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Linkages: A Tool for the Construction of Multivariate Distributions with Given Nonoverlapping Multivariate Marginals

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One of the most useful tools for handling multivariate distributions with given *univariate* marginals is the copula function. Using it, any multivariate distribution function can be represented in a way that emphasizes the separate roles of the marginals and of the dependence structure. The goal of the present paper is to introduce an analogous tool, called the linkage function, that can be used for the study of multivariate distributions with given *multivariate* marginals by emphasizing the separate roles of the dependence structure *between* the given multivariate marginals, and the dependence structure *within* each of the nonoverlapping marginals. Preservation of some setwise positive dependence properties, from the linkage function L to the joint distribution F and vice versa, are studied. When two different distribution functions are associated with the same linkage function (that is, have the same setwise dependence structure) we show that strong stochastic dominance order among the corresponding multivariate marginal distributions implies an overall stochastic dominance between the two underlying distribution functions. © 1996 Academic Press, Inc.

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1. Introduction

One of the most useful tools for handling multivariate distributions with given *univariate* marginals is the copula function. Using it, any multivariate distribution function can be represented in a way that emphasizes the separate roles of the marginals and of the dependence structure. The goal of the present paper is to introduce an analogous tool, called the linkage function, that can be used for the study of multivariate distributions with given *multivariate* marginals by emphasizing the separate roles of the dependence structure *between* the given multivariate marginals, and the dependence structure *within* each of the nonoverlapping marginals.

The linkage function is particularly useful when not all the interrelationships among the random variables are equally important, but rather only the relationships among certain nonoverlapping sets of random variables (i.e., random vectors) are relevant. The need to study relationships among random vectors arises naturally in a variety of circumstances (see, e.g., Chhetry, Sampson, and Kimeldorf [4] and Block and Fang [2]). For example, in a complex engineering system, the relationship among the subsystems can be considered in the framework of this paper, even if the dependence structure within the subsystems is not entirely well understood. Additionally, a framework for studying vector dependencies may lead to further understanding of complicated multivariate distributions.

The present paper is to be contrasted with some previous work in the area of probability distributions with given multivariate marginals. Cohen [5] describes a particular procedure which gives joint distributions with given nonoverlapping multivariate marginals; his procedure depends on the particular set of the given multivariate marginals. Marco and Ruiz–Rivas [12] are concerned with the following problem: Given k (possibly multivariate) marginal distributions $F_1, F_2, ..., F_k$ of dimensions $m_1, m_2, ..., m_k$, respectively, what conditions should a k-dimensional function C satisfy in order for $C(F_1, F_2, ..., F_k)$ to be a $(\sum_{i=1}^{k} m_i)$-dimensional distribution function? They also give a procedure for the construction of such a function C. Cuadras [6] describes a procedure which, under some conditions, yields joint distributions with given nonoverlapping multivariate marginals, such that the resulting regression curves are linear. Rüschendorf [17], and references therein, considered the problem of constructing a joint distribution with given (possibly overlapping) marginals.

The insufficiency of the copula function to handle multivariate distributions with given marginals is illustrated by the following result of Genest, Quesada Molina, and Rodriguez Lallena [8]. They showed that if the function $C: [O, 1]^2 \to [0, 1]$ is such that

$$H(x_1, x_2, ..., x_m, y_1, y_2, ..., y_n)$$
$$= C(F(x_1, x_2, ..., x_m), G(y_1, y_2, ..., y_n))$$

defines a $(m+n)$-dimensional distribution function with marginals F and G for all m and n such that $m+n \geqslant 3$, and for all distribution functions F and G (with dimensions m and n, respectively), then $C(u, v) = uv$. Namely, the only possible copula which works with multidimensional marginals is the independent one.

The approach of the present paper is completely different. Here, given a $(\sum_{i=1}^{k} m_i)$-dimensional distribution function F, with the (possibly multivariate) marginal distributions $F_1, F_2, ..., F_k$ of dimensions $m_1, m_2, ..., m_k$, respectively, we associate with F the so-called linkage function L which contains the information regarding the dependence structure among the underlying random vectors. The dependence structure within the random vectors is not included in L.

After giving some preliminaries we give the definition of the linkage function in Section 3. Preservation of some setwise positive dependence properties (in the sense of Chhetry, Sampson, and Kimeldorf [4], Joag–Dev, Perlman, and Pitt [9], and Chhetry, Kimeldorf, and Zahedi [3]), from the linkage function L to the joint distribution F and vice versa, are studied in Section 4. In some applications two different $(\sum_{i=1}^{k} m_i)$-dimensional distribution functions may be associated with the same linkage function (that is, have the same setwise dependence structure). In Section 5 we show that, in such a case, strong stochastic dominance order among the corresponding multivariate marginal distributions implies an overall stochastic dominance between the two underlying $(\sum_{i=1}^{k} m_i)$-dimensional distribution functions.

2. Some Preliminaries

2.1. *The Standard Construction and Its Inverse*

Let $X_1, X_2, ..., X_n$ be n random variables with a joint distribution F. Denote by $F_1(\cdot)$ the marginal distribution of X_1, and denote by $F_{i+1 \mid 1, 2, ..., i}(\cdot \mid x_1, x_2, ..., x_i)$ the conditional distribution of X_{i+1} given that $X_1 = x_1, X_2 = x_2, ..., X_i = x_i$. The inverse of F_1 will be denoted by $F_1^{-1}(\cdot)$ and the inverse of $F_{i+1 \mid 1, 2, ..., i}(\cdot \mid x_1, x_2, ..., x_i)$ will be denoted by $F^{-1}_{i+1 \mid 1, 2, ..., i}(\cdot \mid x_1, x_2, ..., x_i)$ for every $(x_1, x_2, ..., x_i)$ in the support of $(X_1, X_2, ..., X_i)$, $i = 1, 2, ..., n-1$. Here the inverse F^{-1} of a distribution function F is defined as $F^{-1}(u) = \sup\{x \colon F(x) \leqslant u\}$, $u \in [0, 1]$.

Consider the transformation $\Psi_F\colon \mathbb{R}^n \to [0,1]^n$ (which depends on F) defined by

$$\Psi_F(x_1, x_2, ..., x_n) = (F_1(x_1), F_{2|1}(x_2 \mid x_1), ..., F_{n|1,2,...,n-1}(x_n \mid x_1, x_2, ..., x_{n-1})), \quad (2.1)$$

for all $(x_1, x_2, ..., x_n)$ in the support of $(X_1, X_2, ..., X_n)$.

LEMMA 2.1. *Let $X_1, X_2, ..., X_n$ be n random variables with an absolutely continuous joint distribution F. Define*

$$(U_1, U_2, ..., U_n) = \Psi_F(X_1, X_2, ..., X_n). \quad (2.2)$$

Then $U_1, U_2, ..., U_n$ an are independent uniform $[0,1]$ random variables.

Proof. It is well known that marginally U_1 is a uniform $[0,1]$ random variable. Given $U_1 = u_1$, the value of U_2 can be computed (as a function of X_2 and u_1) as follows: $U_2 = F_{2|1}(X_2 \mid F_1^{-1}(u_1))$. It is thus seen that, given $U_1 = u_1$, the conditional distribution of U_2 is uniform $[0,1]$, independently of the value of U_1. This shows that U_1 and U_2 are independent, and each is a uniform $[0,1]$ random variable. Continuing this procedure we obtain the stated result. ∎

In the univariate case only continuity (rather than absolute continuity) is needed in order to prove the analogous result. That is, if the univariate random variable X has the distribution function F, and F is continuous, then $F(X)$ is a uniform $[0,1]$ random variable. The assumption of absolute continuity in Lemma 2.1 guarantees the continuity of the underlying conditional distributions.

Note that the transformation defined in (2.2) is only one of many transformations which transform the random variables $X_1, X_2, ..., X_n$ into n independent uniform $[0,1]$ random variables. For example, we can permute the indices $1, 2, ..., n$ and get other transformations (see Example 3.1 for a discussion regarding this point).

By "inverting" Ψ_F we can express the X_i's as functions of the independent uniform random variables $U_1, U_2, ..., U_n$ (see, e.g., Rüschendorf and de Valk [18]). Denote

$$x_1 = F^{-1}(u_1), \quad (2.3)$$

and, by induction,

$$x_i = F^{-1}_{i|1,2,...,i-1}(u_i \mid x_1, x_2, ..., x_{i-1}), \qquad i = 2, 3, ..., n. \quad (2.4)$$

Consider the transformation $\Psi_F^*\colon [0, 1]^n \to \mathbb{R}^n$ defined by (here the x_i's are functions of the u_i's as given in (2.3) and (2.4))

$$\Psi_F^*(u_1, u_2, ..., u_n) = (x_1, x_2, ..., x_n), \qquad (u_1, u_2, ..., u_n) \in [0, 1]^n.$$

Let

$$(\hat{X}_1, \hat{X}_2, ..., \hat{X}_n) \equiv \Psi_F^*(U_1, U_2, ..., U_n). \tag{2.5}$$

Then

$$(\hat{X}_1, \hat{X}_2, ..., \hat{X}_n) =_{\text{st}} (X_1, X_2, ..., X_n), \tag{2.6}$$

where "$=_{\text{st}}$" denotes equality in law (note that no continuity assumptions are needed for the validity of (2.6)). In fact, it is well known, and easy to verify, that if F is absolutely continuous then

$$\Psi_F^* \Psi_F(X_1, X_2, ..., X_n) =_{\text{a.s.}} (X_1, X_2, ..., X_n), \tag{2.7}$$

where "$=_{\text{a.s.}}$" denotes equality almost surely under the probability measure associated with F. The construction described in (2.5) is called the *standard construction*; it is a well-known method of multivariate simulation.

2.2. *CIS Random Variables*

Let $X_1, X_2, ..., X_n$ be n random variables with a joint distribution F. In general $\Psi_F^*(u_1, u_2, ..., u_n)$ is not necessarily increasing in $(u_1, u_2, ..., u_n) \in [0, 1]^n$ (here, and throughout this paper, "increasing" means "nondecreasing" and "decreasing" means "nonincreasing"). However, we provide below conditions under which $\Psi_F^*(u_1, u_2, ..., u_n)$ *is* increasing in $(u_1, u_2, ..., u_n) \in [0, 1]^n$.

The random variables $X_1, X_2, ..., X_n$ (or their joint distribution function) are said to be *conditionally increasing in sequence* (CIS) if

$$X_i \uparrow_{\text{st}} (X_1, X_2, ..., X_{i-1}), \qquad i = 2, 3, ..., n,$$

that is, if $E[\phi(X_i) \mid X_1 = x_1, X_2 = x_2, ..., X_{i-1} = x_{i-1}]$ is increasing in $x_1, x_2, ..., x_{i-1}$ for all increasing functions ϕ for which the expectations are defined, $i = 2, 3, ..., n$. The CIS notion is a concept of positive dependence that was studied, e.g., in Lehmann [11] and in Barlow and Proschan [1]. The following result is implicit in Barlow and Proschan [1] and is explicit in Rubinstein, Samorodnitsky, and Shaked [15].

LEMMA 2.2. *Let $X_1, X_2, ..., X_n$ be n random variables with a joint distribution F. If $X_1, X_2, ..., X_n$ are CIS then $\Psi_F^*(u_1, u_2, ..., u_n)$ is increasing in $(u_1, u_2, ..., u_n) \in [0, 1]^n$.*

2.3. *Copulas*

A linkage can be viewed as a multivariate extension of a copula. In this section we recall the definition and the basic properties of copulas. We define linkages in Section 3.

The copula (as named by Sklar [24], or the uniform representation as named by Kimeldorf and Sampson [10], or the dependence function as named by Deheuvels [7]) is one of the most useful tools for handling multivariate distributions with given univariate marginals $F_1, F_2, ..., F_k$. Formally, a copula C is a cumulative distribution function, defined on $[0, 1]^k$, with uniform marginals. Given a copula C, if one defines

$$F(x_1, x_2, ..., x_k) = C(F_1(x_1), F_2(x_2), ..., F_k(x_k)), \qquad (x_1, x_2, ..., x_k) \in \mathbb{R}^k, \tag{2.8}$$

then F is a multivariate distribution with univariate marginals $F_1, F_2, ..., F_k$. Given a continuous F, with marginals $F_1, F_2, ..., F_k$, there corresponds to it a unique copula that can be constructed as

$$C(u_1, u_2, ..., u_k) = F[F_1^{-1}(u_1), F_2^{-1}(u_2), ..., F_k^{-1}(u_k)], \qquad (u_1, u_2, ..., u_k) \in [0, 1]^k. \tag{2.9}$$

Note that different multivariate distributions F may have the same copula. Most of the multivariate dependence structure properties of F are in the copula function, which is independent of the marginals and which is, in general, easier to handle than the original F.

We now list some positive dependence properties that are inherited by F from the corresponding copula. The random vector $\mathbf{X} = (X_1, X_2, ..., X_k)$ (or its distribution function) is said to be positively upper orthant dependent (PUOD) if

$$P\{X_1 > x_1, X_2 > x_2, ..., X_k > x_k\} \geqslant \prod_{i=1}^{k} P\{X_i > x_i\}, \qquad (x_1, x_2, ..., x_k) \in \mathbb{R}^k.$$

It is said to be positively lower orthant dependent (PLOD) if

$$P\{X_1 \leqslant x_1, X_2 \leqslant x_2, ..., X_k \leqslant x_k\} \geqslant \prod_{i=1}^{k} P\{X_i \leqslant x_i\}, \qquad (x_1, x_2, ..., x_k) \in \mathbb{R}^k$$

(see, e.g., Shaked and Shanthikumar [23, Subsection 4.G.1]). It is said to be associated if

$$\mathrm{Cov}(g(\mathbf{X}), h(\mathbf{X})) \geqslant 0, \tag{2.10}$$

for all increasing functions g and h for which the covariance is defined (see, e.g., Barlow and Proschan [1]). Finally, $\mathbf{X}$ (or its distribution function) is said to be positively dependent by mixtures (PDM) if the joint distribution function F of $\mathbf{X}$ can be written as

$$F(x_1, x_2, ..., x_k) = \int_{\Omega} \prod_{i=1}^{k} G^{(\mathbf{w})}(x_i)\, dH(\mathbf{w}),$$

where Ω is a subset of a finite-dimensional Euclidean space, $\{G^{(\mathbf{w})}, \mathbf{w} \in \Omega\}$ is a family of univariate distribution functions, and H is a distribution function on Ω (see Shaked [21]). Note that if $\mathbf{X}$ is (PDM) then $X_1, X_2, ..., X_k$ have a permutation symmetric distribution function. The following results are well known.

PROPOSITION 2.3 [13]. *Let C be a copula, and let F be defined as in* (2.8).

(i) *If C is* PUOD (PLOD) *then F is* PUOD (PLOD).

(ii) *If C is associated then F is associated.*

(iii) *If C is* PDM, *and if $F_1, F_2, ..., F_k$ of* (2.8) *are all equal, then F is* PDM.

PROPOSITION 2.4 [19]. *Let $\mathbf{X} = (X_1, X_2, ..., X_k)$ and $\mathbf{Y} = (Y_1, Y_2, ..., Y_k)$ have the same copula (as defined in* (2.9)*). If $X_i \leqslant_{\text{st}} Y_i$, $i = 1, 2, ..., k$, then $\mathbf{X} \leqslant_{\text{st}} \mathbf{Y}$; that is, $E\phi(\mathbf{X}) \leqslant E\phi(\mathbf{Y})$ for all real increasing functions ϕ for which the expectations are defined.*

3. LINKAGES

Let $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k$ be k random vectors of dimensions $m_1, m_2, ..., m_k$, respectively. We do not necessarily assume that the $\mathbf{X}_i$'s are independent. Let F_i be the (marginal) m_i-dimensional distribution of $\mathbf{X}_i$, $i = 1, 2, ..., k$, and let F be the joint distribution of $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k$ which is, of course, of dimension $\sum_{i=1}^{k} m_i$. For $i = 1, 2, ..., k$, let the transformation $\Psi_{F_i}\colon \mathbb{R}^{m_i} \to [0, 1]^{m_i}$ be defined as in (2.1). Then, by (2.2), if F_i is absolutely continuous, then the vector $\mathbf{U}_i = \Psi_{F_i}(\mathbf{X}_i)$ is a vector of m_i independent uniform [0, 1] random variables. However, since the $\mathbf{X}_i$'s are not necessarily independent, it follows that the $\mathbf{U}_i$'s are not necessarily independent. The joint distribution L of

$$(\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_k) = (\Psi_{F_1}(\mathbf{X}_1), \Psi_{F_2}(\mathbf{X}_2), ..., \Psi_{F_k}(\mathbf{X}_k)) \tag{3.1}$$

will be called the *linkage* corresponding to $(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k)$.

Note that different multivariate distributions F (with marginals of dimensions $m_1, m_2, ..., m_k$) may have the same linkage. Most of the information, regarding the multivariate dependence structure properties *between* the $\mathbf{X}_i$'s is contained in the linkage function, which is independent of the marginals and which may be easier to handle than the original F. Note that the linkage function is not expected to contain any information regarding the dependence properties *within* each of the $\mathbf{X}_i$'s. This information is contained in the m_i-dimensional functions Ψ_{F_i}, and it is erased when we transform the vector $\mathbf{X}_i$, of dependent variables, into the vector $\mathbf{U}_i$, of independent uniform $[0, 1]$ random variables, by $\mathbf{U}_i = \Psi_{F_i}(\mathbf{X}_i)$. Thus, the linkage function can be useful when one is interested in studying the dependence properties *between* the $\mathbf{X}_i$'s, separate from the dependence properties *within* the $\mathbf{X}_i$'s.

If $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k$ have the joint distribution F, and if $\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_k$ have the joint distribution L, where L is the linkage corresponding to F, then it is not hard to show, using (2.7), that $(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2, ..., \hat{\mathbf{X}}_k)$ defined by

$$(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2, ..., \hat{\mathbf{X}}_k) \equiv (\Psi^*_{F_1}(\mathbf{U}_1), \Psi^*_{F_2}(\mathbf{U}_2), ..., \Psi^*_{F_k}(\mathbf{U}_k)), \tag{3.2}$$

is such that

$$(\hat{\mathbf{X}}_1, \hat{\mathbf{X}}_2, ..., \hat{\mathbf{X}}_k) =_{\text{st}} (\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k). \tag{3.3}$$

EXAMPLE 3.1. Consider the case in which $k=2$ and $m_1 = m_2 = 2$. Explicitly we are given now two bivariate marginals F_1 and F_2, say. A linkage in this case is a four-dimensional $(m_1 + m_2 = 4)$ distribution function L, of the random vectors (U_{11}, U_{12}) and (U_{21}, U_{22}), say, where U_{11} and U_{12} are independent uniform $[0, 1]$ random variables, U_{21} and U_{22} are independent uniform $[0, 1]$ random variables, but otherwise L can be any joint distribution. Let $(\hat{X}_{11}, \hat{X}_{12})$ and $(\hat{X}_{21}, \hat{X}_{22})$ be defined as in (3.2), and let F be their joint distribution function. Thus F is a distribution that has the linkage L and the bivariate marginals F_1 and F_2.

For example, let L be such that $P\{U_{11} = U_{21}\} = 1$ (and, of course, U_{12} and U_{22} are both independent of the random variable $U_{11}(=U_{21})$, but they otherwise can have any joint distribution). Then, marginally, the joint distribution of $\hat{X}_{11}$ and $\hat{X}_{21}$ is the Fréchet upper bound with marginals F_{11} and F_{21} (where here F_{ij} denotes the marginal distribution of $\hat{X}_{ij}$). That is, $\hat{X}_{11}$ is an increasing function of $\hat{X}_{21}$ and vice versa (in fact, here we have that $\hat{X}_{11} = F_{11}^{-1}(F_{21}(\hat{X}_{21}))$ or $\hat{X}_{21} = F_{21}^{-1}(F_{11}(\hat{X}_{11}))$).

Assume now, furthermore, that L is such that also $P\{U_{12} = U_{22}\} = 1$ (of course, now $U_{12}(=U_{22})$ is independent of the random variable $U_{11}(=U_{21})$). Do we get then that the joint distribution of $\hat{X}_{12}$ and $\hat{X}_{22}$ is the Fréchet upper bound with marginals F_{12} and F_{22}? The answer is: Not necessarily. This can be seen by computing explicitly

$$\hat{X}_{12} = F^{-1}_{12|11}(F_{22|21}(\hat{X}_{22} \mid F^{-1}_{21}(F_{11}(\hat{X}_{11}))) \mid \hat{X}_{11}),$$

where $F_{ij|ik}$ denote the conditional distribution of $\hat{X}_{ij}$ given $\hat{X}_{ik}$, $i = 1, 2$. That is, given the value of $\hat{X}_{11}$ (or, equivalently, of U_{11}) we see that $\hat{X}_{12}$ is an increasing function of $\hat{X}_{22}$ and vice versa, but this need not be the case when $\hat{X}_{11}$ is not fixed. The fact that we do not necessarily get the Fréchet upper bound for F_{12} and F_{22} is not really surprising; having already the Fréchet upper bound with marginals F_{11} and F_{21} and having the fixed bivariate marginals F_1 and F_2, the latitude that we have in choosing F, with the additional constraint of having to have the univariate marginals F_{12} and F_{22}, is limited.

By choosing L to be such that $P\{U_{11} = 1 - U_{21}\} = 1$ we see that the joint distribution of X_{11} and X_{21} is now the Fréchet lower bound with marginals F_{11} and F_{21}.

If we want to get that the joint distribution of $\hat{X}_{12}$ and $\hat{X}_{22}$ (rather than $\hat{X}_{11}$ and $\hat{X}_{21}$) is the Fréchet upper (or lower) bound with marginals F_{12} and F_{22} then we can apply the above procedure, interchanging the indices 1 and 2 in the proper places. We can even get, if we wish, by the correct choice of indices, that, e.g., the joint distribution of $\hat{X}_{12}$ and $\hat{X}_{21}$ is the Fréchet upper (or lower) bound with marginals F_{12} and F_{21}. The actual choice of indices may depend on the primary and secondary importance of the random variables among $\hat{X}_{11}$, $\hat{X}_{12}$, $\hat{X}_{21}$, and $\hat{X}_{22}$.

EXAMPLE 3.2. Let W and Z be two independent univariate random variables. Define $\mathbf{X} = (\mathbf{X}_1, X_2) = ((Z, Z + W), W)$, so the random vector $\mathbf{X}$ consists of one 2-dimensional vector and one 1-dimensional vector. It is not hard to see, using, e.g. (3.1), that the linkage associated with $\mathbf{X}$ is the joint distribution L of $((U_1, U_2), U_2)$, where U_1 and U_2 are independent uniform $[0, 1]$ random variables. In fact, L is the linkage of $((Z, g(Z, W)), W)$ whenever $g(z, w)$ is strictly increasing in w for all z, even if g is decreasing in z. This illustrates the intuitive idea that the linkage is concerned with the dependence between the underlying random vectors, but need not be affected by the dependence within the vectors. For a similar illustration see Remark 3.5.

EXAMPLE 3.3. Let $\mathbf{X} = ((W_1, W_2), (Z_1, Z_2))$ be a four-dimensional multivariate normal random vector with mean vector $\mathbf{0}$ and correlation matrix

$$\Sigma = \begin{pmatrix} 1 & \rho_W & \rho & \rho \\ \rho_W & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho_Z \\ \rho & \rho & \rho_Z & 1 \end{pmatrix},$$

where $-1<\rho_W<1$, $-1<\rho_Z<1$, and $-1\leqslant\rho\leqslant\frac{1}{2}\sqrt{(1+\rho_W)(1+\rho_Z)}$ (then Σ is a proper correlation matrix; see Corollary 6 of Scarsini and Verdicchio [20]). For simplicity we assume that the variances of all the components of $\mathbf{X}$ are equal to 1. Here the bivariate marginals of $\mathbf{X}$ are bivariate normal random vectors consisting of standard normal variables with correlations ρ_W and ρ_Z, respectively. Denote by Φ the standard normal distribution function, and let "$\sim$" stands for "distributed as." Since $W_1\sim N(0,1)$, $[W_2 \mid W_1=w_1]\sim N(\rho_W w_1, 1-\rho_W^2)$, $Z_1\sim N(0,1)$, and $[Z_2 \mid Z_1=z_1]\sim N(p_Z z_1, 1-\rho_Z^2)$, we see that the linkage of $\mathbf{X}$ is the distribution L of

$$\begin{pmatrix} U_1\\ U_2\\ V_1\\ V_2\end{pmatrix}=\begin{pmatrix}\Phi(W_1)\\ \Phi\left(\dfrac{W_2-\rho_W W_1}{\sqrt{1-\rho_W^2}}\right)\\ \Phi(Z_1)\\ \Phi\left(\dfrac{Z_2-\rho_Z Z_1}{\sqrt{1-\rho_Z^2}}\right)\end{pmatrix}.$$

We note that the transformation from $\mathbf{X}$ to $(\mathbf{U},\mathbf{V})$ does not depend on ρ, although L depends on ρ since the distribution of $\mathbf{X}$ depends on it. In order to get some insight into L we note that it is the linkage of $(\Phi^{-1}(U_1), \Phi^{-1}(U_2), \Phi^{-1}(V_1), \Phi^{-1}(V_2))$, that is, the linkage of

$$\begin{pmatrix} W_1\\ \dfrac{W_2-\rho_W W_1}{\sqrt{1-\rho_W^2}}\\ Z_1\\ \dfrac{Z_2-\rho_Z Z_1}{\sqrt{1-\rho_Z^2}}\end{pmatrix}$$

$$\sim N\left(0,\begin{pmatrix} 1 & 0 & \rho & \sqrt{\dfrac{1-\rho_Z}{1+\rho_Z}}\rho\\ 0 & 1 & \sqrt{\dfrac{1-\rho_W}{1+\rho_W}}\rho & \sqrt{\dfrac{1-\rho_W}{1+\rho_W}}\sqrt{\dfrac{1-\rho_Z}{1+\rho_Z}}\rho\\ \rho & \sqrt{\dfrac{1-\rho_W}{1+\rho_W}}\rho & 1 & 0\\ \sqrt{\dfrac{1-\rho_Z}{1+\rho_Z}}\rho & \sqrt{\dfrac{1-\rho_W}{1+\rho_W}}\sqrt{\dfrac{1-\rho_Z}{1+\rho_Z}}\rho & 0 & 1\end{pmatrix}\right).$$

It is known that if a random vector$(X_1, X_2, ..., X_k)$, with continuous marginals, has the copula C, then the random vector $(g_1(X_1), g_2(X_2), ..., g_k(X_k))$ has the same copula C whenever the g_i's are strictly increasing real univariate functions; that is, the copula is preserved under strictly increasing univariate transformations. The following result is a generalization of this fact.

THEOREM 3.4. *Let* $\mathbf{X} = (\mathbf{X}_1, ..., \mathbf{X}_k) = ((X_{11}, ..., X_{1m_1}), ..., (X_{k1}, ..., X_{km_k}))$ *be random vector with an absolutely continuous function. Let* $\mathbf{Y} = (\mathbf{Y}_1, ..., \mathbf{Y}_m) = ((Y_{11}, ..., Y_{1m_1}), ..., (Y_{k1}, ..., Y_{km_k}))$ *be another random vector such that*

$$((Y_{11}, ..., Y_{1m_1}), ..., (Y_{k1}, ..., Y_{km_k})) \\ =_{\mathrm{st}} ((g_{11}(X_{11}), ..., g_{1m_1}(X_{1m_1})), ..., (g_{k1}(X_{k1}), ..., g_{km_k}(X_{km_k}))), \quad (3.4)$$

where the g_{ij}*'s are strictly increasing real univariate functions. Then* $\mathbf{X}$ *and* $\mathbf{Y}$ *have the same linkage.*

Proof. Let $F_{ij \mid i1, ..., i(j-1)}(\cdot \mid x_{i1}, ..., x_{i(j-1)})$ denote the conditional distribution of X_{ij} given that $X_{i1} = x_{i1}, ..., X_{i(j-1)} = x_{i(j-1)}$. Similarly, let $G_{ij \mid i1, ..., i(j-1)}(\cdot \mid y_{i1}, ..., y_{i(j-1)})$ denote the conditional distribution of Y_{ij} given that $Y_{i1} = y_{i1}, ..., Y_{i(j-1)} = y_{i(j-1)}$. From (3.4) it follows that

$$G_{ij \mid i1, ..., i(j-1)}(y_{ij} \mid y_{i1}, ..., y_{i(j-1)}) \\ = F_{ij \mid i1, ..., i(j-1)}(g_{ij}^{-1}(y_{ij}) \mid g_{i1}^{-1}(y_{i1}), ..., g_{i(j-1)}^{-1}(y_{i(j-1)})). \quad (3.5)$$

Denote by $F_i[G_i]$ the marginal distribution of $\mathbf{X}_i[\mathbf{Y}_i]$, $i = 1, 2, ..., k$. Define $\hat{Y}_{ij} = g_{ij}(X_{ij})$, $j = 1, 2, ..., m_i$, $i = 1, 2, ..., k$. Thus

$$\hat{\mathbf{Y}} =_{\mathrm{st}} \mathbf{Y}.$$

Using (3.5) we obtain

$$\begin{aligned}
(\mathbf{V}_1, ..., \mathbf{V}_k) &\equiv (\Psi_{G_1}(\hat{\mathbf{Y}}_1), ..., \Psi_{G_k}(\hat{\mathbf{Y}}_k)) \\
&= (\Psi_{G_1}(\hat{Y}_{11}, ..., \hat{Y}_{1m_1}), ..., \Psi_{G_k}(\hat{Y}_{k1}, ..., \hat{Y}_{km_k})) \\
&= (\Psi_{F_1}(g_{11}^{-1}(\hat{Y}_{11}), ..., g_{1m_1}^{-1}(\hat{Y}_{1m_1})), ..., \\
&\qquad \Psi_{F_k}(g_{k1}^{-1}(\hat{Y}_{k1}), ..., g_{km_k}^{-1}(\hat{Y}_{km_k}))) \\
&= (\Psi_{F_1}(X_{11}, ..., X_{1m_1}), ..., \Psi_{F_k}(X_{k1}, ..., X_{km_k})) \\
&\equiv (\mathbf{U}_1, ..., \mathbf{U}_k).
\end{aligned}$$

Therefore $\mathbf{X}$ and $\hat{\mathbf{Y}}$ have the same linkage. Hence $\mathbf{X}$ and $\mathbf{Y}$ have the same linkage. ∎

One may ask whether, under the assumption of absolute continuity, $(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k)$ and $(\mathbf{g}_1(\mathbf{X}_1), \mathbf{g}_2(\mathbf{X}_2), ..., \mathbf{g}_k(\mathbf{X}k))$ have the same linkage, where $\mathbf{g}_i$ is a strictly increasing function from $\mathbb{R}^{m_i}$ to $\mathbb{R}^{m_i}$, $i = 1, 2, ..., k$ (here when we say that $\mathbf{g}_i$ is strictly increasing, we mean that each of the m_i coordinates of $\mathbf{g}_i$ is strictly increasing in each of its m_i arguments). we do not believe so, but we do not have a counterexample. However, for increasing (rather than strictly increasing) functions $\mathbf{g}_i$'s it is easy to show that $(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k)$ and $(\mathbf{g}_1(\mathbf{X}_1), \mathbf{g}_2(\mathbf{X}_2), ..., \mathbf{g}_k(\mathbf{X}_k))$ need not have the same linkage. To see it, let Z and W be two independent random variables and let $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2) = ((Z, W), (Z, W))$. Let $\mathbf{g}_1$ be the identity function, and let $\mathbf{g}_2(x_1, x_2) = (x_2, x_1)$. Then $(\mathbf{g}_1(\mathbf{X}_1), \mathbf{g}_2(\mathbf{X}_2)) = ((Z, W), (W, Z))$ which has a different linkage than $((Z, W), (Z, W))$.

Remark 3.5. The proof of Theorem 3.4 can actually be used to prove a slightly stronger result than the one stated in the theorem. If in (3.4), $g_{ij}(X_{ij})$ is replaced by $g_{ij}(X_{i1}, X_{i2}, ..., X_{ij})$, where $g_{ij}(x_{i1}, x_{i2}, ..., x_{ij})$ is strictly increasing in the last variable x_{ij} when the other variables are held fixed, $j = 1, 2, ..., m_i$, $i = 1, 2, ..., k$, then the conclusion of Theorem 3.4 still holds. Note that here we do not require $g_{ij}(x_{i1}, x_{i2}, ..., x_{ij})$ to be increasing in x_{il}, $l < j$. This shows that the linkage may not be affected by the dependence within the underlying random vectors as it is affected by the dependence between them.

The following result is a corollary of Theorem 3.4. It shows that if $\mathbf{X}$ and $\mathbf{Y}$ have the same copula then they have the same linkage. This should be contrasted with Remark 3.7 in which it will be shown that if $\mathbf{X}$ and $\mathbf{Y}$ have the same linkage then they need not have the same copula.

COROLLARY 3.6. *If* $\mathbf{X} = (\mathbf{X}_1, ..., \mathbf{X}_k) = ((X_{11}, ..., X_{1m_1}), ..., (X_{k1}, ..., X_{km_k}))$ *and* $\mathbf{Y} = (\mathbf{Y}_1, ..., \mathbf{Y}_k) = ((Y_{11}, ..., Y_{1m_1}), ..., (Y_{k1}, ..., Y_{km_k}))$ *have the same unique copula then they have the same linkage.*

Proof. It is easy to prove that if $((X_{11}, ..., X_{1m_1}), ..., (X_{k1}, ..., X_{km_k}))$ and $((Y_{11}, ..., Y_{1m_1}), ..., (Y_{k1}, ..., Y_{km_k}))$ have the same unique copula then (3.4) holds. The result now follows from Theorem 3.4.

Remark 3.7. If two random vectors $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k)$ and $\mathbf{Y} = (\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_k)$ have the same linkage L it does not necessarily follow that they have the same copula C. This is so because the copula is affected by all the dependencies among the $\sum_{i=1}^{k} m_i$ underlying random variables, whereas the linkage is affected only by the dependencies between the k underlying random vectors. To see it, let W and Z be two independent univariate random variables as in Example 3.2. Define $\mathbf{X} = (\mathbf{X}_1, X_2) = ((Z, Z + W), W)$ and $\mathbf{Y} = (\mathbf{Y}_1, Y_2) = ((Z, 2Z + W), W)$. From Example 3.2 it follows that $\mathbf{X}$ and $\mathbf{Y}$ have the same linkage L, but it is easy to see, for example, when W and Z are standard normal random variables, that $\mathbf{X}$ and $\mathbf{Y}$ do not have the same copula C.

4. Preservation of Positive Dependence Properties

The following extensions of the PUOD and the PLOD concepts, applied to sets of random variables, were introduced in Chhetry, Sampson, and Kimeldorf [4] (see also [9, 3]). The k random vectors $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k$ of dimensions $m_1, m_2, ..., m_k$, respectively (or their joint distribution function), are said to be setwise positively upper set dependent (SPUSD) if

$$P\left[\bigcap_{i=1}^{k} \{\mathbf{X}_i \in U_i\}\right] \geqslant \prod_{i=1}^{k} P[\mathbf{X}_i \in U_i],$$

for all upper sets U_i in $\mathbb{R}^{m_i}$, $i=1, 2, ..., k$ (a set $U[B]$ is an upper [lower] set in $\mathbb{R}^m$ if $(x_1, x_2, ..., x_m) \in U[B]$ and $(x_1, x_2, ..., x_m) \leqslant [\geqslant]$ $(y_1, y_2, ..., y_m)$ imply that $(y_1, y_2, ..., y_m) \in U[B]$). The random vectors $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k$ (or their joint distribution function) are said to be setwise positively lower set dependent (SPLD) if

$$P\left[\bigcap_{i=1}^{k} \{\mathbf{X}_i \in B_i\}\right] \geqslant \prod_{i=1}^{k} P[\mathbf{X}_i \in B_i],$$

for all lower sets B_i in $\mathbb{R}^{m_i}$, $i=1, 2, ..., k$. It is not hard to verify that $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k$ are SPUSD if, and only if,

$$E\left[\prod_{i=1}^{k} g_i(\mathbf{X}_i)\right] \geqslant \prod_{i=1}^{k} E[g_i(\mathbf{X}_i)],$$

for all nonnegative increasing m_i-dimensional functions g_i, $i=1, 2, ..., k$, for which the expectations exist. Similarly, $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k$ are SPLSD if, and only if,

$$E\left[\prod_{i=1}^{k} h_i(\mathbf{X}_i)\right] \geqslant \prod_{i=1}^{k} E[h_i(\mathbf{X}_i)],$$

for all nonnegative decreasing m_i-dimensional functions h_i, $i=1, 2, ..., k$, for which the expectations exist. In particular, when $k=2$, then $\mathbf{X}_1$ and $\mathbf{X}_2$ are SPUSD if, and only if,

$$\operatorname{Cov}(g_1(\mathbf{X}_1), g_2(\mathbf{X}_2)) \geqslant 0,$$

for all nonnegative increasing functions g_1 and g_2 (of the proper dimensions) for which the covariance is well defined. A similar statement holds also for pairs of SPLSD random vectors.

These setwise positive dependence properties are often inherited from the linkage by the resulting joint distribution. This is shown in the next result. Note that no continuity assumptions are needed for the validity of this theorem.

THEOREM 4.1. *Let $\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_k$ be distributed according to a linkage L, and let $F_1, F_2, ..., F_k$ be k (possibly multivariate) distributions. Let F be a distribution that has the linkage L and marginals $F_1, F_2, ..., F_k$ (that is, F is the distribution of*

$$(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k) = (\Psi^*_{F_1}(\mathbf{U}_1), \Psi^*_{F_2}(\mathbf{U}_2), ..., \Psi^*_{F_k}(\mathbf{U}_k)); \tag{4.1}$$

see (3.2) *and* (3.3)). *If L is* SPUPD (SPLSD) *and if each F_i is* CIS, *then F is* SPUSD (SPLSD).

Proof. We only prove the SPUSD part of the theorem; the proof of the SPLSD is similar. Let g_i, $i=1, 2, ..., k$, be nonnegative increasing functions of the proper dimension. From Lemma 2.2 we see that $\Psi^*_{F_i}(\mathbf{u}_i)$ is increasing in $\mathbf{u}_i$ (since F_i is CIS), and therefore $g_i(\Psi^{-1}_{F_i}(\mathbf{u}_i))$ is increasing in $\mathbf{u}_i$. Now,

$$\begin{aligned} E\left[\prod_{i=1}^{k} g_i(\mathbf{X}_i)\right] &= E\left[\prod_{i=1}^{k} g_i(\Psi^*_{F_i}(\mathbf{U}_i))\right] \\ &\geqslant \prod_{i=1}^{k} E[g_i(\Psi^*_{F_i}(\mathbf{U}_i))] \\ &= \prod_{i=1}^{k} E[g_i(\mathbf{X}_i)], \end{aligned}$$

where the first equality follows from (4.1), the inequality follows from the assumption that L is SPUSD, and the last equality follows from (4.1) again. This completes the proof for the SPUSD case. ∎

The property of association (see (2.10)) is also often inherited from the linkage by the resulting joint distribution. This is shown in the next result. We say that a linkage L is associated if $\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_k$ have the joint distribution L and the vector $(\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_k)$ (of dimension $\sum_{i=1}^{k} m_i$) is associated in the sense that

$$\text{Cov}(g(\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_k), h(\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_k)) \geqslant 0,$$

for all increasing functions g and h (of dimension $\sum_{i=1}^{k} m_i$) for which the covariance is defined. Note that, although each $\mathbf{U}_i$ consists of independent random variables, the whole vector $(\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_k)$ can be positively dependent because of some positive relationship *between* the $\mathbf{U}_i$'s. Again, note that no continuity assumptions are needed for the validity of the next theorem.

THEOREM 4.2. *Let $L, F_1, F_2, ..., F_k$ and F be in Theorem* 4.1. *If L is associated, and if each F_i is* CIS, *then F is associated.*

Proof. In the proof of Theorem 4.1 we argued that, under the CIS assumption, $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k$ of (4.1) can be represented as increasing functions of $\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_k$, respectively, where the $\mathbf{U}_i$'s are described in Theorem 4.1. It is well known that increasing functions of associated random variables are associated (see, e.g., [1]). This observation gives the stated result. ∎

Note that in Theorem 4.2 the assumption that each F_i is CIS implies at once that each vector $\mathbf{X}_i$ is associated from *within* (see, e.g., [1]). The association of the linkage gives us then the positive dependence (within and between) of all the $\sum_{i=1}^{k} m_i$ underlying random variables.

Chhetry, Sampson, and Kimeldorf [4] extended the notion of PDM to the multivariate case as follows. The random vector $\mathbf{X} = (\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k)$ (or its distribution function), where each $\mathbf{X}_i$ is m-dimensional, is said to be setwise dependent by mixture (SDM) if the joint distribution function F of $(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k)$ has the representation

$$F(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k) = \int_{\Omega} \prod_{i=1}^{k} G^{(\mathbf{w})}(\mathbf{x}_i)\, dH(\mathbf{w}), \tag{4.2}$$

where Ω is some subset of a finite-dimensional Euclidean space, $\{G^{(\mathbf{w})}, \mathbf{w} \in \Omega\}$ is some family of m-variate distribution functions, and H is a distribution function on Ω. Note that if $\mathbf{X}$ is SDM then $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k$ all have the same marginal distribution functions.

In the next result it is shown that the property of SDM is inherited from the linkage by the resulting distribution function, under proper dimensionality conditions combined with the requirement that the marginal distribution functions are all equal. Again, no continuity assumptions are needed for the validity of the next theorem. Also, note that the marginals here are not required to be CIS.

THEOREM 4.3. *Consider a linkage L and let $F_1 = F_2 = \cdots = F_k$ be k m-dimensional distributions, and consider the distribution F of the vector defined in* (4.1). *If Lis* SDM *then F is* SDM.

Proof. Since $F_1 = F_2 = \cdots = F_k$ it follows that $\Psi_{F_1} = \Psi_{F_2} = \cdots = \Psi_{F_k} = \Psi$, say. If $(\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_k)$ is distributed according to L then, from the fact that L has the representation (4.2) it follows that, for all collections $\{A_1, A_2, ..., A_k\}$ of k Borel sets in $\mathbb{R}^m$,

$$P\{\mathbf{U}_i \in A_i, i=1, 2, ..., k\} = \int_\Omega \prod_{i=1}^{k} P^{(\mathbf{w})}\{A_i\}\, dH(\mathbf{w}), \tag{4.3}$$

where Ω is some subset of a finite-dimensional Euclidean space, $\{P^{(\mathbf{w})}, \mathbf{w} \in \Omega\}$ is some family of probability measures on $\mathbb{R}^m$, and H is a distribution function on Ω. Thus, if $(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k)$ is distributed according to F, then, for any collection $\{B_1, B_2, ..., B_k\}$ of k Borel sets in $\mathbb{R}^m$,

$$\begin{aligned} P\{\mathbf{X}_i \in B_i, i=1, 2, ..., k\} &= P\{\Psi^*(\mathbf{U}_i) \in B_i, i=1, 2, ..., k\} \\ &= P\{\mathbf{U}_i \in (\Psi^*)^{-1}(B_i), i=1, 2, ..., k\} \\ &= \int_\Omega \prod_{i=1}^{k} \Omega^{(\mathbf{w})}\{B_i\}\, dH(\mathbf{w}), \end{aligned}$$

where the first equality follows from (4.1), the third equality follows from (4.3), and $\Omega^{(\mathbf{w})}\{\cdot\} = P^{(\mathbf{w})}\{(\Psi^*)^{-1}(\cdot)\}$ for all $\mathbf{w} \in \Omega$. Thus F has the representation (4.2) and therefore it is SDM. ▌

5. Stochastic Comparisons

In this section we extend Proposition 2.4 to the multivariate case.

Let $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k$ and $\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_k$ be two sets of (possibly dependent) random vectors. In order to be able to stochastically compare these sets, it is necessary that $\mathbf{X}_i$ and $\mathbf{Y}_i$ have the same dimension, m_i, say, $i=1, 2, ..., k$. Let F be the joint distribution of $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k$, with marginals F_i, $i=1, 2, ..., k$, and let G be the joint distribution of $\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_k$, with marginals G_i, $i=1, 2, ..., k$. We will assume below that $(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k)$ and $(\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_k)$ have the same linkage. That is, we will assume that

$$(\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_k) = (\Psi_{F_1}(\mathbf{X}_1), \Psi_{F_2}(\mathbf{X}_2), ..., \Psi_{F_k}(\mathbf{X}_k))$$

and

$$(\mathbf{V}_1, \mathbf{V}_2, ..., \mathbf{V}_k) = (\Psi_{G_1}(\mathbf{Y}_1), \Psi_{G_2}(\mathbf{Y}_2), ..., \Psi_{G_k}(\mathbf{Y}_k))$$

satisfy

$$((\mathbf{U}_1, \mathbf{U}_2, ..., \mathbf{U}_k) =_{\text{st}} (\mathbf{V}_1, \mathbf{V}_2, ..., \mathbf{V}_k) \tag{5.1}$$

with the common linkage L. One would expect in this case, in light of Proposition 2.4, that if $\mathbf{X}_i \leqslant_{\text{st}} \mathbf{Y}_i$, $i=1, 2, ..., k$, then $(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k) \leqslant_{\text{st}}$

$(\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_k)$. By just assuming $\mathbf{X}_i \leqslant_{\mathrm{st}} \mathbf{Y}_i$, $i=1, 2, ..., k$, we were not able to obtain the conclusion $(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k) \leqslant_{\mathrm{st}} (\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_k)$. We need to assume a little more (see Remark 5.3 and Theorem 5.4 below). Before stating Theorem 5.4 we first state and prove a special case of it which corresponds to the choice of $k=2$ and $m_1=m_2=2$.

THEOREM 5.1. *Let (X_{11}, X_{12}) and (X_{21}, X_{22}) be two possibly dependent bivariate random vectors with an absolutely continuous joint distribution F and marginal distributions F_1 and F_2, respectively. Let (Y_{11}, Y_{12}) and (Y_{21}, Y_{22}) be two other (possibly dependent) bivariate random vectors with an absolutely continuous joint distribution G and marginal distributions G_1 and G_2, respectively. Suppose that $((X_{11}, X_{12}), (X_{21}, X_{22}))$ and $((Y_{11}, Y_{12}), (Y_{21}, Y_{22}))$ have the same linkage in the sense that*

$$\begin{aligned}
&((U_{11}, U_{12}), (U_{21}, U_{22}))\\
&\quad = (\Psi_{F_1}(X_{11}, X_{12}), \Psi_{F_2}(X_{21}, X_{22}))\\
&\quad = ((F_{11}(X_{11}), F_{12|11}(X_{12} \mid X_{11})), (F_{21}(X_{21}), F_{22|21}(X_{22} \mid X_{21})))
\end{aligned}$$

and

$$\begin{aligned}
&((V_{11}, V_{12}), (V_{21}, V_{22}))\\
&\quad = (\Psi_{G_1}(Y_{11}, Y_{12}), \Psi_{G_2}(Y_{21}, Y_{22}))\\
&\quad = ((G_{11}(Y_{11}), G_{12|11}(Y_{12} \mid Y_{11})), (G_{21}(Y_{21}), G_{22|21}(Y_{22} \mid Y_{21})))
\end{aligned}$$

satisfy

$$((U_{11}, U_{12}), (U_{21}, U_{22})) =_{\mathrm{st}} ((V_{11}, V_{12}), (V_{21}, V_{22})),$$

with a common distribution (= linkage) $L((u_{11}, u_{12}), (u_{21}, u_{22}))$, say, where here $F_{ij}[G_{ij}]$ denote the univariate distribution of $X_{ij}[Y_{ij}]$, and $F_{ij \mid ik}[G_{ij \mid ik}]$ denote the conditional distribution of $X_{ij}[Y_{ij}]$ given $X_{ik}[Y_{ik}]$. If

$$X_{11} \leqslant_{\mathrm{st}} Y_{11}, \tag{5.2}$$

$$[X_{12} \mid X_{11} = x_{11}] \leqslant_{\mathrm{st}} [Y_{12} \mid Y_{11} = y_{11}] \qquad \textit{whenever} \quad x_{11} \leqslant y_{11}, \tag{5.3}$$

$$X_{21} \leqslant_{\mathrm{st}} Y_{21}, \tag{5.4}$$

$$[X_{22} \mid X_{21} = x_{21}] \leqslant_{\mathrm{st}} [Y_{22} \mid Y_{21} = y_{21}] \qquad \textit{whenever} \quad x_{21} \leqslant y_{21}, \tag{5.5}$$

then

$$((X_{11}, X_{12}), (X_{21}, X_{22})) \leqslant_{\mathrm{st}} ((Y_{11}, Y_{12}), (Y_{21}, Y_{22})). \tag{5.6}$$

Proof. Let U_{11}, U_{12}, U_{21}, and U_{22} be four jointly distributed uniform $[0, 1]$ random variables, defined on some probability space and having the joint distribution $L((u_{11}, u_{12}), (u_{21}, u_{22}))$ mentioned above. Define on the same probability space as the U_{ij}'s

$$\begin{aligned}&((\hat{X}_{11}, \hat{X}_{12}), (\hat{X}_{21}, \hat{X}_{22}))\\&=(\Psi^*_{F_1}(U_{11}, U_{12}), \Psi^*_{F_2}(U_{21}, U_{22}))\\&=((F_{11}^{-1}(U_{11}), F_{12|11}^{-1}(U_{12} \mid F_{11}^{-1}(U_{11}))), (F_{21}^{-1}(U_{21}), F_{22|21}^{-1}(U_{22} \mid F_{21}^{-1}(U_{21}))))\end{aligned} \tag{5.7}$$

and

$$\begin{aligned}&((\hat{Y}_{11}, \hat{Y}_{12}), (\hat{Y}_{21}, \hat{Y}_{22}))\\&=(\Psi^*_{G_1}(U_{11}, U_{12}), \Psi^*_{G_2}(U_{21}, U_{22}))\\&=((G_{11}^{-1}(U_{11}), G_{12|11}^{-1}(U_{12} \mid G_{11}^{-1}(U_{11}))), (G_{21}^{-1}(U_{21}), G_{22|21}^{-1}(U_{22} \mid G_{21}^{-1}(U_{21}))))\end{aligned} \tag{5.8}$$

Then, from (3.2) and (3.3), using the assumption that $((X_{11}, X_{12}), (X_{21}, X_{22}))$ and $((Y_{11}, Y_{12}), (Y_{21}, Y_{22}))$ have the same linkage L, we get

$$((\hat{X}_{11}, \hat{X}_{12}), (\hat{X}_{21}, \hat{X}_{22})) =_{\text{st}} ((X_{11}, X_{12}), (X_{21}, X_{22})),$$

$$((\hat{Y}_{11}, \hat{Y}_{12}), (\hat{Y}_{21}, \hat{Y}_{22})) =_{\text{st}} ((Y_{11}, Y_{12}), (Y_{21}, Y_{22})).$$

Thus, in order to obtain (5.6) we just need to show that

$$((\hat{X}_{11}, \hat{X}_{12}), (\hat{X}_{21}, \hat{X}_{22})) \leqslant_{\text{a.s.}} ((\hat{Y}_{11}, \hat{Y}_{12}), (\hat{Y}_{21}, \hat{Y}_{22}))$$

From (5.2) it follows that

$$\hat{X}_{11} = F_{11}^{-1}(U_{11}) \leqslant_{\text{a.s.}} G_{11}^{-1}(U_{11}) = \hat{Y}_{11}.$$

Using the fact that $F_{11}^{-1}(U_{11}) \leqslant G_{11}^{-1}(U_{11})$, it follows from (5.3) that

$$\hat{X}_{12} = F_{12|11}^{-1}(U_{12} | F_{11}^{-1}(U_{11})) \leqslant_{\text{a.s.}} G_{12|11}^{-1}(U_{12} | G_{11}^{-1}(U_{11})) = \hat{Y}_{12}.$$

Similarly, using (5.4) and (5.5), it can be shown that

$$\hat{X}_{21} \leqslant_{\text{a.s.}} \hat{Y}_{21}$$

and that

$$\hat{X}_{22} \leqslant_{\text{a.s.}} \hat{Y}_{22}.$$

This completes the proof of (5.9). ∎

Remark 5.2. The assumption of absolute continuity in Theorem 5.1 is not needed in the following sense. If $((U_{11}, U_{12}), (U_{21}, U_{22}))$ is such that U_{11} and U_{12} are independent uniform $[0, 1]$ random variables and U_{21} and U_{22} are independent uniform $[0, 1]$ random variables, and if $((X_{11}, X_{12}), (X_{21}, X_{22}))$ and $((Y_{11}, Y_{12}), (Y_{21}, Y_{22}))$ have the same distribution as $((\hat{X}_{11}, \hat{X}_{12}), (\hat{X}_{21}, \hat{X}_{22}))$ and $((\hat{Y}_{11}, \hat{Y}_{12}), (\hat{Y}_{21}, \hat{Y}_{22}))$ of (5.7) and (5.8), then (5.6) holds, even if $((X_{11}, X_{12}), (X_{21}, X_{22}))$ and $((Y_{11}, Y_{12}), (Y_{21}, Y_{22}))$ do not have continuous distribution functions. The same remark applies also to Theorem 5.4 below.

Remark 5.3. From a result of Veinott [25] it follows that if (5.2) and (5.3) hold then $(X_{11}, X_{12}) \leqslant_{\text{st}} (Y_{11}, Y_{12})$. Similarly, (5.4) and (5.5) together is a stronger assumption than merely assuming $(X_{21}, X_{22}) \leqslant_{\text{st}} (Y_{21}, Y_{22})$. In fact, the result of Veinott [25] says, for random vectors $\mathbf{W} = (W_1, W_2, ..., W_n)$ and $\mathbf{Z} = (Z_1, Z_2, ..., Z_n)$, that if

$$W_1 \leqslant_{\text{st}} Z_1 \tag{5.10}$$

and if

$$[W_i \mid W_1 = w_1, W_2 = w_2, ..., W_{i-1} = w_{i-1}] \leqslant_{\text{st}} [Z_i \mid Z_1 = z_1, Z_2 = z_2, ..., Z_{i-1} = z_{i-1}] \tag{5.11}$$

whenever $w_j \leqslant z_j, j = 1, 2, ..., i-1, i = 2, 3, ..., n$, then $\mathbf{W} \leqslant_{\text{st}} \mathbf{Z}$. Below, if $\mathbf{W}$ and $\mathbf{Z}$ satisfy (5.10) and (5.11), we will denote it by $\mathbf{W} \leqslant_{\text{sst}} \mathbf{Z}$. Note that $\leqslant_{\text{sst}}$ is not an order in the usual sense. In fact, it is obvious that

$$\mathbf{W} \leqslant_{\text{sst}} \mathbf{W} \Leftrightarrow (\mathbf{W} \text{ is CIS}). \tag{5.12}$$

This representation of the positive dependence notion of CIS is reminiscent of the characterizations of some other positive dependence orders discussed in Shaked and Shanthikumar [22].

The proof of the next theorem is a straightforward extension of the proof of Theorem 5.1, and therefore it is omitted.

THEOREM 5.4. *Let* $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k$ *and* $\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_k$ *be two sets of* (*possibly dependent*) *random vectors.* (*We assume that* $\mathbf{X}_i$ *and* $\mathbf{Y}_i$ *have the same dimension,* m_i, *say,* $i = 1, 2, ..., k$.) *If* $(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k)$ *and* $(\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_k)$ *have the same linkage* (*in the sense of* (5.1)) *and if*

$$\mathbf{X}_i \leqslant_{\text{sst}} \mathbf{Y}_i, \qquad i = 1, 2, ..., k \tag{5.13}$$

then

$$(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k) \leqslant_{\text{st}} (\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_k). \tag{5.14}$$

One may ask whether, under the conditions of Theorem 5.4, it is possible to obtain the conclusion $(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k) \leqslant_{\text{sst}} (\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_k)$, which is stronger than (5.14). It turns out that, in general, that is not the case. In order to see it, take $\mathbf{X}_i = \mathbf{Y}_i$, $i = 1, 2, ..., k$, in Theorem 5.4, where each $\mathbf{X}_i$ is CIS. Then, by (5.12), we have that (5.13) holds. However, if the conclusion of Theorem 5.4 were $(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k) \leqslant_{\text{sst}} (\mathbf{Y}_1, \mathbf{Y}_2, ..., \mathbf{Y}_k)$, that is, $(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k) \leqslant_{\text{sst}} (\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k)$, then it would have followed, again by (5.12), that $(\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_k)$ is CIS. But in general this need not be true. For example, let $\mathbf{X}_1 = (W, W + Z)$ and $X_2 = Z$, where W and Z are independent random variables. Then $\mathbf{X}_1 \leqslant_{\text{sst}} \mathbf{X}_1$ (since $\mathbf{X}_1$ is CIS), and $X_2 \leqslant_{\text{sst}} X_2$ (for univariate random variables $\leqslant_{\text{sst}}$ and $\leqslant_{\text{st}}$ are the same), but $((W, W+Z), Z) = (W, W+Z, Z)$ is not CIS. The latter claim can be seen from the fact that $[Z \mid W = a_1, W + Z = a_2] = [Z \mid W = a_1, Z = a_2 - a_1] =_{\text{st}} [Z | Z = a_2 - a_1]$, and $[Z | Z = a_2 - a_1]$ is stochastically decreasing (rather than increasing) in a_1.

6. Discussion

In this paper we introduced the linkage function and derived some basic properties of it. We also indicated, through some examples, how the linkage function can be computed, or at least described by means of the associated uniform [0, 1] random variables (see, e.g., Example 3.3); other similar examples can be routinely worked out. However, many questions regarding the linkage function are still unanswered.

Perhaps the most befuddling aspect of the linkage function is its dependence on the order of the random variables within each vector that is transformed to independent uniform random variables. Thus, when we say that "most of the information regarding the dependence structure *between* the $\mathbf{X}_i$'s is contained in the linkage function, whereas the information regarding the dependence structure *within* the $\mathbf{X}_i$'s is erased by it," we say it intuitively and not quantitatively. The question of how the order of the random variables influences the amount of information about the dependence structure between the $\mathbf{X}_i$'s, that is contained in the resulting (different) linkage functions, is still untouched and unanswered. Therefore, we do not know what is the best choice of order to use, if there is no natural indexing. In the permutation symmetric case it seems that the order of the random variables is irrelevant, but, even then, there is a loss of some symmetry. For instance, in Example 3.3 all the pairs of variables W_i and

Z_j are equally correlated, but each pair U_i and V_j of the resulting uniform $[0, 1]$ variables has a different correlation.

Another related question regarding the particular definition of the linkage function in this paper is the use of the *standard* construction. There exist several other constructions that can transform independent uniform random variables to a desired vector $\mathbf{X}$. A study that parallels the present paper, but which applies to a construction that is different than the standard construction, can be undertaken, and then compared to the present study.

Our study of the linkage function somewhat parallels previous studies of the copula function. However, during the last 35 years many properties of the copula function have been discovered, and we could not investigate all the possible analogous properties of the linkage function in the present paper. For example, a referee has asked us whether there is an analog of the Fréchet upper bound for linkage functions. Since a linkage function is a proper multivariate distribution with uniform $[0, 1]$ marginal one can use known results that give bounds on multivariate distributions. Let L be a linkage function, that is, the joint distribution of $\mathbf{U} = (\mathbf{U}_1, ..., \mathbf{U}_k) = ((U_{11}, ..., U_{1m_1}), ..., (U_{k1}, ..., U_{km_k}))$, where within each $\mathbf{U}_i$ the random variables are independent. Then, by Rüschendorf [16], (1981),

$$\max\left(\sum_{n=1}^{k}\prod_{l=1}^{m_n} u_{nl} - (k-1), 0\right) \leqslant L((u_{11}, ..., u_{1m_1}), ..., (u_{k1}, ..., u_{km_k}))$$

$$\leqslant \min\left(\prod_{l=1}^{m_1} u_{1l}, ..., \prod_{l=1}^{m_k} u_{kl}\right),$$

and for fixed u_{ij}'s, $j = 1, 2, ..., m_i$, $i = 1, 2, ..., k$, the bounds are sharp.

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