We study Hessian K3 surfaces of non-Sylvester form. They are obtained as toric hypersurfaces, and their periods satisfy the Lauricella’s hypergeometric differential equation $F_C$. The period domain is the Siegel upper half-space of degree 2. We construct modular forms on it using results of Ibukiyama.

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of classical invariants $I_n$ that are given as polynomials of $q_i$. On the other hand, Hessian K3 surfaces $H_{ns1}(a)$ of $S_{ns1}(a)$ are classified by the period domain which is isomorphic to the Siegel upper half-space $\mathcal{S}_2$ of degree 2. The main result of this paper is Theorem 4.5, where we give an isomorphism

$$\widetilde{\mathcal{S}_2}/\mathcal{P}^*_0(2)_2 \cong \mathbb{P}(1, 2, 3, 4)$$

with explicit correspondence between invariants $I_n$ and modular forms on $\mathcal{S}_2$ constructed by theta constants. This result will be a foothold to construct modular forms associated general Hessian K3 surfaces by theta constants.

The family $\{H_{ns1}(a)\}$ is obtained also as toric hypersurfaces, and it is a mirror partner $[C1,D1]$ of anti-canonical classes of $(\mathbb{P}^1)^3$. Since explicit examples of the period and mirror maps for 3-parameter families are not so known, this family is interesting in the viewpoint of mirror symmetry of K3 surfaces. By explicit calculation of the Poincaré residue as in $[PS]$, we show that periods of $H_{ns1}(a)$ satisfy Lauricella’s hypergeometric differential equation $\mathcal{F}_C(1, \frac{1}{2}; 1, 1, 1)$. According to the Deligne–Mostow–Terada’s theory, Lauricella’s $\mathcal{F}_D$ for certain parameters give period maps to complex ball, but uniformizations related to $\mathcal{F}_C$ are not studied as far as author knows.

1. **Hessian K3 surfaces of non-Sylvester type**

1.1. It is classically known that the ring of $SL_4(\mathbb{C})$-invariants of quaternary cubic forms is

$$\mathbb{C}[I_8, I_{16}, I_{24}, I_{32}, I_{40}, I_{100}] \quad (\deg I_n = n)$$

where $I_8, \ldots, I_{40}$ are algebraically independent and $I_{100}^2 \in \mathbb{C}[I_8, I_{16}, I_{24}, I_{32}, I_{40}]$ $[H, Sa]$. Hence the moduli space of cubic surfaces $\mathcal{M}_I$ is isomorphic to the weighted projective space

$$\text{Proj } \mathbb{C}[I_8, I_{16}, I_{24}, I_{32}, I_{40}] = \mathbb{P}(1, 2, 3, 4, 5)_I.$$  

A general cubic surface is written as a complete intersection

$$S_\lambda: \quad X_0 + \cdots + X_4 = 0, \quad \lambda_0 X_0^3 + \cdots + \lambda_4 X_4^3 = 0$$

in $\mathbb{P}^4$ with $\lambda_0, \ldots, \lambda_4 \neq 0$, which is called the Sylvester form. Let $\sigma_i$ be the $i$-th elementary symmetric polynomial in $\lambda_0, \ldots, \lambda_4$. They give invariants of $S_\lambda$, and we have

$$I_8 = \sigma_4^2 - 4\sigma_3\sigma_5, \quad I_{16} = \sigma_3^2\sigma_1, \quad I_{24} = \sigma_4^4\sigma_4, \quad I_{32} = \sigma_6^4\sigma_2, \quad I_{40} = \sigma_5^8.$$  

This correspondence gives a birational map

$$\mathbb{P}(1, 2, 3, 4, 5)_I \longrightarrow \mathbb{P}(1, 2, 3, 4, 5)_I$$  

with the base locus $\sigma_5 = \sigma_4 = 0$.

Next we recall definition of Hessian K3 surface. Let $X$ be a hypersurface in $\mathbb{P}^n$ given by $F(t_0, \ldots, t_n) = 0$. The Hessian hypersurface $H(X)$ of $X$ is defined by $\det(\partial^2 F/\partial t_i \partial t_j) = 0$. If $X$ is a cubic surface, then $H(X)$ is a quartic surface which is classically known as a symmetroid, and the minimal desingularization of $H(X)$ is a K3 surface. The Hessian of $S_\lambda$ is given by

$$H_\lambda: \quad X_0 + \cdots + X_4 = 0, \quad \frac{1}{\lambda_0} X_0^3 + \cdots + \frac{1}{\lambda_4} X_4^3 = 0.$$  

The Picard lattice of the desingularization of a general $H_\lambda$ is $U \oplus U(2) \oplus A_2(2)$ (see $[DK]$).
1.2. Dardanelli–van Geemen’s stratification

The following facts on $\mathcal{M}$ were proved in [DvG].

(I) The subvariety of $\mathcal{M}$ parametrizing cubic surfaces which do not admit a Sylvester form is defined by $I_{24} = 0$. In general, such surfaces are given by

$$S_{ns1}(a) = X_1^3 + X_2^3 + X_3^3 - X_0^2(a_0 X_0 + 3a_1 X_1 + 3a_2 X_2 + 3a_3 X_3) = 0.$$ 

If we denote the $i$-th elementary symmetric polynomial in $a_1^2, a_2^2, a_3^2$ by $\rho_i$, then we have

$$[S_{ns1}(a)] = [-4 \rho_1 + a_0^2 : \rho_2 : 2 \rho_3 : \rho_4 : 0] \in \mathbb{P}(1, 2, 3, 4, 5)_1.$$ 

The Hessian of $S_{ns1}(a)$ is given by

$$H_{ns1}(a) = X_0 X_1 X_2 X_3 \left( a_1^2 X_0 + a_2^2 X_2 + a_3^2 X_3 + a_0 + a_1^2 X_0^2 + a_2^2 X_0 X_1 + a_3^2 X_0 X_2 + a_3^2 X_0 X_3 \right) = 0$$

and the transcendental lattice of the desingularization of a general $H_{ns1}(a)$ is $T_{ns1} = \mathbb{U} \oplus \mathbb{U}(2) \oplus (-4)$. In affine coordinates $[X_0 : X_1 : X_2 : X_3] = [1 : x/a_1 : y/a_2 : z/a_3]$, the equation of $H_{ns1}(a)$ is

$$xyz \left( x + y + z + a_0 + a_1^2 \frac{1}{x} + a_2^2 \frac{1}{y} + a_3^2 \frac{1}{z} \right) = 0.$$ 

(II) The subvariety of $\mathcal{M}$ parametrizing cyclic cubic surfaces

$$S_{ns2}(b) = X_1^3 + X_2^3 + 2b_0 X_3^3 - 3X_3(b_1 X_1 + x_2 X_3 + X_0^2) = 0$$

is defined by $I_{24} = I_{40} = 0$, and we have

$$[S_{ns2}(b)] = [-8b_0 : 1 + b_1^2 : 0 : b_2^3 : 0] \in \mathbb{P}(1, 2, 3, 4, 5)_1.$$ 

The Hessian of $S_{ns2}(b)$ is given by

$$H_{ns2}(b) = X_1 X_2 X_3(-2b_0 X_3 + b_1 X_1 + X_2) + X_3^3(X_1 + b_1^2 X_2) - X_0^2 X_1 X_2 = 0$$

and the transcendental lattice of the desingularization of a general $H_{ns2}(b)$ is $T_{ns2} = \mathbb{U} \oplus \mathbb{U}(2)$.

(III) The subvariety of $\mathcal{M}$ parametrizing “cyclic cubic surfaces”

$$S_{cyc}(a) = a_4 X_4^3 - a_3(X_0 + X_1 + X_2)^3 + a_0 X_0^3 + a_1 X_1^3 + a_2 X_2^3 = 0$$

is defined by $I_{24} = I_{32} = I_{40} = 0$, and we have

$$[S_{cyc}(a)] \in [\mu_3^2 - 4 \mu_2 \mu_4 : \mu_4^3 : 0 : 0 : 0] \in \mathbb{P}(1, 2, 3, 4, 5)_1$$

where $\mu_i$ is the $i$-th symmetric polynomial of $a_0, \ldots, a_3$. The Hessian of $S_{cyc}(a)$ is reducible.

(IV) The strictly semi-stable surface $t^3 = xyz$ corresponds to the point

$$[8 : 1 : 0 : 0 : 0],$$

and the Fermat cubic surface corresponds to the point

$$[1 : 0 : 0 : 0 : 0].$$
1.3. Batyrev’s mirror construction

Hessian surfaces \( \{ H_{ns} \} \) are obtained also as toric hypersurfaces. Let \( \Delta \) be the octahedron in \( \mathbb{R}^3 \) with vertices

\[
(\pm 1, 0, 0), \quad (0, \pm 1, 0), \quad (0, 0, \pm 1).
\]

It is a simplicial reflexive polytope, and its dual polytope \( \Delta^* \) is the cube with vertices \((\pm 1, \pm 1, \pm 1)\). Considering faces of \( \Delta \) as simplicial cones, we obtain a toric variety \( X(\Delta) = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \). The linear system of anti-canonical classes of \( X(\Delta) \) (that is, K3 surfaces of degree \((2, 2, 2)\) in \( (\mathbb{P}^1)^3 \)) is given by

\[
F(\Delta^*) = \left\{ \sum a_{ijk} x^i y^j z^k = 0 \mid (i, j, k) \in \Delta^* \cap \mathbb{Z}^3 \right\} \quad \left( (x, y, z) \in (\mathbb{C}^\times)^3 \subset (\mathbb{P}^1)^3 \right).
\]

Similarly, we have the dual family of K3 surfaces

\[
F(\Delta) = \left\{ c_1 x + c_2 y + c_3 z + c_4 + c_5 \frac{1}{x} + c_6 \frac{1}{y} + c_7 \frac{1}{z} = 0 \right\}
\]

as hypersurfaces of \( X(\Delta^*) \). It is obvious that this family is birationally equivalent to the family \( \{ H_{ns} \} \). Note that the Picard lattice of a general member of \( F(\Delta^*) \) is

\[
P = \begin{bmatrix}
0 & 2 & 2 \\
2 & 0 & 2 \\
2 & 2 & 0
\end{bmatrix} \cong \mathbb{U}(2) \oplus (-4),
\]

and we have \( T_{ns} = \mathbb{U} \oplus P \). Hence \( F(\Delta) \) is the mirror partner of \( F(\Delta^*) \) (see [B,C1,D1] and [GN]). Note also that \( F(\Delta) \) is a subfamily of \( F(\Delta^*) \). In the following, we regard \( H_{ns} \) as hypersurfaces in \( (\mathbb{P}^1)^3 \), and we replace coefficients \( a_0, a_1, a_2, a_3 \) of \( H_{ns} \) by \( u_1, u_2, u_3 \):

\[
H(u): \quad f_u = xyz(x + y + z + 1) + (u_1 y z + u_2 x z + u_3 x y) = 0 \quad (x, y, z) \in (\mathbb{P}^1)^3.
\]

1.4. Remark. From the 1-parameter family

\[
H_{PS}(u): \quad xyz(x + y + z + 1) + u(xy + yz + zx) = 0,
\]

by the base change \( u = (t + t^{-1})^{-2} \), we obtain the family

\[
x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + t + \frac{1}{t} = 0
\]

studied by Peters and Stienstra in [PS]. They studied the Picard–Fuchs equation and modular forms. The transcendental lattice of a general member is \( \mathbb{U} \oplus (12) \). This K3-fibration is considered as a (singular) Calabi–Yau hypersurface in \( (\mathbb{P}^1)^4 \) (see [V]).
1.5. Singularities

Let us assume \( u_1u_2u_3 \neq 0 \). Then \( H(u) \cap (\mathbb{C}^\times)^3 \) is smooth if and only if
\[
\Delta_{\text{sing}}(u) = \prod (1 \pm 2\sqrt{u_1} \pm 2\sqrt{u_2} \pm 2\sqrt{u_3}) \neq 0.
\]

Therefore we define the parameter space
\[
\mathcal{U} = \{ u = (u_1, u_2, u_3) \mid u_1u_2u_3\Delta_{\text{sing}}(u) \neq 0 \}.
\]

For any \( u \in \mathcal{U} \), we see that \( H(u) \cap ((\mathbb{P}^1)^3 - (\mathbb{C}^\times)^3) \) is decomposed into twelve lines
\[
L_{x00} = \mathbb{P}^1 \times \{0\} \times \{0\}, \ldots, L_{\infty\infty\infty} = \{\infty\} \times \{\infty\} \times \mathbb{P}^1.
\]

They intersect at eight points
\[
(0, 0, 0), \quad (0, 0, \infty), \quad (0, \infty, 0), \quad (0, \infty, \infty),
\]
\[
(\infty, 0, 0), \quad (\infty, 0, \infty), \quad (\infty, \infty, 0), \quad (\infty, \infty, \infty),
\]

that are singular points of \( H(u) \), and all of them are \( A_1 \)-singularities. Blowing up eight singular points of \( H(u) \), we obtain a K3 surface \( \tilde{H}(u) \). Let \( N_u \subset H^2(\tilde{H}(u), \mathbb{Z}) \) be a sublattice generated by twelve lines \( L_{x00}, \ldots, L_{\infty\infty\infty} \) and eight exceptional curves \( E_{000}, \ldots, E_{\infty\infty\infty} \) that are blown down to \( (0, 0, 0), \ldots, (\infty\infty\infty) \).

1.6. Proposition.

(1) For a general \( u \in \mathcal{U} \), the lattice \( N_u \) is the Picard lattice \( \text{Pic}(\tilde{H}(u)) \).

(2) We have three involutions
\[
\epsilon_x : (x, y, z) \mapsto \left( \frac{u_1}{x}, y, z \right), \quad \epsilon_y : (x, y, z) \mapsto \left( x, \frac{u_2}{y}, z \right), \quad \epsilon_z : (x, y, z) \mapsto \left( x, y, \frac{u_3}{z} \right)
\]
on \( \tilde{H}(u) \), and the product \( \epsilon = \epsilon_x\epsilon_y\epsilon_z \) is an Enriques involution.

(3) Let \( N_u^+ \subset N \otimes \mathbb{Q} \) be the dual lattice of \( N_u \), and \( q_N : N_u^+ / N_u \to \mathbb{Q} / 2\mathbb{Z} \) be the discriminant form \( [N] \). Then we have \( \epsilon = \epsilon_x = \epsilon_y = \epsilon_z \) as elements of the finite orthogonal group \( O(q_N) \). Moreover, we have \( O(q_N) = S_3 \times \langle \epsilon \rangle \), where \( S_3 \) is realized as symmetry of \((x, y, z)\).

Proof. (1) The self-intersection numbers of \( L_{***} \) and \( E_{***} \) are \(-2\), and we have \( E_{abc} \cdot L_{stu} = 1 \) if two of three equalities \( a = s, b = t \) or \( c = u \) hold. Other intersection numbers are zero. Using a computer, we can show that the rank of the intersection matrix of them is 17. In fact, we have equalities
\[
E_{000} = E_{000} + E_{000} + 3E_{000} - 3E_{000} - E_{000} - E_{000} + E_{000} - 2L_{x00} + 2L_{x00} + 2L_{0y0} - 2L_{0y0} + 2L_{0y0} - 2L_{0y0},
\]
\[
L_{\infty\infty\infty} = 2E_{000} + 2E_{000} - 2E_{000} - 2E_{000} - L_{x00} - L_{x00} + L_{x00} + L_{x00} + L_{x00} + 2L_{0y0} + 2L_{0y0} - 2L_{0y0},
\]
\[
L_{\infty\infty\infty} = 2E_{000} + 2E_{000} - 2E_{000} - 2E_{000} - L_{x00} + L_{x00} - L_{x00} + L_{x00} + L_{x00} + L_{x00} + 2L_{0y0} - 2L_{0y0},
\]
\[
+ L_{x00} + 2L_{0y0} + L_{0y0} - L_{x00} + 2L_{0y0} - 2L_{0y0}.
\]
as elements of $N_u$. Therefore $E_{000}$, $L_{\infty y\infty}$ and $L_{\infty z\infty}$ are redundant. Since the determinant of the intersection matrix of other 17 curves is 16, we see that they span the orthogonal complement of $T_{ns1} = U \oplus U(2) \oplus (-4)$.

(2) As an involution of $(\mathbb{P}^1)^3$, fixed points of $\epsilon$ are $(\pm \sqrt{\nu_1}, \pm \sqrt{\nu_2}, \pm \sqrt{\nu_3})$. If $u \in \mathcal{U}$, then such points are not on $H(u)$.

(3) We have $N_u/N_u \cong T^*_u/T_{ns1} \cong (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/4\mathbb{Z})$, and it is generated by

$$
\ell_1 = \frac{1}{2}(L_{0y0} + L_{0y\infty} + L_{00z} + L_{0\infty z}),
$$

$$
\ell_2 = \frac{1}{2}(L_{x00} + L_{x0\infty} + L_{00z} + L_{\infty 0z}),
$$

$$
m = \frac{1}{4}(2E_{0\infty\infty} + 2E_{\infty 0\infty} + 2E_{\infty\infty 0} + 2E_{\infty\infty\infty} + 2L_{x00}
+ 3L_{x0\infty} + 3L_{x\infty 0} + 2L_{0y0} + L_{0y\infty} + L_{\infty 0y} + 3L_{00z} + L_{\infty 0z}).
$$

By machine computation, we see that

$$
\epsilon_x(\ell_i) = \ell_i \quad (i = 1, 2), \quad \epsilon_x(m) = -m,
$$

and the same for $\epsilon_y$ and $\epsilon_z$. The 2-torsion subgroup of $N^*_u/N_u$ is generated by $\ell_1$, $\ell_2$ and $\ell_3 = 2m + \ell_1 + \ell_2$, and these are all of elements $x \in N^*_u/N_u$ of order 2 such that $q_N(x) = 0$. We have a split exact sequence

$$
1 \longrightarrow \langle \epsilon \rangle \longrightarrow O(q_N) \longrightarrow \text{permutations of } \ell_1, \ell_2, \ell_3 \longrightarrow 1
$$

since permutations of $(x, y, z)$ give permutations of $\ell_i$'s. □

2. The period mapping and modular groups

2.1. The period mapping

The period domain of the family $[\tilde{H}(u)]_{u \in \mathcal{U}}$ is the bounded symmetric domain

$$
\mathbb{D}_{ns} = \left\{ z \in \mathbb{P}^4 \mid {}^t z Q z = 0, \ {}^t z Q \bar{z} > 0 \right\}, \quad Q = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \oplus [-4]
$$

of type IV defined by the lattice $T_{ns1}$. More explicitly, we have

$$
[1 : z_2 : \cdots : z_5] \in \mathbb{D}_{ns} \iff \begin{cases} 
\bar{z}_2 = -2(z_3 z_4 - z_5^2) \\
y_3 y_4 - y_5^2 > 0 \quad (y_i = \text{Im } z_i)
\end{cases}
$$

and $\mathbb{D}_{ns} = \bigcup_{u \in \mathcal{U}} \mathbb{D}_{ns}^u$ where $\mathbb{D}_{ns}^u = \{ z \in \mathbb{D}_{ns} : \pm y_3 > 0 \}$. Let us define the orthogonal group

$$
O^+_{ns} = \left\{ g \in \text{GL}_5(\mathbb{Z}) \mid {}^t g \, Q \, g = Q, \, g(\mathbb{D}_{ns}^+) = \mathbb{D}_{ns}^+ \right\}
$$

on the lattice $T_{ns1}$, which acts on $\mathbb{D}_{ns}^+$. We define also the discriminant form

$$
q_{ns1} : T^*_u/T_{ns1} \longrightarrow \mathbb{Q}/2\mathbb{Z}
$$
and the orthogonal group $O(q_{ns})$. Let $O_{ns}^+(2)_\epsilon$ be the kernel of the natural homomorphism

$$O_{ns}^+ \longrightarrow O(q_{ns}) \cong O(q_N) \cong S_3 \times \langle \epsilon \rangle,$$

and $O_{ns}^+(2)$ be the kernel of the composition map

$$O_{ns}^+(2)_\epsilon \longrightarrow S_3 \times \langle \epsilon \rangle \longrightarrow S_3.$$

We have $-1 \in O_{ns}^+(2)$ and $-1 \notin O_{ns}^+(2)_\epsilon$. Since $[O_{ns}^+(2)_\epsilon : O_{ns}^+(2)] = 2$, we see that

$$D_{ns}^+ / O_{ns}^+(2)_\epsilon = D_{ns}^+ / O_{ns}^+(2).$$

Let $S_u \subset H_2(\tilde{H}(u), \mathbb{Z})$ be the sublattice generated by $L_{\ast \ast}$'s and $E_{\ast \ast}$'s, that is, the Poincaré dual of $N_u \subset H^2(\tilde{H}(u), \mathbb{Z})$. Taking suitable 2-cycles

$$\gamma_1(u), \ldots, \gamma_5(u) \in (S_u)^{\perp} \cong T_{ns1},$$

that are uniquely determined up to $O_{ns}^+$-action, we can define the period mapping

$$\text{Per} : \mathcal{U} \longrightarrow D_{ns}^+, \quad u = (u_1, u_2, u_3) \mapsto \left[ \int_{\gamma_1(u)} \omega_u : \cdots : \int_{\gamma_5(u)} \omega_u \right]$$

where $\omega_u \in H^{2,0}(\tilde{H}(u))$.

2.2. Proposition. The multi-valued map $\text{Per}$ induces an injective $S_3$-equivariant map $\mathcal{U} \to D_{ns}^+ / O_{ns}^+(2)_\epsilon$ and the map $\mathcal{U} / S_3 \to D_{ns}^+ / O_{ns}^+(2)$ for $S_3$-quotients.

Proof. Note that

(1) the monodromy action of $\pi_1(\mathcal{U}, u)$ on $S_u \subset H_2(\tilde{H}(u), \mathbb{Z})$ is trivial,
(2) we can lift $g \in O_{ns}^+$ to an isometry $\tilde{g}$ on $H_2(\tilde{H}(u), \mathbb{Z})$ such that $\tilde{g} | S_u = \text{id}$ iff $g \in O_{ns}^+(2)_\epsilon$.

From these facts together with Proposition 1.6, we see that the map is injective as the period map of $N_u$-polarized K3-surfaces (see [D1]).

2.3. Proposition. The period map $\text{Per}$ is given by the developing map of the Lauricella's hypergeometric differential equation for

$$F_C \left( 1, \frac{1}{2}; 1, 1, 1; -2u_1, -2u_2, -2u_3 \right)$$

(see [Y]).

Proof. Indeed, we obtain a period of $H(u)$ as follows.

$$I(u_1, u_2, u_3) = \iiint_{[x]=|y|=|z| = \epsilon} \frac{dx \wedge dy \wedge dz}{f_u}$$
Therefore, we obtain

\[
\psi: \mathbb{D}_{\text{ns}}^+ \to \mathfrak{G}_2 = \left\{ \tau \in \text{GL}_2(\mathbb{C}) \mid \text{Im} \tau > 0 \right\}, \quad [1 : z_2 : \cdots : z_5] \mapsto \begin{pmatrix} z_3 & z_5 \\ z_5 & z_4 \end{pmatrix}.
\]

The symplectic group

\[
\text{Sp}_{2g}(\mathbb{R}) = \left\{ g \in \text{GL}_{2g}(\mathbb{R}) \mid {}^t g J g = J \right\}, \quad J = \begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix}
\]
We consider the congruence subgroup \( \mathcal{G}_k \) by \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \tau = (A\tau + B)(C\tau + D)^{-1} \). Let \( \Gamma_g \) be the Siegel modular group \( \text{Sp}_2(\mathbb{R}) \cap \text{GL}_2(\mathbb{Z}) \).
We consider the congruence subgroup
\[
\Gamma_0(2)_g = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma_g \middle| C \equiv 0 \mod 2 \right\}.
\]
and the extension \( \Gamma_0^*(2)_2 \) of \( \Gamma_0(2)_2 \) by a normalizer \( W = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \).

2.5. Proposition. Then we have an isomorphism \( \mathbb{D}_{ns}^+ / \{ \pm 1 \} \cong \Gamma_0^*(2)_2 / \{ \pm 1 \} \) as automorphisms of \( \mathbb{D}_{ns}^+ \cong \mathfrak{S}_2 \).

Proof. This is an easy consequence of Theorem 3.1 in [Ko], and we omit the proof. We give just explicit correspondences of generators:

(1) The map \( g : \text{GL}_2(\mathbb{Z}) \to \mathbb{D}_{ns}^+ \):
\[
\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \mapsto I_2 \oplus \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_1a_3 & a_2a_4 & a_1a_4 + a_2a_3 \end{bmatrix}
\]
is a homomorphism such that \( \text{Ker } g = \{ \pm 1 \} \) and \( \Psi(g(A) \cdot z) = \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} \cdot \Psi(z) \).

(2) Let \( B_2 \) be the additive group of integral symmetric matrices of degree 2. Then the map \( h : B_2 \to \mathbb{D}_{ns}^+ \):
\[
\begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2m_1m_2 + 2m_3^2 & 1 & -2m_1 & -4m_3 \\ m_1 & 0 & 1 & 0 \\ m_2 & 0 & 0 & 1 \\ m_3 & 0 & 0 & 1 \end{bmatrix}
\]
is a homomorphism such that \( \Psi(h(B) \cdot z) = \begin{bmatrix} b_2 & b_3 \\ 0 & b_2 \end{bmatrix} \cdot \Psi(z) \).

(3) For \( w = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 1 \end{bmatrix} \oplus \begin{bmatrix} -1 \end{bmatrix} \in \mathbb{D}_{ns}^+ \), we have \( \Psi(w \cdot z) = -\frac{1}{2} \Psi(z)^{-1} = W \cdot \Psi(z) \).

2.6. Proposition.

(1) If \( x, y \in \mathbb{Z}^5 \) satisfy \( x^TQx = y^TQy = 0 \) and \( x^TQy = 1 \), then there exists a transformation \( \gamma \in \mathbb{D}_{ns}^+ \) such that \( \gamma \cdot x = e_1 \) and \( \gamma \cdot y = e_2 \), where \( e_i \) is the \( i \)-th unit vector.

(2) For any primitive sublattice \( M \cong U \oplus (12) \) of \( T_{ns1} \), there exists \( \gamma \in \mathbb{D}_{ns}^+ \) such that \( \gamma(M) \) is either
\[
M_1 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}(e_3 + 3e_4) \quad \text{or} \quad M_2 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}(2e_3 + 2e_4 + e_5).
\]

For any primitive sublattice \( M' \cong U \oplus U(2) \) of \( T_{ns1} \), there exists \( \gamma' \) \in \( \mathbb{D}_{ns}^+ \) such that
\[
\gamma'(M') = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_4.
\]

(3) We have the following table for periods of special subfamilies:

<table>
<thead>
<tr>
<th>lattice</th>
<th>( \mathbb{D}<em>{ns}^+ / \mathbb{O}</em>{ns}^+ )</th>
<th>( \mathfrak{S}_2 / \Gamma_0^*(2)_2 )</th>
</tr>
</thead>
</table>
| \( H_{ns2}(b) \) | \( U \oplus U(2) \) \[ 1 : z_2 : z_3 : z_4 : 0 \] \[
\begin{bmatrix} z_1 & 0 \\ 0 & z_4 \end{bmatrix}
\] | |
| \( H_{PS}(u) \) | \( U \oplus (12) \) \[ 1 : z_2 : 2z_5 : 2z_5 : z_5 \] \[
\begin{bmatrix} 2z_5 & z_5 \\ z_5 & 2z_5 \end{bmatrix}
\] | |
3.2. Theorem

Let \( \gamma \in \mathbb{O}_{n}^{+} \) and \( x, y, z \in \mathbb{Z} \) such that

\[
\gamma(M) = \mathbb{Z}e_{1} \oplus \mathbb{Z}e_{2} \oplus \mathbb{Z}(xe_{3} + ye_{4} + ze_{5}), \quad xy - z^{2} = 3.
\]

Now the assertion for \( M \) follows from the facts:

(i) the integer solutions of the system of equations

\[
xy = z^{2} + 3, \quad |x| > |z|, \quad |y| > |z|
\]

are \((2, 2, \pm 1)\) or \((-2, -2, \pm 1),\)

(ii) if \(|x| < |z|\) or \(|y| < |z|\), then multiplying

\[
I_{2} \oplus \begin{bmatrix} 1 & 1 & \pm 2 \\ 0 & 1 & 0 \\ \pm 1 & 0 & 1 \end{bmatrix}, \quad I_{2} \oplus \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & \pm 2 \\ 0 & \pm 1 & 1 \end{bmatrix} \in \mathbb{O}_{n}^{+},
\]

we can decrease the value of \(|z|\).

The assertion for \( M' \) is easily shown by the same way.

(3) By (2), periods of \( H_{n2}(b) \) belong to the divisor \(|z_{5} = 0| \in \mathbb{D}_{n5}^{+}. \) Because surfaces \( H_{PS}(u) \) don't belong to the family \( \{H_{n2}(b)\}, \) their periods don't belong to \( \mathbb{P}(M_{1} \otimes \mathbb{C}) \subset \{z_{5} = 0\}. \) Hence periods of \( H_{PS}(u) \) belong to \( \mathbb{P}(M_{2} \otimes \mathbb{C}) \). \( \square \)

3. Graded ring of theta constants

3.1. Let \( \Gamma' \) be a subgroup of \( \text{Sp}_{4}(\mathbb{R}). \) A holomorphic function \( f(\tau) \) on \( \mathbb{H}_{2} \) is a modular form of weight \( k \) with respect to \( \Gamma' \) if it holds

\[
f((A\tau + B)(C\tau + D)^{-1}) = \det(C\tau + D)^{k} f(\tau)
\]

for any \( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \Gamma'. \) Let \( M_{k}(\Gamma') \) be the vector space of such functions, and \( A(\Gamma')_{\text{even}} \) be the graded ring \( \bigoplus_{k=0}^{\infty} M_{2k}(\Gamma'). \) The generators of the graded ring \( A(\Gamma_{0}(2)_{2})_{\text{even}} \) are given by theta constants

\[
\theta_{a,b}(\tau) = \sum_{n \in \mathbb{Z}^{2}} \exp[\pi i^{t} (n + a) \tau (n + a) + 2\pi i^{t} (n + a)b], \quad \tau \in \mathbb{H}_{2}
\]

(see [Ig2]). For simplicity, we denote \( \theta_{a,b} \) by \( \theta_{xyzw} \) if \( a = \tau(x/2, y/2) \) and \( b = \tau(z/2, w/2). \)

3.2. Theorem (Ibukiyama). (See [Ib].) The graded ring \( A(\Gamma_{0}(2)_{2})_{\text{even}} \) is a free algebra \( \mathbb{C}[\vartheta, \phi_{1}, \phi_{2}, \chi], \) where

\[
\vartheta = (\theta_{0000}^{\vartheta} + \theta_{0001}^{\vartheta} + \theta_{0010}^{\vartheta} + \theta_{0011}^{\vartheta})/4, \quad \phi_{1} = (\theta_{00000}^{\phi_{1}}\theta_{00010}^{\phi_{1}}\theta_{00100}^{\phi_{1}}\theta_{00111}^{\phi_{1}})^{2},
\]

\[
\phi_{2} = (\theta_{0100}^{\phi_{2}} - \theta_{0110}^{\phi_{2}})^{2}/16384, \quad \chi = (\theta_{01000}\theta_{01100}\theta_{10000}\theta_{10101}\theta_{11001}\theta_{11111})^{2}/4096
\]

are modular forms of weight 2, 4, 4 and 6. Hence we have

\[
\text{Proj} A(\Gamma_{0}(2)_{2})_{\text{even}} \cong \mathbb{P}(2, 4, 4, 6).
\]
3.3. Lemma. The zero divisor of the function \( \chi(\tau) \) is \( \Gamma_0(2)_2 \)-orbit of
\[
\mathbb{H} \times \mathbb{H} = \left\{ \tau = \begin{bmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{bmatrix} \in \mathbb{C} \mid \tau_2 = 0 \right\}
\]
with multiplicity 1, and \( \chi(\tau) \) is the unique non-trivial function in \( M_6(\Gamma_0(2)_2) \) vanishing there.

Proof. The first assertion is proved by exactly the same way as in [Kl, pp. 116–118]. By the equality of theta constants of one variable \( \theta_{00}^4 = \theta_{01}^4 + \theta_{00}^4 \), we see that
\[
\vartheta(\tau) = \left( \theta_{00}^4(\tau_1) + \theta_{01}^4(\tau_1) \right) \left( \theta_{00}^4(\tau_3) + \theta_{01}^4(\tau_3) \right) / 4,
\]
\[
\phi_1 = \theta_{00}^4(\tau_1) \theta_{01}^4(\tau_1) \theta_{00}^4(\tau_3) \theta_{01}^4(\tau_3),
\]
\[
\phi_2 = \left( \theta_{00}^4(\tau_1) - \theta_{01}^4(\tau_1) \right)^2 \left( \theta_{00}^4(\tau_3) - \theta_{01}^4(\tau_3) \right)^2 / 16384
\]
for \( \tau \in \mathbb{H} \times \mathbb{H} \). Therefore \( \vartheta^3, \vartheta \phi_1, \vartheta \phi_2 \) are linearly independent on \( \mathbb{H} \times \mathbb{H} \). \( \square \)

3.4. Proposition. The involution \( W = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \) acts on \( A(\Gamma_0(2)_2)_{even} \) as follows
\[
\vartheta(W \cdot \tau) = (2 \det \tau)^2 \vartheta(\tau), \quad \phi_1(W \cdot \tau) = 1024(2 \det \tau)^4 \phi_2(\tau),
\]
\[
\phi_2(W \cdot \tau) = (2 \det \tau)^4 \phi_1(\tau)/1024, \quad \chi(W \cdot \tau) = (2 \det \tau)^6 \chi(\tau).
\]
Therefore we have
\[
A(\Gamma_0^*(2)_2)_{even} = \mathbb{C}[\vartheta, \phi, \chi, \psi], \quad \text{Proj} \ A(\Gamma_0^*(2)_2)_{even} \cong \mathbb{P}(2 : 4 : 6 : 8)
\]
where \( \phi = \phi_1 + 1024 \phi_2 \) and \( \psi = \phi_1 \phi_2 \).

Proof. By the following formula [Ig1, p. 408]
\[
\theta_{0000}^2(\tau/2) = \theta_{0000}^2(\tau) + \theta_{1000}^2(\tau) + \theta_{0100}^2(\tau) + \theta_{1100}^2(\tau),
\]
\[
\theta_{0001}^2(\tau/2) = \theta_{0000}^2(\tau) + \theta_{1000}^2(\tau) - \theta_{0100}^2(\tau) - \theta_{1100}^2(\tau),
\]
\[
\theta_{0010}^2(\tau/2) = \theta_{0000}^2(\tau) - \theta_{1000}^2(\tau) + \theta_{0100}^2(\tau) - \theta_{1100}^2(\tau),
\]
\[
\theta_{0011}^2(\tau/2) = \theta_{0000}^2(\tau) - \theta_{1000}^2(\tau) - \theta_{0100}^2(\tau) + \theta_{1100}^2(\tau),
\]
\[
\theta_{0100}^2(\tau/2) = 2(\theta_{0000} \theta_{0100} + \theta_{1000} \theta_{1100})(\tau),
\]
\[
\theta_{0110}^2(\tau/2) = 2(\theta_{0000} \theta_{0100} - \theta_{1000} \theta_{1100})(\tau),
\]
\[
\theta_{1000}^2(\tau/2) = 2(\theta_{0000} \theta_{1000} + \theta_{1000} \theta_{1100})(\tau),
\]
\[
\theta_{1001}^2(\tau/2) = 2(\theta_{0000} \theta_{1000} - \theta_{1000} \theta_{1100})(\tau),
\]
\[
\theta_{1100}^2(\tau/2) = 2(\theta_{0000} \theta_{1100} + \theta_{1000} \theta_{1100})(\tau),
\]
\[
\theta_{1111}^2(\tau/2) = 2(\theta_{0000} \theta_{1100} - \theta_{1000} \theta_{1000})(\tau)
\]
we see that
Applying the inversion formula, we obtain

$$\vartheta(W \cdot \tau) = 4(\det \tau)^2 \vartheta(\tau).$$

By the same way, we can show that

$$\phi_2(-\tau^{-1}/2) = (\det \tau)^4 \phi_1(\tau)/64,$$

and replacing $\tau$ by $-\tau^{-1}/2$, we see that

$$\phi_1(-\tau^{-1}/2) = 16384(\det \tau)^4 \phi_2(\tau).$$

For the modular form $\chi(\tau)$, we have

$$\chi(-\tau^{-1}/2) = \frac{1}{64}(\det \tau)^6 \chi(\tau) = \frac{1}{64}(\det \tau)^4 \vartheta(\tau).$$

Since the right hand side vanishes on $H \times H$, it coincides with $c \chi(\tau)$ for some constant $c$. Comparing Fourier coefficients, we see that $c = 1$. 

### 4. Boundary

#### 4.1. Let us study the extension of the period map $\text{Per}: U \to \mathbb{D}^+_{ns}$ to the locus $\{u_3 = 0\}$. Note that we have

$$F_C\left(1, \frac{1}{2}; 1, 1, 1; -2u_1, -2u_2, 0\right) = F_4\left(1, \frac{1}{2}; 1, 1; -2u_1, -2u_2\right)$$

where $F_4$ is Appell’s hypergeometric series, and we have

$$F_4\left(1, \frac{1}{2}; 1, 1; \frac{-x}{(1-x)(1-y)}, \frac{-y}{(1-x)(1-y)}\right) = (1-x)^{1/2}(1-y)^{1/2} F_1\left(\frac{1}{2}, \frac{1}{2}; 1; xy\right)$$

(see [E]). It is known that Gauss’s hypergeometric series $2F_1(1, \frac{1}{2}; 1; t)$ has an elliptic integral representation. Indeed, the same computation as in Proposition 2.2 shows that $F_4(1, \frac{1}{2}; 1, 1; -2u_1, -2u_2)$ is a period integral of a curve

$$C(u): xy(x + y + 1) + u_1y + u_2x = 0$$

of degree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$. The relation between this family and Appell’s $F_4$ was already studied by Stienstra in [St]. Here we study the relation between the invariants of $C(u)$ and the degeneration of the map

$$DvG: \mathcal{U} \to \mathbb{P}(1, 2, 3, 4), \quad (u_1, u_2, u_3) \mapsto [-4s_1 + 1 : s_2 : 2s_3 : s_1 s_3]$$
defined by invariants of cubic surfaces, where \( s_i \) is the \( i \)-th symmetric polynomial of \( u_1, u_2, u_3 \). For this map, we have

\[
\lim_{u_3 \to 0} [-4s_1 + 1 : s_2 : 2s_3 : s_1s_3] = [1 - 4u_1 - 4u_2 : u_1u_2 : 0 : 0].
\]

On the other hand, the curve \( C(u) \) is birationally equivalent to an elliptic curve

\[
E(u): \quad y^2 = f_u(X) = X^4 + X^3 + \left(-u_2 + \frac{u_1}{2} + \frac{1}{4}\right)X^2 + \frac{u_1}{4}X + \frac{u_1^2}{16}
\]

by the transformation

\[
(x, y) = \left(2X, \frac{4Y - 4X^2 - 2X - u_1}{4X}\right).
\]

**4.2. Lemma.** The classical invariants of the quartic equation \( f_u(X) = 0 \) are

\[
g_2(u) = \frac{1}{192}((1 - 4u_1 - 4u_2)^2 - 48u_1u_2),
\]

\[
g_3(u) = -\frac{1}{13824}((1 - 4u_1 - 4u_2)((1 - 4u_1 - 4u_2)^2 - 72u_1u_2),
\]

\[
\Delta_E(u) = g_2(u)^3 - 27g_3(u)^2 = \frac{1}{4096}u_1^2u_2^2((1 - 4u_1 - 4u_2)^2 - 64u_1u_2).
\]

Therefore \([1 - 4u_1 - 4u_2 : u_1u_2] \in \mathbb{P}(1, 2)\) corresponds to a singular \( E(u) \) iff

\[
[1 - 4u_1 - 4u_2 : u_1u_2] = [1 : 0] \quad \text{or} \quad [8 : 1].
\]

Moreover we have \( \Delta_{\text{sing}}(u_1, u_2, 0) = (4096\Delta_E(u_1, u_2)/u_1^2u_2^2)^2 \).

**Proof.** This is obtained from the definition

\[
g_2 = ae - 4bd + 3c^2, \quad g_3 = \text{det} \begin{bmatrix} a & b & c \\ b & c & d \\ c & d & e \end{bmatrix}
\]

for \( aX^4 + 4bX^3 + 6cX^2 + 4dX + e = 0. \quad \square \)

4.3. Now we can define a degenerated period map

\[
\text{Per}_{12} : U_{12} = \{(u_1, u_2) \in \mathbb{C}^2 \mid \Delta_E(u_1, u_2) \neq 0\} \longrightarrow \mathbb{H},
\]

and construct the inverse map

\[
\mathbb{H} \longrightarrow [1 - 4u_1 - 4u_2 : u_1u_2] \in \mathbb{P}(1, 2)
\]

by the Siegel \( \Phi \)-operator \( \Phi(f)(\tau_1) = \lim_{t \to \infty} f(\begin{bmatrix} \tau_1 & 0 \\ 0 & t \end{bmatrix}) \). Let us define modular forms
of weight 2 and 4 with respect to $\Gamma_0(2)_1$.

4.4. Lemma. Modular forms $h_1$ and $h_2$ satisfy same relations for $1 - 4u_1 - 4u_2$ and $u_1u_2$ in Lemma 4.2. Indeed, we have

$$h_1(\tau) = \Phi(8 \vartheta) = 4(\theta^4_{00} + \theta^4_{01}),$$
$$h_2(\tau) = \Phi(\vartheta^2 - \varphi) = \frac{1}{4}(\theta^4_{00} + \theta^4_{01})^2 - 4\theta^4_{00}\theta^4_{01} = \frac{1}{4}(\theta^4_{00} - \theta^4_{01})^2$$

of weight 2 and 4 with respect to $\Gamma_0(2)_1$.

**Proof.** By the formula

$$\theta^2_{00}(2\tau) = \frac{1}{2}(\theta^2_{00}(\tau) + \theta^2_{01}(\tau)), \quad \theta^2_{01}(2\tau) = \theta_{00}(\tau)\theta_{01}(\tau), \quad \theta^2_{10}(2\tau) = \frac{1}{2}(\theta^2_{00}(\tau) - \theta^2_{01}(\tau)),$$

we have

$$E_4(2\tau) = [\theta^8_{00} - \theta^4_{00}\theta^4_{01} + \theta^8_{01}](2\tau)$$
$$= \frac{1}{16}[(\theta^2_{00} + \theta^2_{01})^4 - 4(\theta^2_{00} + \theta^2_{01})^2\theta^2_{00}\theta^2_{01} + 16\theta^4_{00}\theta^4_{01}](\tau)$$
$$= \frac{1}{64}[h^2_1 - 48h_2](\tau)$$

and

$$h_1(\tau)^2 - 48h_2(\tau) = 64E_4(2\tau) = 64 \left(1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^{2n}\right),$$
$$h_1(\tau)(h_1(\tau)^2 - 72h_2(\tau)) = -512E_6(2\tau) = -512 \left(1 + 504 \sum_{n=1}^{\infty} \sigma_5(n)q^{2n}\right),$$
$$h_2(\tau)^2(h_1(\tau)^2 - 64h_2(\tau)) = 2^{18}\eta(2\tau) = 2^{18}q^2 \prod_{n=1}^{\infty} (1 - q^{2n})^{24},$$
$$h_2(\tau)/(h_1(\tau)^2 - 64h_2(\tau)) = \frac{\eta(2\tau)}{\eta(\tau)}$$

where $q = \exp(2\pi i \tau)$, and

$$\lim_{\tau \to \infty} [h_1(it) : h_2(it)] = [1 : 0], \quad \lim_{\tau \to \infty} [h_1(-1/2it) : h_2(-1/2it)] = [8 : 1] \in \mathbb{P}(2, 4).$$

**Proof.** By the formula

$$\theta^2_{00}(2\tau) = \frac{1}{2}(\theta^2_{00}(\tau) + \theta^2_{01}(\tau)), \quad \theta^2_{01}(2\tau) = \theta_{00}(\tau)\theta_{01}(\tau), \quad \theta^2_{10}(2\tau) = \frac{1}{2}(\theta^2_{00}(\tau) - \theta^2_{01}(\tau)),$$

we have

$$E_4(2\tau) = [\theta^8_{00} - \theta^4_{00}\theta^4_{01} + \theta^8_{01}](2\tau)$$
$$= \frac{1}{16}[(\theta^2_{00} + \theta^2_{01})^4 - 4(\theta^2_{00} + \theta^2_{01})^2\theta^2_{00}\theta^2_{01} + 16\theta^4_{00}\theta^4_{01}](\tau)$$
$$= \frac{1}{64}[h^2_1 - 48h_2](\tau)$$

and

Since $\eta(2\tau)/\eta(\tau)$ is the Hauptmodul for $\Gamma_0(2)_1$ (see [C2,D2]), we see that the map

$$\mathbb{H}/\Gamma_0(2)_1 \cup \{0, \infty\} \to \mathbb{P}(1, 2), \quad \tau \mapsto [h_1(\tau) : h_2(\tau)] = [1 - 4u_1 - 4u_2 : u_1u_2]$$

is an isomorphism.
\[ E_6(2\tau) = -\frac{1}{2} \left[ (\theta_{00}^4 + \theta_{01}^4)(2\theta_{00}^4 - \theta_{01}^4)(\theta_{00}^4 - 2\theta_{01}^4) \right](2\tau) \]

\[ = -\frac{1}{64} \left[ (\theta_{00}^4 + 6\theta_{00}^2\theta_{01}^2 + \theta_{01}^4)(\theta_{00}^4 + \theta_{01}^4)(\theta_{00}^4 - 6\theta_{00}^2\theta_{01}^2 + \theta_{01}^4) \right](\tau) \]

\[ = -\frac{1}{512} \left[ h_1(h_1^2 - 72h_2) \right](\tau). \]

Other assertions are shown by a similar calculation. □

4.5. Theorem. Let us define an embedding \( \Theta : \mathbb{S}_2/\Gamma_0^*(2)_2 \rightarrow \mathbb{P}(1, 2, 3, 4) \) by

\[ \tau \mapsto \left[ 8\vartheta : \vartheta^2 - \phi : 1024\chi : 1024(\psi - \vartheta\chi) \right]. \]

Then we have the commutative diagram

\[ \begin{array}{ccc}
\mathcal{U} & \xrightarrow{\text{Per}} & \mathbb{D}^+_{nS} \\
DvG \downarrow & & \downarrow \Theta \\
\mathbb{P}(1, 2, 3, 4) & \leftarrow & \mathbb{S}_2/\Gamma_0^*(2)_2 \cong \mathbb{D}^+_{nS}/O_{nS}^+ \\
\end{array} \]

and \( \Theta \) induces an isomorphism \( \mathbb{S}_2/\Gamma_0^*(2)_2 \cup \mathbb{H}/\Gamma_0(2)_1 \cup \{0, \infty\} \cong \mathbb{P}(1, 2, 3, 4) \).

Proof. Indeed, the map \( \Theta \) is the unique map

\[ \mathbb{S}_2/\Gamma_0^*(2)_2 \longrightarrow \mathbb{P}(1, 2, 3, 4), \quad \tau \mapsto \left[ F_2(\tau) : F_4(\tau) : F_6(\tau) : F_8(\tau) \right] \quad (F_k \in M_k(\Gamma_0^*(2)_2)) \]

such that

(i) \( F_6 \) vanishes on \( \mathbb{H} \times \mathbb{H} \),

(ii) \( \lim_{t \to \infty} \left[ F_2 : F_4 : F_6 : F_8 \right] \left( \begin{bmatrix} \tau & 0 \\ 0 & it \end{bmatrix} \right) = [h_1(\tau) : h_2(\tau) : 0 : 0] \),

(iii)

\[ [F_2 : F_4 : F_6 : F_8] \left( \begin{bmatrix} 2\tau & \tau \\ \tau & 2\tau \end{bmatrix} \right) = [-4s_1 + 1 : s_2 : 2s_3 : s_1s_3]_{u_1 = u_2 = u_3 = u} \]

\[ = [-12u + 1 : 3u^2 : 2u^3 : 3u^4], \]

that is, \( F_4^2 = 3F_8 \) and \( 9F_6^2 = 4F_4F_8 \).

For (i), we see that \( F_6 = c\chi \) by Proposition 2.6 and Lemma 3.3. For (ii), note that

\[ \lim_{t \to \infty} \left( c_1\vartheta^4 + c_2\vartheta^2\phi + c_3\vartheta^2 + c_4\vartheta\chi + c_5\psi \right) \left( \begin{bmatrix} \tau & 0 \\ 0 & it \end{bmatrix} \right) = 0 \]

\[ \iff c_1(\theta_{00}^4 + \theta_{01}^4)^4/16 + c_2(\theta_{00}^4 + \theta_{01}^4)^2\theta_{00}\theta_{01}^4/4 + c_3(\theta_{00}\theta_{01}^4)^2 = 0 \]

\[ \iff c_1 = c_2 = c_3 = 0. \]
Therefore we have

\[ [F_2 : F_4 : F_6 : F_8] = [8\vartheta^2 - \phi : c\chi : c_4\vartheta\chi + c_5\psi]. \]

Now the condition (iii) implies

\[ (\vartheta^2 - \phi)^2(\tau) = 3(c_4\vartheta\chi + c_5\psi)(\tau), \quad 9(c\chi)^2(\tau) = \left[ 4(\vartheta^2 - \phi)(c_4\vartheta\chi + c_5\psi) \right](\tau) \]

for \( \tau = \begin{bmatrix} 2\tau_1 & \tau_1 \\ \tau_1 & \tau_1 \end{bmatrix} \). Comparing Fourier coefficients

\[ \vartheta(\tau) = 1 + 72q^8 + 192q^{12} + 504q^{16} + 576q^{20} + 2280q^{24} + \cdots, \]
\[ \phi(\tau) = q^8 - 4q^{12} - 2q^{16} + 20q^{20} + 5q^{24} + \cdots, \]
\[ \chi(\tau) = q^{12} - 6q^{16} + 3q^{20} + 40q^{24} + \cdots, \]
\[ \psi(\tau) = q^{12} + 6q^{16} - 21q^{20} - 56q^{24} + \cdots, \]

we obtain \( c = 1024, c_4 = -1024 \) and \( c_5 = 1024 \). \( \square \)

References


