denote by \( K \) the Banach space \( L^1(\mathcal{A}, \tau) + \mathcal{A}_0 \) equipped with the norm

\[
\|x\|_K := \int_0^1 \mu_\mathcal{A}(x) \, dt, \quad x \in K.
\]

Here we denote by \( \mathcal{A}_0 \) the set of all \( x \in \mathcal{A} \) such that \( \lim_{t \to \infty} \mu_\mathcal{A}(x) = 0 \). We remark that in the special case that \( \mathcal{A} \) is \( \mathcal{B}(\mathcal{H}) \) with the canonical trace,
then \( K \) is precisely the class \( \mathcal{C}_c \) of all compact operators on \( \mathcal{H} \) equipped with the operator norm.

We can now state the following extension of Macaev’s theorem.

**Theorem 4.4 (Generalised Macaev Theorem).** If \( E \) is a separable symmetric Banach function space on \( \mathbb{R}^+ \) then the following statements are equivalent.

(i) \( E \) has non-trivial Boyd indices.

(ii) There exists a positive constant \( c(E) > 0 \), depending only on \( E \) such that for every semifinite von Neumann algebra \( (\mathcal{M}, \tau) \), for every \( G \)-flow \( \{U_g\}_{g \in G} \) on \( (\mathcal{M}, \tau) \) and for all closed semigroups \( S \subseteq \hat{G} \) such that \( S \cup (-S) = \hat{G} \) and \( S \cap (-S) = \{0\} \), whenever \( x \in K^{L^E}_{\Sigma(0)} \) and \( \mathfrak{R}x \in E(\mathcal{M}, \tau) \), then

\[
\|\mathfrak{R}x\|_{E(\mathcal{M}, \tau)} \leq c(E) \|\mathfrak{A}x\|_{E(\mathcal{M}, \tau)}.
\]

(iii) There exists a positive constant \( c(E) > 0 \) which depends only on \( E \) such that

\[
\left\| \sum_{1 \leq m < n \leq N} p_m x_{p_n} \right\|_{E(\mathcal{M}, \tau)} \leq c(E) \|x\|_{E(\mathcal{M}, \tau)}
\]

for all \( x \in E(\mathcal{M}, \tau) \), for every semifinite von Neumann algebra \( (\mathcal{M}, \tau) \), for all finite sequences \( p_1, p_2, ..., p_N \) of mutually orthogonal projections in \( \mathcal{M} \).

**Proof.** (i) \(\Rightarrow\) (ii). By Theorem 4.1 and \([Zs, \text{Lemma 3.11}]\), there exist \( x_0 \in E(\mathcal{M}, \tau)^{L^E}_{\{0\}} \), \( x_1 \in E(\mathcal{M}, \tau)^{L^E}_{\Sigma \setminus \{0\}} \), \( x_2 \in E(\mathcal{M}, \tau)^{L^E}_{(-\Sigma \setminus \{0\})} \) such that

\[
\mathfrak{A}x = x_0 + x_1 + x_2.
\]

It is clear that

\[
E(\mathcal{M}, \tau)^{L^E}_{\{0\}} \subseteq K^{L^E}_{\{0\}}, \quad E(\mathcal{M}, \tau)^{L^E}_{\Sigma \setminus \{0\}} \subseteq K^{L^E}_{\Sigma \setminus \{0\}},
\]

\[
E(\mathcal{M}, \tau)^{L^E}_{(-\Sigma \setminus \{0\})} \subseteq K^{L^E}_{(-\Sigma \setminus \{0\})}.
\]

Since \( x \in K^{L^E}_{\Sigma \setminus \{0\}} \), it follows from Lemma 4.3 (ii) that \( x^* \in K^{L^E}_{(-\Sigma \setminus \{0\})} \).

Observing that

\[
\mathfrak{A}x - (x_1 + x_2) = x_0,
\]

it follows from Lemma 4.3 (i) that \( x_0 = 0 \). We obtain

\[
\frac{1}{2i} x - x_1 = \frac{1}{2i} x^* + x_2,
\]
and it follows again from Lemma 4.3 (i) that
\[ x = 2t(x_1 = 2iP_{\Sigma\setminus\{0\}}^x) \cap x. \]
An appeal to Theorem 4.1 now suffices to complete the proof of the implication.

(ii) \( \implies \) (iii). Without loss of generality, we may assume that \( p_1 + p_2 + \cdots + p_N = 1 \). For \( t \in \mathbb{T} \), define the unitary operator \( U_t \in \mathcal{H} \) by setting
\[ U_t = \sum_{m=1}^{N} e^{\alpha_{mt}} p_m, \]
and the isometry
\[ U_t^*: E(\mathcal{H}, \tau) \rightarrow E(\mathcal{H}, \tau) \]
by setting
\[ U_t^*(x) = u_t^* x u_t, \quad x \in E(\mathcal{H}, \tau). \]
It is clear that the mapping \( t \mapsto U_t^* \) is a strongly continuous representation of \( \mathbb{T} \) on \( E(\mathcal{H}, \tau) \) whose restriction to \( \mathcal{H} \) is a \( \mathbb{T} \)-flow. We take \( \Sigma \) to be the natural cone for the usual ordering of the dual group \( \mathbb{Z} \). It is simple to verify that
\[ E(\mathcal{H}, \tau)^U_{\Sigma\setminus\{0\}} = \left\{ \sum_{1 \leq n \leq m \leq N} p_m x p_n; \ x \in E(\mathcal{H}, \tau) \right\}. \]
Suppose now that \( x \in E(\mathcal{H}, \tau) \) and assume without loss of generality that \( x = x^* \). Applying the assertion of (ii), we obtain that
\[ \left\| \sum_{1 \leq n \leq m \leq N} p_m x p_n - \sum_{1 \leq m \leq n \leq N} p_m x p_n \right\|_{E(\mathcal{H}, \tau)} \leq c(E) \left\| \sum_{1 \leq n \leq m \leq N} p_m x p_n + \sum_{1 \leq m \leq n \leq N} p_m x p_n \right\|_{E(\mathcal{H}, \tau)}. \]
Observing that
\[ \sum_{1 \leq n \leq m \leq N} p_m x p_n \]
\[ = \frac{1}{2} \left( x + \sum_{1 \leq n \leq m \leq N} p_m x p_n - \sum_{1 \leq m \leq n \leq N} p_m x p_n - \sum_{n=1}^{N} p_n x p_n \right). \]
and using the fact that
\[ \sum_{n=1}^{N} p_n x p_n \|_{E(\mathfrak{A}, \tau)} \leq \|x\|_{E(\mathfrak{A}, \tau)}, \]
we obtain that
\[ \sum_{1 \leq n < m \leq N} p_m x p_n \|_{E(\mathfrak{A}, \tau)} \leq (1 + \frac{1}{2} c(E)) \|x\|_{E(\mathfrak{A}, \tau)}, \]
and this suffices to prove the implication (ii) \(\Rightarrow\) (iii).

(iii) \(\Rightarrow\) (i). This is just the assertion of Theorem 3.3 (iii) \(\Rightarrow\) (i).

If \(\mathcal{H}\) is a separable Hilbert space and if \(x \in \mathcal{L}(\mathcal{H})\) is a compact quasi-nilpotent operator, then it is shown in [Zs, Theorem 5.5 and Corollary 5.6] (see also [BGM3, ABG]) that there exists a uniformly continuous one-parameter group \(\{U_t\}_{t \in \mathbb{R}}\) of unitaries on \(\mathcal{H}\) such that if \(\{U_t\}_{t \in \mathbb{R}}\) denotes the \(\mathbb{R}\)-flow on \(\mathcal{L}(\mathcal{H})\) defined by setting
\[ U_t(y) = u_t^* y u_{-t}, \quad y \in \mathcal{L}(\mathcal{H}) \]
then \(x \in K^{\Sigma x}_{1 \setminus \{0\}}\), where \(\Sigma\) is the natural cone of non-negative reals. The classical Macaev theorem now follows directly from the implication (i) \(\Rightarrow\) (ii). On the other hand, the implication (i) \(\Rightarrow\) (ii) also contains as a special case the harmonic conjugation theorem of M. Riesz, as pointed out in [BGM3]. In fact, if \(G\) is the circle group \(\mathbb{T}\) and the flow \(U\) is given by right translation on the abelian von Neumann algebra \(L^\infty(\mathbb{T}, dm)\), we obtain immediately that if \(1 < p < \infty\), then there exists a constant \(c(p) > 0\), depending only on \(p\) such that if \(\phi\) is an analytic trigonometric polynomial then
\[ \|R\phi\|_{L^p(\mathbb{T}, dm)} \leq c(p) \|P\phi\|_{L^p(\mathbb{T}, dm)}. \]
The equivalence (i) \(\Leftrightarrow\) (iii) contains the well-known assertion that the operator of triangular truncation is bounded on the Schatten \(p\)-classes \(\mathcal{C}_p\), if and only if \(1 < p < \infty\) and is due to Arazy [Ar] in the more general setting of trace ideals. For the case that \(G\) is a compact connected group, the implication (i) \(\Rightarrow\) (iii) above is (essentially) due to Ferleger and Sukochev [FS]. We note, however, that in [FS], the assumption that \(x \in K^{\Sigma x}_{1 \setminus \{0\}}\) is replaced by the assumption that \(x \in \mathcal{H}_{1 \setminus \{0\}}^{\Sigma x}\).
5. SHELL DECOMPOSITIONS IN SYMMETRIC SPACES

Let $E$ be a symmetric Banach function space in $\mathbb{R}^+$, and suppose that \( \{q_n\} \) is a sequence of orthogonal projections in $\mathcal{M}$ such that $q_n \uparrow 1$, where for convenience, we set $q_0 = 0$. For $n = 1, 2, \ldots$, we define the projections $Q_n$ on $\mathcal{M}$ by setting

$$Q_n x = q_n x q_n - q_{n-1} x q_{n-1}, \quad x \in E(\mathcal{M}, \tau).$$

Observe that $Q_n Q_m = Q_m Q_n = 0$, whenever $m \neq n$ and that $\sum_{n=1}^\infty Q_n x = q_n x q_n$ for all $x \in \mathcal{M}$. If the sequence \( \{Q_n\} \) determines a Schauder decomposition of $E(\mathcal{M}, \tau)$, that is, if $x = \sum_{n=1}^\infty Q_n x$ with convergence in the norm topology for all $x \in E(\mathcal{M}, \tau)$, then we shall refer to this decomposition as the shell decomposition of $E(\mathcal{M}, \tau)$ determined by the sequence $\{q_n\}$. We remark that if $E$ is a separable symmetric Banach function space on $\mathbb{R}^+$, then any sequence $\{q_n\}$ of orthogonal projections in $\mathcal{M}$ such that $q_n \uparrow 1$ determines a shell decomposition of $E(\mathcal{M}, \tau)$, since separability of $E$ (and therefore order continuity of the norm on $E(\mathcal{M}, \tau)$) implies $\|x - q_n x q_n\|_{E(\mathcal{M}, \tau)} \to 0$ for all $x \in E(\mathcal{M}, \tau)$.

Suppose that $q_n \uparrow 1$ in $\mathcal{M}$ determines a shell decomposition in the space $E(\mathcal{M}, \tau)$. This shell decomposition will be called unconditional if the series $\sum_{n=1}^\infty Q_n x$ is unconditionally convergent for every $x \in E(\mathcal{M}, \tau)$. The main result of this section is the following theorem.

**Theorem 5.1.** Let $E$ be a symmetric Banach function space on $\mathbb{R}^+$ with Fatou norm. The following statements are equivalent.

(i) $E$ has non-trivial Boyd indices.

(ii) There exists a constant $K(E) > 0$ which depends only on $E$ such that

$$\left\| \sum_{n=1}^N e_n (e_n x e_n - e_{n-1} x e_{n-1}) \right\|_{E(\mathcal{M}, \tau)} \leq K(E) \|x\|_{E(\mathcal{M}, \tau)}$$

for all semifinite von Neumann algebras $(\mathcal{M}, \tau)$, for all $x \in E(\mathcal{M}, \tau)$, for all finite sequences of projections $\{e_n\}_{n=0}^N$ such that $0 = e_0 \leq e_1 \leq \cdots \leq e_N \leq 1$ and all choices of signs $e_n = \pm 1$, $1 \leq n \leq N$.

(iii) For all semifinite von Neumann algebras $(\mathcal{M}, \tau)$, every shell decomposition of $E(\mathcal{M}, \tau)$ is unconditional.

**Proof.** (i) $\Rightarrow$ (ii). We observe that it suffices to prove the implication for the case that $E = L^p(\mathbb{R}^+, dm)$, $1 < p < \infty$. The stated result then follows by
applying the Boyd interpolation theorem, in combination with [DDP2, Theorem 3.4], to the operators

\[ x \rightarrow \sum_{n=1}^{N} \varepsilon_n(x_n x_{n-1} - e_{n-1} x_{n-2}), \quad x \in \mathcal{H}(\tau). \]

We set \( p_n = e_n - e_{n-1}, \, 1 \leq n \leq N \) and define the projection \( P \) on \( L^p(\mathcal{H}, \tau) \) \((1 < p < \infty)\) by setting

\[ P x = \sum_{1 \leq n \leq m \leq N} p_n x_p, \quad x \in L^p(\mathcal{H}, \tau). \]

Again using the fact that

\[ \left\| \sum_{n=1}^{N} p_n x_p \right\|_p \leq \|x\|_p, \]

it follows from Theorem 3.3(iii) that there exists a constant \( C_p \) depending only on \( p \) such that \( \|P\| \leq C_p \). If \( \varepsilon_n = \pm 1, \, 1 \leq n \leq N \) is an arbitrary choice of signs, then

\[
\begin{align*}
\sum_{n=1}^{N} \varepsilon_n(e_n x_n - e_{n-1} x_{n-2}) & = \sum_{n=2}^{N} \varepsilon_n \left[ \sum_{i=1}^{n} p_n x_{p_i} + \sum_{k=1}^{n-1} p_k x_{p_n} \right] + \varepsilon_1 p_1 x_{p_1} \\
& = \sum_{n=1}^{N} \varepsilon_n p_n \left( \sum_{1 \leq i < k \leq N} p_k x_{p_i} \right) + \sum_{n=2}^{N} \varepsilon_n \left( \sum_{1 \leq k < j \leq N} p_k x_{p_j} \right) p_n \\
& = \left( \sum_{n=1}^{N} \varepsilon_n p_n \right) P x + (x - P x) \left( \sum_{n=2}^{N} \varepsilon_n p_n \right).
\end{align*}
\]

Since \( \|\sum_{n=1}^{N} \varepsilon_n p_n \|_\infty < 1 \), for \( i = 1, 2 \), we conclude that

\[
\left\| \sum_{n=1}^{N} \varepsilon_n(e_n x_n - e_{n-1} x_{n-2}) \right\|_p \leq \|P x\|_p + \|x - P x\|_p \\
\leq (2C_p + 1) \|x\|_p,
\]

and this proves the implication \( (i) \Rightarrow (ii) \).
(ii) ⇒ (iii). Suppose that $q_n \uparrow 1$ in $\mathcal{M}$ determines a shell decomposition in the space $E(\mathcal{M}, \tau)$. Let $N \in \mathbb{N}$, $x \in E(\mathcal{M}, \tau)$ be given and set $e_0 = q_0 = 0$, $e_j = q_j$, $1 \leq j \leq N$. Applying the assertion of (ii) to the element $\sum_{n=1}^{N} Q_n x$ now yields the estimate

$$\left\| \sum_{n=1}^{N} e_n Q_n x \right\|_{E(\mathcal{M}, \tau)} \leq K(E) \left\| \sum_{n=1}^{N} Q_n x \right\|_{E(\mathcal{M}, \tau)}.$$ 

Since the series $\sum_{n=1}^{\infty} Q_n x$ is norm convergent in $E(\mathcal{M}, \tau)$ for all $x \in E(\mathcal{M}, \tau)$, the preceding inequality implies that the series $\sum_{n=1}^{\infty} e_n Q_n x$ is norm convergent as well for all choices of signs $e_n = \pm 1$ and all $x \in E(\mathcal{M}, \tau)$.

(iii) ⇒ (i). The proof of this implication follows the lines of that of the implication (iii) ⇒ (i) of Theorem 3.4. We suppose first that $\mathcal{A} \subseteq \mathcal{M}$ be $L^\infty(\mathbb{R}^+) \otimes \mathcal{L}(F)$ with trace $\tau$ given by $\lambda \otimes \text{tr}$, where $\lambda$ denotes the trace on $L^\infty(\mathbb{R}^+)$ induced by Lebesgue measure on $\mathbb{R}^+$ and $\text{tr}$ denotes the canonical trace on $\mathcal{L}(F)$. For each $n \geq 1$, let $p_n$ denote the projection onto the first $n$ coordinates of $F$. We set $e_0 = 0$ and $e_n = Z_{(0,n)} \otimes p_n$ for all $n \geq 1$.

By assumption (iii), it follows that there exists a constant $K = K(E, \mathcal{M}, \{e_n\})$ such that

$$\left\| \sum_{n=1}^{N} e_n (z \otimes A) e_n - e_{n-1} (z \otimes A) e_{n-1} \right\|_{E(\mathcal{M}, \tau)} \leq K \|z \otimes A\|_{E(\mathcal{M}, \tau)}$$

for all $N \geq 1$, all choices of signs $e_n = \pm 1$, $1 \leq n \leq N$, for all $z \in E$ and for all (infinite) matrices $A$ with at most finitely many non-zero entries. Using the fact that the natural shell decomposition (with respect to some fixed basis) is not unconditional in the Schatten class $\mathcal{C}_1$ (see, for example [Ar, Lemma 4.5]) there exists a natural number $N$, a choice $\{e_i\}_{i=1}^{N}$ of signs and an $N \times N$-matrix $A = \{a_{ij}\}_{i,j=1}^{N}$ such that if $B = \{b_{ij}\}_{i,j=1}^{N}$ is the $N \times N$-matrix defined by setting

$$b_{ij} = e_{\max(i,j)} a_{ij}, \quad 1 \leq i, j \leq N,$$

then

$$\|A\|_1 = 1, \quad \|B\|_1 \geq 2K.$$

By [LT, 2.b.6], there exist non-negative, disjointly supported, equidistributed functions $\{f_i\}_{i=1}^{N}$ with $\|f_i\|_E = 1$, $1 \leq i \leq N$ and

$$\frac{1}{N} \sum_{i=1}^{N} |a_i| \leq \left\| \sum_{i=1}^{N} a_i f_i \right\|_E.$$
We observe that
\[ \sum_{i=1}^{n} e_i(f_i \otimes A) e_i - e_{i-1}(f_i \otimes A) e_{i-1} = f_i \otimes B. \]

The assumption (iii) implies that \( \|f_i \otimes B\|_{E, \alpha, \tau} \leq K \). On the other hand, the definition of the matrices \( A, B \) implies that \( \|f_i \otimes B\|_{E, \alpha, \tau} \geq 4/3K \), and this is a contradiction. The case that \( \sum K = 0 \) is similar and we omit the details. This completes the proof of the theorem. [Ar] 

The preceding theorem in the setting of trace ideals is due to Arazy [Ar].

REFERENCES


LIPSCHITZ CONTINUITY


