Virtual Line-Mass Harmonic Analysis Based on Satellite Gravity Gradients*

Wu Xing\textsuperscript{a}, Zhang Chuanding\textsuperscript{b}, Wang Kai\textsuperscript{c}, Feng Wei\textsuperscript{d}

\textsuperscript{a}Beijing Special Type Engineering Design and Research Institute Beijing, China
\textsuperscript{b}College of Earth Science Graduate University of Chinese Academy of Sciences Beijing, China
\textsuperscript{c}Institute of Surveying and Mapping Information Engineering University Zhengzhou, Henan Province, China
\textsuperscript{d}Global Information Application and Development Center of Beijing Beijing, China

wuxing1979@163.com, wsdhld@yahoo.com, cn pladsps@sina.com

Abstract

Targeting at the new type data called gravity gradient, the basic formulae and equations of the global virtual line mass model and harmonic analysis are made in this paper. The virtual line mass models based on radius real component, horizontal complex component and full tensor are studied. So are the least squares solutions. The harmonic analysis formulae of virtual line mass models are shown completely, which have various advantages such as eliminating the disperse errors and dealing with multi-types data compared with virtual point mass harmonic analysis. With consideration of different depths, the virtual line mass method is better than the virtual point mass method with various frequencies. Finally, the availability and validity are made through simulation experiments.

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1. Foreword

Point mass method, put forward after the Bjerhammar method, is a numerical approximation of the exterior earth gravity field, of which the principle is that based on the non-uniqueness of the field source of field, virtual point mass distributed on the Bjerhammar sphere are chose to be acted as the abnormal source of the earth. It has various advantages in both theory and practice [2-5], and is wildly used in the calculation of the trajectory disturbing gravitation [2], determination of the local truncated geoid [3] and

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so on. Besides, as the discrete data are treated as continuous data in the method, point mass harmonic analysis was put forward in [4], and numerical simulations show that it could overcome the data discrete error in traditional harmonic analysis.

With the successful launch of GOCE (Gravity Field and Steady-State Ocean Circulation Explorer), it is reliable to obtain global uniform distributed gravity gradient data with high precision. Then, how to construct a high precision global gravity field model from these data is the key factor in the processing of GOCE data. We studied single layer point mass method based on gravity gradient data in [5] and found that it could not recover multi-frequent gravity model coefficients. Besides, multi-layers point mass method overcame the above problem with extremely complicated calculation. In order to solve the problems mentioned above, line mass method based on gravity gradient data is put forward in this paper, numerical results show that its theory is correct with available method.

2. Point mass Model and the Harmonic Spherical Expression

The determination of the exterior earth gravity field based on gravity gradient tensor is equal to the determination of disturbing potential $T$, which satisfies:

$$
\begin{cases}
\Delta T = 0; & \text{on } \Sigma \text{ and exterior} \\
B_q T|_\Sigma = f_q; & \text{on } \Sigma, q = 0, 1, 2 \\
T = o(r^{-3}), & r \to \infty
\end{cases}
$$

(1)

where, $\Sigma$ is the boundary sphere of the observational gravity gradient, $B_q$ mean various arithmetic operators of various gravity field cells. $\nabla_q \nabla_\rho(\alpha, \beta = -1, 0, 1)$ are local complex gradient arithmetic operators, which could be expressed as:

$$\nabla_q \nabla_\rho = 2 \nabla_z \nabla_\rho = \partial_{z^2} = \partial_{rr}
$$

(2)

$$\nabla_z \nabla_0 = \nabla_0 \nabla_z = \pm (\partial_{zz} \pm j \partial_{z z})/\sqrt{2}
$$

(3)

$$\nabla_z \nabla_z = [\partial_{z z} \pm 2 j \partial_{zz}]/2
$$

(4)

where, $j = \sqrt{-1}$, $(x, y, z)$ is in local north-east-up coordinate system, $z$ points the radius direction, $x$ points the north, $y$ together with $x$, and $z$ constituent a left hand coordinate system, $(r, \psi, \gamma)$ is in polar coordinate system of which the origin is calculation point $P$, $\psi$ is the spherical argument of $P$ and fluxional point $Q$, $\gamma$ is the azimuth angle from $P$ to $Q$, $f_q$ is the corresponding value.

Bjerhammar layer is in inner earth with the radius of $R_g$, of which the density is $\rho$, then, the potential at $P$ from the sphere layer $T^*$ is a harmonic function:

$$T^*(P) = G \int_{\rho} \rho |_{\rho Q} \, d\sigma
$$

(5)

where $G$ is the gravitational constant, $l_{\rho Q}$ is the distance between $P$ and $Q$.

With the equation satisfied:

$$B_q T^*|_\Sigma = B_q T|_\Sigma
$$

(6)
due to the uniqueness of the boundary value problem, in the entire exterior field of the earth:

$$T(P) = T^*(P)$$  \hspace{1cm} (7)

This is ensured by the Runge theorem [8]. With the discrete observational gravity data on the boundary, the discrete approximate numerical solution of the integral equation is made in usual.

Divided the whole boundary sphere into $N \times 2M$ equiangular grids with $N$ and $2M$ indicating the grid numbers in latitude and longitude directions separately and $\overline{f}_{q, k}$ indicating the mean disturbing gravity gradient of each grid as the observational data. Meanwhile, with the assumption that the Bjerhammar sphere $S$ is divided into grids as the same, of which the densities are approximately constants. Then, (7) is expressed approximately

$$T(P) = G \sum_{i=0}^{N-1} \sum_{j=0}^{2M-1} \rho_{i,j} \mid l_{PQ_i} \Delta \sigma_{i,j}$$  \hspace{1cm} (8)

where, $l_{PQ_i}$ is the special distance between $P$ and $Q$, which is the centre point of $\Delta \sigma_{i,j}$. Let $\delta M_{i,j} = \rho_{i,j} \Delta \sigma_{i,j}$ be the mass of $\Delta \sigma_{i,j}$, then, (8) can be rewritten as

$$T(P) = G \sum_{i=0}^{N-1} \sum_{j=0}^{2M-1} \delta M_{i,j} \mid l_{PQ_i}$$  \hspace{1cm} (9)

Then, the discretion of the sphere layer gravitational potential into gravitational potentials of point massed is made with $\delta M_{i,j}$ indicating point mass.

The basic point mass harmonic analysis is:

$$\bar{C}_{mn} + j\bar{S}_{mn} = \frac{1}{2 n + 1} \sum_{i=0}^{n-1} \sum_{j=0}^{2M-1} \rho_{i,j} \mid l_{PQ_i} \delta M_{i,j}$$  \hspace{1cm} (10)

From (10) we could find that different buried depths result in different coefficient matrix structure. Thus, the selection of buried depths is the key factor in point mass harmonic analysis. However, because of the relationship between the depth and the spectral characteristics of the gravity field, the single layer point mass model can recover the whole spectral frequent gravity field.

3. Line mass Model and the Harmonic Spherical Expression

Assume that there is a sphere layer in inner earth with $R_{g1} - R_{g2}$ thickness, of which the radius density is $\rho'(\theta, \lambda)$. Then, the gravitational potential at $P$ from the sphere layer could be obtained from (5):

$$T^*(P) = G \int_{R_{g1}}^{R_{g2}} \int_{0}^{2\pi} \rho'(\theta, \lambda) \mid l_{PQ_i} d\sigma dR_g$$  \hspace{1cm} (11)

With the satisfaction of (6), (7) is also the same as in the point mass analysis.

With the whole boundary and Bjerhammar spheres divided into $N \times 2M$ equiangular grids as that in the point mass analysis and the densities of each cube be constants, (7) can approximately be expressed as:

$$T(P) = G \sum_{i=0}^{N-1} \sum_{j=0}^{2M-1} \rho_{i,j}' \Delta \sigma_{i,j} \mid l_{PQ_i}$$  \hspace{1cm} (12)

Let $\delta m'_{i,j} = \rho_{i,j}' \Delta \sigma_{i,j} \mid (R_{g1} - R_{g2})$ which is called as line mass, $\rho_{i,j}'$, line density and
\[1/T_{pq}\sigma = 1/(R_{R1} - R_{R2}) \int_{R_{R2}}^{R_{R1}} 1/T_{pq}\sigma dR_{q}\]  

(13)

Then, (7) can be rewritten as

\[T(P) = G \sum_{n=0}^{N} \sum_{m=-n}^{n} \delta m_{n}\sigma \int_{T_{pq}\sigma} (14)

\]

, which is the same as that of the point mass model.

Spherical spectral expression of \(1/T_{pq}\sigma\) can be obtained from that of \(1/T_{pq}\sigma\):

\[1/T_{pq}\sigma = \sum_{n=0}^{N} \int (2n+1) \left( \begin{array}{c} R_{R1}^{n+1} - R_{R2}^{n+1} \\ (R_{R1} - R_{R2})(n+1)R^{n+1} \end{array} \right) \cdot \sum_{m=-n}^{n} \int (2-\rho_{0}^{\omega})(\cos\theta) \bar{P}_{m}\sigma (\cos\theta_{i}) e^{i\lambda(\lambda-\lambda_{i})} \]

\[(15)\]

Compare the spherical harmonic function expansion coefficients of \(1/T_{pq}\sigma\) and \(T(P)\) with (14) considered, the line mass harmonic analysis formula is obtained:

\[C_{m n} + jS_{m n} = \frac{1}{2n+1} \left( \frac{R_{R1}^{n+1} - R_{R2}^{n+1}}{(R_{R1} - R_{R2})(n+1)R^{n+1}} \right) e^{\frac{m\lambda_{i}}{2}} \cdot \sum_{x=0}^{N} \bar{P}_{n}\sigma (\cos\theta_{i}) \delta m_{x}(m) \]

\[(16)\]

With comparison of (10) and (16), there is only one control factor in point mass analysis while there are two in the line mass analysis. Line mass harmonic analysis method could overcome the problem brought from the buried depth and improve the stability of the harmonic solution.

4. Observational Equation and the Solution of SGG Based on Line Mass Model

Based on (14) and line mass model, the discrete spherical boundary value conditional equation of the mean gravity gradient grid is

\[T_{q',s} = G \sum_{s=0}^{N} \sum_{r=0}^{M} \bar{T}_{r,s} \delta m_{r} \]

\[(17)\]

in which \(\bar{T}_{r,s}\) is

\[\bar{T}_{r,s} = \iint l_{r,s}^{n} d\sigma_{r} / \iint d\sigma_{s} \]

\[(18)\]

where,

\[l_{r,s}^{n} = B_{q} (1/T_{w}) \]

\[(19)\]

As shown in Fig. 1, \(R_{r}\) is the radius of the boundary sphere, \(l_{r,s}\) and \(\psi_{r,s}\) indicate the distance and the gnomonic argument between calculation point \(P_{r}\) and fluxional point \(Q_{s}\) with

\[l_{r,s} = \sqrt{R_{r}^{2} + R_{s}^{2} - 2R_{r}R_{s}\cos\psi_{r,s}} \]

\[(20)\]

\[\cos\psi_{r,s} = \cos\theta_{r}\cos\theta_{s} + \sin\theta_{r}\sin\theta_{s}\cos(\lambda_{r} - \lambda_{s}) \]

\[(21)\]

where, \(\gamma_{r,s}\) is the spherical azimuth from \(P_{r}\) to \(Q_{s}\), which could be evaluated from
\[
\sin \gamma_{dk} = \sin^{-1} \psi_{dk} \sin \theta_{s} \sin(\lambda_{i} - \lambda_{s}) \quad (22)
\]

With equations (2)~(4), \( \hat{L}_{sk}^{nt} \) can be calculated from (19). Substitution of \( \hat{L}_{sk}^{nt} \) into (18), \( \tilde{L}_{sk}^{nt} \) can be obtained from ordinary numerical integral methods.

Equation (17) is the basic function of the construction of line mass model from disturbing gravity gradient data. The solving method is the same with that of the point mass model [6]. The least squares solutions in frequency domain from the observational gravity gradient data such as radius component, horizontal component or the whole tensor are listed separately:

\[
\hat{\mathbf{m}}'(m) = \mathbf{R}_0^{-1}(m) \mathbf{f}_0'(m), \quad q = 0
\]

\[
\hat{\mathbf{m}}'(m) = \left[ \mathbf{R}_0^{-1}(m) \mathbf{R}_q^{-1}(m) \right]^{-1} \mathbf{R}_q^{-1}(m) \mathbf{f}_q'(m), \quad q = 1, 2
\]

\[
\hat{\mathbf{m}}'(m) = \left[ \mathbf{R}_0^{-1}(m) \mathbf{R}_0^{-1}(m) + \frac{1}{2} \mathbf{R}_1^{-1}(m) \mathbf{R}_1^{-1}(m) + \frac{1}{2} \mathbf{R}_2^{-1}(m) \mathbf{R}_2^{-1}(m) \right]^{-1}
\]

\[
\times \left[ \mathbf{R}_0^{-1}(m) \mathbf{f}_0'(m) + \frac{1}{2} \mathbf{R}_1^{-1}(m) \mathbf{f}_1'(m) + \frac{1}{2} \mathbf{R}_2^{-1}(m) \mathbf{f}_2'(m) \right]
\]

where \( \mathbf{R}_q(m) \) can be obtained from the combination of the latent root of the coefficient matrix, \( \mathbf{f}_q'(m) \) is consisted of the observational gravity gradient in frequent domain. Then, substitution of the above equations into (16) gives global gravity field model. The method in this paper is called line mass harmonic analysis.

5. Simulation Experiment of the Line Mass Harmonic Analysis

Take the gravity gradient radius component as the example in this experiment. Firstly, observational data of gravity gradient radius component at the resolution of 60’×60’ are obtained through the spherical harmonic synthesis from EGM2008. Secondly, the estimations of gravity field model coefficients are obtained from point mass analysis and line mass analysis separately. The degree variance error of the coefficient estimation is calculated by (26) as the precision evaluation.

\[
\sigma_n = \sqrt{\frac{1}{2n+1} \sum_{n=1}^{M} \left( \hat{C}_{nm}^{est} - C_{nm}^{EGM} \right)^2 + \left( \hat{S}_{nm}^{est} - S_{nm}^{EGM} \right)^2}
\]

where, \( (\hat{C}_{nm}^{est}, \hat{S}_{nm}^{est}) \) are the estimation of the coefficients from the harmonic analysis while \( (C_{nm}^{est}, S_{nm}^{est}) \) are the disturbing gravity model obtained from the reduction of reference model from EGM2008.

In order to be compared with point mass harmonic analysis, due to the experimental resolution at 60’×60’, the buried depth is set to 100 kilometers which is approximately equal to the point mass interval. The degree variance of the point mass harmonic analysis in [4] is shown in Fig. 2 (solid line).

In the calculation of the line mass harmonic analysis, the buried depth ranges of the line mass are 100~300 kilometers, 100~500 kilometers and 100~800 kilometers of which the results are shown in Fig. 2.

From Fig. 2, we could find that the recovery effect of the line mass harmonic analysis is better than that of the single point mass harmonic analysis. The reason is that the line mass has two control parameters. With certain buried depth, it could take different buried depths into account, i.e., the coefficients at various sub-frequencies are considered.
6. Conclusion

The basic function of solution of global disturbing line mass model based on global gravity gradient is constructed here.

Line mass method has the same advantages like the point mass method. It implies the interpolation of the observational data on ground, which overcomes the data discrete errors of general harmonic analysis. Multi-gravity gradient components could be dealt with at the same time. What’s more, that line mass method reflects various buried depth and sub-frequencies, of which the precision is better than that of the point mass method is validated by the numerical experiment in this paper.

References


Fig.1 Presentations of the line mass both in sphere coordinate and in sphere polar coordinate
Fig. 2 Comparison of the point mass harmonic analysis and the line mass harmonic analysis