Duality Theory in Fuzzy Mathematical Programming Problems with Fuzzy Coefficients

CHENG ZHANG
College of Information Engineering
Dalian University, Dalian 116622, P.R. China

XUE-HAI YUAN
Department of Mathematics
Liaoning Normal University, Dalian 116029, P.R. China

E. STANLEY LEE*
Department of Industrial & Manufacturing Systems Engineering
Kansas State University, Manhattan, KS 66506, U.S.A.
eslee@ksu.edu

(Received November 2003, accepted December 2004)

Abstract—In this paper, the notions of subgradient, subdifferential, and differential with respect to convex fuzzy mappings are investigated, which provides the basis for the fuzzy extremum problem theory. We consider the problems of minimizing or maximizing a convex fuzzy mapping over a convex set and develop the necessary and/or sufficient optimality conditions. Furthermore, the concept of saddle-points and minmax theorems under fuzzy environment is discussed. The results obtained are used to formulate the Lagrangian dual of fuzzy programming. Under certain fuzzy convexity assumptions, KKT conditions for fuzzy programming are derived, and the “perturbed” convex fuzzy programming is considered. Finally, these results are applied to fuzzy linear programming and fuzzy quadratic programming. © 2005 Elsevier Ltd. All rights reserved.

Keywords—Convex fuzzy mapping, Subdifferential, Saddle-point, Minmax theorem, Fuzzy programming, Lagrangian dual, KKT conditions, Fuzzy linear programming, Fuzzy quadratic programming.

1. INTRODUCTION

Fuzzy mathematical programming was developed for treating real world problems where the problems are usually vague and not well defined. The use of fuzzy models not only avoids various unrealistic assumptions, but also retains the original realistic information. Fuzzy concepts are used not only to define the objectives and constraints in mathematical programming, but also to reflect the aspiration levels given by the decision maker.

Bellman and Zadeh [1] first proposed the basic concepts of fuzzy decision making. Based on this concept, Zimmermann [2] formulated fuzzy linear programming problems by the use of both the minimum operator, which is noncompensatory, and the product operator, which is compensatory. Since then, many papers have appeared to investigate fuzzy decision making problems.

*Author to whom all correspondence should be addressed.
The collection of papers on fuzzy optimization edited by Rommelfanger and Slowinski [3] and Delgado et al. [4] summarize the main ideas on this topic. Lai and Hwang [5] also give an insightful survey. However, in contrast with the vast literature on modeling and solution procedures for a linear programming in a fuzzy environment, few studies on the duality of fuzzy programming problems have appeared. The duality of fuzzy linear programming was first studied by Rodder and Zimmermann [6] who considered the economic interpretation of the dual variables. Verdegay [7] defined the fuzzy dual problem with the help of a parametric linear programming problem and showed that the fuzzy primal and dual problems both have the same fuzzy solution under some suitable conditions. Sakawa and Yano [8] proposed a fuzzy dual decomposition method for large-scale multiobjective nonlinear programming problems with block angular structure, the Lagrangian function and Lagrangian multipliers in the dual problem were considered. Liu et al. [9] proposed the fuzzy primal and dual problems by considering the fuzzy-max and fuzzy-min in the objective functions as the usual pattern of the linear programming problems. Richardt et al. [10] gave some interesting connections between fuzzy theory, simulated annealing, and convex duality. Bector and Chandra [11] constructed a modified pair of fuzzy primal-dual linear programming problems. Zhong and Shi [12] presented a parametric approach for duality in fuzzy multicriteria and multiconstraint level linear programming which extended fuzzy linear programming approaches. Inuguchi et al. [13] defined the concepts of feasibility and satisfying solutions, the concept of duality, and proved weak and strong duality theorems. Recently, Wu [14–16] gave the insightful survey for duality theory of fuzzy mathematical programming problems. In [14], the fuzzy primal and dual linear programming problems with fuzzy coefficients were formulated by using fuzzy scalar product, weak and strong duality theorems were then proved. In [15], the saddle-point optimality conditions in fuzzy optimization problems were discussed by introducing the fuzzy scalar product and a solution concept that is essentially similar to the notion of Pareto solution in multiobjective optimization problems. In [16], the fuzzy-valued Lagrangian function for the fuzzy mathematical programming problem via the concept of the fuzzy scalar product was proposed, a solution concept of fuzzy optimization problems was also introduced by ranking the fuzzy numbers using the necessity indices. Under these settings, the weak and strong duality theorem could be elicited.

In this paper, the duality theory of fuzzy programming or the saddle-point problem are formulated based on the concept of convexity and convex fuzzy mapping. The concepts of subgradient, subdifferential, and differentials in terms of fuzzy convex mapping were introduced. Based on these concepts, the theory of fuzzy extremum problems was considered and KKT conditions for fuzzy programming were obtained. These results were applied to fuzzy linear and fuzzy quadratic programming problems.

The paper is organized as follows. After some preliminary summaries on the fuzzy numbers and convex fuzzy mappings in Section 2, the notions of subgradient, subdifferential, and differential with respect to convex fuzzy mappings are investigated in Section 3. We consider the problems of minimizing or maximizing a convex fuzzy mapping over a convex set and develop the necessary and/or sufficient optimality conditions. In Section 4, we discuss the concept of saddle-points and minimax theorems under fuzzy environment, the results obtained are used to the Lagrangian dual of fuzzy programming. In Section 5, KKT conditions for fuzzy programming are derived under certain fuzzy convexity assumptions, and the “perturbed” convex fuzzy programming is considered. Finally, the results are applied to fuzzy linear programming and fuzzy quadratic programming.

2. PRELIMINARIES

Let $R$ denote the set of all real numbers. In this paper, a fuzzy number (see [17]) will be a fuzzy set $u : R \rightarrow [0, 1]$ with the following properties (1)–(4).

1. $u$ is upper semicontinuous.
(2) $u$ is normal, i.e., there exists an $x_0 \in R$, with $u(x_0) = 1$.
(3) $u$ is fuzzy convex, i.e., $u((1-\lambda)x+\lambda y) \geq \min\{u(x), u(y)\}$ whenever $x, y \in R$ and $\lambda \in [0, 1]$.
(4) $\text{cl}(\text{supp} \ u) = \text{cl}\{x \in R : u(x) > 0\}$ is a compact set.

The family of all fuzzy numbers will be denoted by $F_0$. Obviously, the $\alpha$-level sets of a fuzzy number $u \in F_0$ is a closed interval (denoted as $[u_*(\alpha), u^*(\alpha)]$).

$$[u_*(\alpha), u^*(\alpha)] = [u]_\alpha = \begin{cases} \{x \in R : u(x) \geq \alpha\}, & \text{if } 0 < \alpha < 1, \\
\text{cl}(\text{supp} \ u), & \text{if } \alpha = 0. \end{cases}$$

**Theorem 2.1. Representation Theorem.** (See [18].) If $u \in F_0$, then $u_*(r)$ and $u^*(r)$ are functions on $[0, 1]$ satisfying conditions (1)-(4).

(1) $u_*(r)$ is a nondecreasing function on $[0, 1]$.
(2) $u^*(r)$ is a nonincreasing function on $[0, 1]$.
(3) $u_*(r)$ and $u^*(r)$ are bounded and left continuous on $(0, 1]$, and right continuous at $r = 0$.
(4) $u_*(1) < u^*(1)$.

Conversely, if functions $u_*(r)$ and $u^*(r)$ on $[0, 1]$ satisfy conditions (1)-(4), then there exist a unique $u \in F_0$ such that $[u]_\alpha = [u_*(\alpha), u^*(\alpha)]$ for $\alpha \in [0, 1]$.

Since each $a \in R$ can be considered as a fuzzy number $\tilde{a}$ defined by

$$\tilde{a}(t) = \begin{cases} 1, & t = a, \\
0, & t \neq a, \end{cases}$$

$R$ can be embedded in $F_0$.

For notational convenience in this paper, we write $\tilde{a}$ as $a$.

From Theorem 2.1, we see that a fuzzy number $u \in F_0$ is determined by the endpoints of the interval $[u]_\alpha$. Thus, we can identify a fuzzy number $u$ with the parameterized triples

$$\{(u_*(\alpha), u^*(\alpha), \alpha) : \alpha \in [0, 1]\},$$

where $u_*(\alpha)$ and $u^*(\alpha)$ denote the left-hand and the right-hand endpoints of $[u]_\alpha$, respectively. Suppose that $u, v \in F_0$ are fuzzy numbers represented by $\{(u_*(\alpha), u^*(\alpha), \alpha) : \alpha \in [0, 1]\}$ and $\{(v_*(\alpha), v^*(\alpha), \alpha) : \alpha \in [0, 1]\}$, respectively. Define a partial ordering $\leq$ in $F_0$ by $u \leq v$, if and only if $[u]_\lambda \leq [v]_\lambda$ for any $\lambda \in [0, 1]$, i.e.,

$$u_*(\lambda) \leq v_*(\lambda), \quad u^*(\lambda) \leq v^*(\lambda).$$

We say that $u < v$, if $u \leq v$ and there exists $\lambda_0 \in [0, 1]$ such that $u_*(\lambda_0) < v_*(\lambda_0)$ or $u^*(\lambda_0) < v^*(\lambda_0)$.

We see that $u = v$, if $u \leq v$ and $u \geq v$. It is often convenient to write $v \geq u$ (respectively, $v > u$) in place of $u \leq v$ (respectively, $u < v$).

A subset $S$ of $F_0$ is said to be bounded from above if there exists a fuzzy number $v \in F_0$, called an upper bound of $S$, such that $u \leq v$ for any $u \in F_0$. $v_0 \in F_0$ is called the supremum of $S$ if $v_0$ is an upper bound of $S$ and satisfies $v_0 \leq v$ for any upper bound $v$ of $S$, and we denote it as $v_0 = \sup u \in A u$. A lower bound and the infimum of $S$ are defined similarly. $S$ is said to be order bounded if it is both bounded from above and bounded from below.

In this paper, the fuzzy numbers that we considered are finite.

Using the extension principle presented by Zadeh [19-21], the binary operation $\ast$ between two fuzzy numbers is defined as follows:

$$(u \ast v)(x) = \sup_{y, z = x} \min\{u(y), v(z)\},$$
where \( \ast \in \{+,-,\times,\land,\lor\} \). That is,

\[
(u + v)(x) = \sup_{y+z=x} \min\{u(y), v(z)\},
\]

\[
(u - v)(x) = \sup_{y-z=x} u(y) \land v(z),
\]

\[
(ku)(x) = \begin{cases} 
  u(k^{-1}x), & \text{if } k \neq 0, \\
  0, & \text{if } k = 0,
\end{cases}
\]

\[
(u \cdot v)(x) = \sup_{yz=x} \min\{u(y), v(z)\},
\]

\[
(u \land v)(x) = \sup_{y \land z=x} \min\{u(y), v(z)\},
\]

\[
(u \lor v)(x) = \sup_{y \lor z=x} \min\{u(y), v(z)\},
\]

for \( u, v \in F_0, k \in R \).

If the fuzzy numbers \( u, v \in F_0 \) represented by \( \{(u_*(\alpha), u^*(\alpha), \alpha) : \alpha \in [0, 1]\} \) and \( \{(v_*(\alpha), v^*(\alpha), \alpha) : \alpha \in [0, 1]\} \), respectively, then the above addition and scalar multiplication can be represented as follows:

\[
\begin{align*}
(u + v)(x) &= \{(u_*(\alpha) + v_*(\alpha), u^*(\alpha) + v^*(\alpha), \alpha) : \alpha \in [0, 1]\}, \\
(ku)(x) &= \{(ku_*(\alpha), ku^*(\alpha), \alpha) : \alpha \in [0, 1]\}, \quad k \geq 0, \\
(ku)(x) &= \{(ku_*(\alpha), ku^*(\alpha), \alpha) : \alpha \in [0, 1]\}, \quad k < 0, \\
[u \cdot v](x) &= [u \cdot v_*(\alpha), (u \cdot v)^*(\alpha)], \\
[u - v](x) &= [u_*(\alpha) - v_*(\alpha), u^*(\alpha) - v^*(\alpha)], \\
(u \land v)(x) &= \{(u_*(\alpha) \land v_*(\alpha), u^*(\alpha) \land v^*(\alpha), \alpha) : \alpha \in [0, 1]\}, \\
(u \lor v)(x) &= \{(u_*(\alpha) \lor v_*(\alpha), u^*(\alpha) \lor v^*(\alpha), \alpha) : \alpha \in [0, 1]\},
\end{align*}
\]

where

\[
(u \cdot v)_*(\alpha) = \min \{u_*(\alpha)v_*(\alpha), u_*(\alpha)v^*(\alpha), u^*(\alpha)v_*(\alpha), u^*(\alpha)v^*(\alpha)\},
\]

\[
(u \cdot v)^*(\alpha) = \max \{u_*(\alpha)v_*(\alpha), u_*(\alpha)v^*(\alpha), u^*(\alpha)v_*(\alpha), u^*(\alpha)v^*(\alpha)\}.
\]

Using the above definitions, it is not difficult to prove the following results.

**Lemma 2.1.** If \( a, b, c, d \in F_0 \), then

1. \( a \leq b \Leftrightarrow a + c \leq b + c \);
2. \( a \leq b \Rightarrow ka \leq kb, \mu a \geq \mu b, (k \geq 0, \mu < 0) \);
3. \( a \leq b \Leftrightarrow -a \geq -b \);
4. \( a \leq b, c \leq d \Rightarrow a + c \leq b + d, a < b, c \leq d \Rightarrow a + c < b + d \);
5. \( b = c \Leftrightarrow a + b = a + c \);
6. \( a \leq b, c \leq d, a + c = b + d \Rightarrow a = b, c = d \).

**Lemma 2.2.** Let \( S \) be a subset of \( F_0 \), if \( \sup S(\inf S) \) exists, then

\[
\begin{align*}
\sup S &= \bigcup_{\lambda \in [0,1]} \lambda \left[ \sup_{u \in S} u_*(\lambda), \sup_{u \in S} u^*(\lambda) \right] = \lor_{u \in S} u, \\
\inf S &= \bigcup_{\lambda \in [0,1]} \lambda \left[ \inf_{u \in S} u_*(\lambda), \inf_{u \in S} u^*(\lambda) \right] = \land_{u \in S} u.
\end{align*}
\]
Lemma 2.3. Let $S$, $T$ be subsets of $F_0$, if $\sup S(\inf S)$ and $\sup T(\inf T)$ exist, then

$$
\sup\{u + v : u \in S, v \in T\} \leq \sup\{u : u \in S\} + \sup\{v : v \in T\},
$$

$$
\inf\{u + v : u \in S, v \in T\} \geq \inf\{u : u \in S\} + \inf\{v : v \in T\}.
$$

$\mathbf{A}$ is said to be an $n$-dimensional fuzzy vector if the components of $\mathbf{A}$ are composed by $n$ fuzzy numbers, denoted by $\mathbf{A} = (x_1, x_2, \ldots, x_n)^T$. The set of all $n$-dimensional fuzzy vectors is denoted by $F_0(R^n)$.

A $\lambda$-level vector of fuzzy vector $\mathbf{A} = (x_1, x_2, \ldots, x_n)^T$ is defined as

$$
[A]_\lambda := ([x_1]_\lambda, [x_2]_\lambda, \ldots, [x_n]_\lambda)^T
$$

and

$$
\mathbf{A}_+ (\lambda) := (x_1(\lambda), x_2(\lambda), \ldots, x_n(\lambda))^T,
$$

$$
\mathbf{A}^* (\lambda) := (x_1^*(\lambda), x_2^*(\lambda), \ldots, x_n^*(\lambda))^T,
$$

where the symbol " := " means that the expression on the left is defined as equal to the expression on the right.

The addition and the scalar multiplication of fuzzy vectors $\mathbf{A} = (x_1, x_2, \ldots, x_n)^T$ and $\mathbf{B} = (y_1, y_2, \ldots, y_n)^T$ are defined as

$$
\mathbf{A} + \mathbf{B} := (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)^T,
$$

$$
k\mathbf{A} := (kx_1, kx_2, \ldots, kx_n)^T (k \in R)
$$

Obviously, if $k > 0$, then

$$
kx_i = k \{ (x_i^*(r), x_i^*(r), r) : r \in [0, 1]\} = \{(kx_i^*(r), kx_i^*(r), r) : r \in [0, 1]\},
$$

(i = 1, 2, \ldots, n).

Let $\mathbf{A} = (x_1, x_2, \ldots, x_n)^T$, $\mathbf{B} = (y_1, y_2, \ldots, y_n)^T \in F_0(R^n)$, then the inner product of fuzzy vectors $\mathbf{A}$ and $\mathbf{B}$, denoted by $\mathbf{A} \circ \mathbf{B}$ or $\langle \mathbf{A}, \mathbf{B} \rangle$, is defined as

$$
x_1 \cdot y_1 + \cdots + x_n \cdot y_n
$$

It is a fuzzy number.

Let $R^n_+ = \{x = (x_1, x_2, \ldots, x_n)^T : x_i \geq 0 (i = 1, 2, \ldots, n)\}$, $\tilde{A} = (x_1, x_2, \ldots, x_n)^T \in F_0(R^n)$, and $c \in R^n_+$, then $c \circ \tilde{A} \in F_0$.

A fuzzy matrix whose $(i, j)^{th}$ entry is a fuzzy number, the operations are defined by operations between fuzzy numbers.

Let $A$ be an $m \times n$ fuzzy matrix, then $A$ has the following form:

$$
A = ([a_{ij}^*(r), a_{ij}^*(r), r] \in [0, 1])_{m \times n}.
$$

For any $r \in [0, 1]$, we have following two matrices:

$$
A(r)_+ := (a_{ij}^*(r))_{m \times n}, \quad A(r)^* := (a_{ij}^*(r))_{m \times n},
$$

where $a_{ij} = \{(a_{ij}^*(r), a_{ij}^*(r), r) : r \in [0, 1]\}$ is a fuzzy number ($i = 1, \ldots, m$; $j = 1, \ldots, n$).

By the above definitions, it is easy to prove, that is, $Ac \in F_0(R^m)$, $c \in R^n_+$.

Definition 2.1. An $n \times n$ fuzzy matrix $A = ([a_{ij}^*(r), a_{ij}^*(r), r] : r \in [0, 1])_{n \times n}$ is said to be positive semidefinite and symmetric if $A(r)_+ = (a_{ij}^*(r))_{n \times n}$ and $A(r)^* = (a_{ij}^*(r))_{n \times n}$ are $n \times n$ positive semidefinite and symmetric for any $r \in [0, 1]$. 
LEMMA 2.4. If fuzzy matrix $Q$ is positive semidefinite and symmetric, $x \in R^n$, then $x^TQx$ is a fuzzy number, and
\[
x^TQx = \left( \left\{ \left( \sum_{i=1}^{n} x_i x_j q_{ij}(r), \sum_{j=1}^{n} x_j x_j q_{ij}(r), r \right) : r \in [0, 1] \right\} \right)_{n \times n},
\]
where $x = (x_1, \ldots , x_n)^T$, $Q = \{ (q_{ij}(r), q_{ij}(r), r) : r \in [0, 1] \}^n$.

Let $S$ be a nonempty set in $R^n$ (denoted by $E$), a mapping $f : S \rightarrow F_0$ is said to be a fuzzy mapping. The epigraph of $f$, denoted by $\text{epi} f$, is a subset of $E \times F_0$ defined by
\[
\text{epi} f = \{(x, u) : x \in S, u \in F_0, u \geq f(x)\}.
\]
The hypograph of $f$, denoted by $\text{hyp} f$, is a subset of $E \times F_0$ defined by
\[
\text{hyp} f = \{(x, u) : x \in S, u \in F_0, u \leq f(x)\}.
\]
The lower-level set of $f$, denoted by $S_u(f)$, is a subset of $E$ defined by
\[
S_u(f) = \{x \in S : f(x) \leq u\} \quad (u \in F_0).
\]

A subset $C$ of $E$ is said to be convex if $(1 - \lambda)x + \lambda y \in C$ whenever $x \in C$, $y \in C$, and $0 < \lambda < 1$. We define $f$ to be a convex fuzzy mapping on $S$ if $\text{epi} f$ is convex as a subset of $E \times F_0$ and, equivalently, $f$ is a concave fuzzy mapping on $S$ if $\text{hyp} f$ is convex as a subset of $E \times F_0$.

THEOREM 2.2. (See [22, 23].) A fuzzy mapping $f : C \rightarrow F_0$ defined on a convex subset $C$ in $E$ is convex (respectively, concave) if and only if
\[
f((1 - \lambda)x + \lambda y) \leq (\text{respectively, } \geq) (1 - \lambda)f(x) + \lambda f(y), \quad \lambda \in [0, 1],
\]
for every $x$ and $y$ in $C$.

A fuzzy mapping $f : C \rightarrow F_0$ defined on a convex subset $C$ in $E$ is called strictly convex if
\[
f((1 - \lambda)x + \lambda y) < (1 - \lambda)f(x) + \lambda f(y), \quad \lambda \in (0, 1),
\]
for every $x$ and $y$ in $C$, and $x \neq y$.

THEOREM 2.3. (See [22].) Let $C$ be a convex subset in $E$ and $f : C \rightarrow F_0$ be a convex fuzzy mapping, then $S_u(f)$ is a convex subset in $E$.

According to the parametric representation of fuzzy number, a fuzzy mapping $f(x)$ can be written as follows:
\[
f(x) = \{(f(x)_x(r), f(x)^*(r), r) : r \in [0, 1]\}.
\]

THEOREM 2.4. (See [24].) Let $C$ be a convex subset in $E$ and $f : C \rightarrow F_0$ be a fuzzy mapping, then $f$ is convex if and only if $f(x)_x(r)$ and $f(x)^*(r)$ are all convex functions of $x$ for any fixed $r \in [0, 1]$.

THEOREM 2.5. (See [25].) If $f_1, f_2, \ldots , f_m$ are convex (respectively, concave) fuzzy mappings defined on $C \subseteq E$, $\lambda_i \geq 0 \quad (i = 1, \ldots , m)$, then $\lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_m f_m$ is a convex (respectively, concave) fuzzy mappings on $C$.

EXAMPLE 2.1. A fuzzy mapping
\[
f(x) = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n + b = (a, x) + b
\]
is a convex and concave, where $a_1, a_2, \ldots , a_n \in F_0$ (i.e., $a \in F_0(R^n)$), $x \in R^n$, $b \in R$.  


Example 2.2. Let fuzzy matrix $Q$ be positive semidefinite and symmetric, $x \in \mathbb{R}^n$, then $h(x) = x^T Q x$ is a convex fuzzy mapping.

As a matter of fact, it is known from the Lemma 2.4 that $h(x) = x^T Q x$ is a fuzzy number. Hence, $h(x) = x^T Q x$ is a convex fuzzy mapping.

Let $x = (x_1, \ldots, x_n)^T$, $Q = \{(q_{ij}(r), q_{ij}^*(r), r) : r \in [0, 1]\}$, $x_1^T Q x = \{(x_1^T Q(r)x, x_1^T Q(r)^* x, r) : r \in [0, 1]\}$. By Lemma 2.4, we have

$$x^T Q x = \{(x^T Q(r)x, x_1^T Q(r)^* x, r) : r \in [0, 1]\}.$$

According to Definition 2.1, $Q(r)_*$ and $Q(r)^*$ are all $n \times n$ positive semidefinite and symmetric for any $r \in [0, 1]$, therefore, $x^T Q(r)x$ and $x^T Q(r)^* x$ are convex functions in $x$. By Theorem 2.4, we know that $h(x)$ is a convex fuzzy mapping.

3. SUBDIFFERENTIAL AND ITS APPLICATIONS TO FUZZY EXTREMUM PROBLEMS

Definition 3.1. An $n$-dimensional fuzzy vector $\xi$ is said to be a subgradient of a convex fuzzy mapping $f : C \to F_0$ at $x \in C$ if

$$f(x) \geq f(z) + \langle \xi, z - x \rangle, \quad \text{for any } z \in C.$$

This condition is called the subgradient inequality.

According to the parametric representation of fuzzy number, a fuzzy mapping $f(x)$ can be written as follows:

$$f(x) = \{(f(x)_*(r), f(x)^*(r), r) : r \in [0, 1]\}.$$

The above subgradient inequality is equivalent to the following two inequalities:

$$f(x)_*(r) \geq f(x)_*(r) + \langle \xi_*(r), z - x \rangle,$n

$$f(x)^*(r) \geq f(x)^*(r) + \langle \xi^*(r), z - x \rangle,$n

for any $r \in [0, 1]$. By Theorem 2.4, $f(x)_*(r)$ and $f(x)^*(r)$ are convex functions defined on $C$, this leads to a conclusion that $\xi_*(r)$ and $\xi^*(r)$ are subgradients of convex functions $f(x)_*(r)$ and $f(x)^*(r)$ at $x \in C$, respectively. (See [10].)

Theorem 3.1. Let $C$ be a convex subset in $E$, $f : C \to F_0$ be a convex fuzzy mapping, then fuzzy vector $\xi \in F_0(\mathbb{R}^n)$ is a subgradient of $f$ at $x \in C$ if and only if $\xi_*(r)$ and $\xi^*(r)$ are subgradients of convex functions $f(x)_*(r)$ and $f(x)^*(r)$ at $x \in C$ for any fixed $r \in [0, 1]$, respectively.

The set of all subgradients of $f$ at $x$ is called the subdifferential of $f$ at $x$ and is denoted by $\partial f(x)$. If $\partial f(x)$ is not empty, $f$ is said to be subdifferential at $x$.

Lemma 3.1. Let $f, g$ be convex fuzzy mappings defined on $C \subseteq E$, and $\text{int} C \neq \phi$, then

$$\partial(\lambda f)(x) = \lambda \partial f(x) (\lambda > 0), \quad \partial(f + g)(x) = \partial f(x) + \partial g(x).$$

Proof. By Theorem 2.5, $\lambda f$ and $f + g$ are convex fuzzy mappings. It is immediate from the definition of subgradient that

$$\partial(\lambda f)(x) = \lambda \partial f(x) (\lambda > 0).$$

By Theorem 23.8 in [12], we have

$$\partial(f + g)(x)_*(r) = \partial f(x)_*(r) + \partial g(x)_*(r),$$

$$\partial(f + g)(x)^*(r) = \partial f(x)^*(r) + \partial g(x)^*(r),$$

for any fixed $r \in [0, 1]$. 


According to Theorem 3.1, one has that
\[ \partial(f + g)(x) = \partial f(x) + \partial g(x), \]
and this is what we wanted to prove.

\( f \) is said to be a differential at \( x \) if \( f \) has a unique subgradient \( \xi \) at \( x \) and is denoted by \( \nabla f(x) \).

A convex fuzzy mapping \( f \) is a differential at \( x_0 \) means that \( \xi = \nabla f(x) \) is the unique element of \( \partial f(x) \). By Theorem 3.1, this condition leads to the conclusion that \( \partial f(x_0)_+(r) = \{ \xi_+(r) \} \) and \( \partial f(x_0)_-(r) = \{ \xi_-(r) \} \) are all singletons for any fixed \( r \in [0, 1] \). Therefore, \( f(x)_+(r) \) and \( f(x)_-(r) \) are differentials at \( x_0 \). Conversely, if \( f(x)_+(r) \) and \( f(x)_-(r) \) are differentials at \( x_0 \) for any fixed \( r \in [0, 1] \), then \( \partial f(x_0)_+(r) \) and \( \partial f(x_0)_-(r) \) are singletons, assuming that their singletons are \( \{ \xi_+(r) \} \) and \( \{ \xi_-(r) \} \), respectively. If the corresponding components of \( \xi_+(r) \) and \( \xi_-(r) \) satisfy the conditions of Theorem 2.1, then there exist a unique fuzzy vector \( \xi \) such that

\[ \xi_+(r) = (\xi_{1_+}(r), \xi_{2_+}(r), \ldots, \xi_{n_+}(r))^T, \]
\[ \xi_-(r) = (\xi_{1_-}(r), \xi_{2_-}(r), \ldots, \xi_{n_-}(r))^T, \]

where \( \xi_i = \{(\xi_{i_+}(r), \xi_{i_-}(r), r) : r \in [0, 1] \} \) is a fuzzy number. This means that \( \partial f(x_0) \) is a singleton, therefore, the convex fuzzy mapping \( f \) is differential at \( x_0 \).

**Theorem 3.2.** Let convex fuzzy mapping \( f, g \) defined on \( C \subseteq E \), and \( \text{int}C \neq \phi \), if \( f, g \) are differential at \( x^* \), then \( \lambda f \) and \( f + g \) are also differential at \( x^* \), and
\[ \nabla(\lambda f)(x^*) = \lambda \nabla f(x^*), \quad \nabla(f + g)(x^*) = \nabla f(x^*) + \nabla g(x^*). \]

**Proof.** By Theorem 2.5, \( \lambda f \) and \( f + g \) are convex fuzzy mappings. Since \( f, g \) are differential at \( x^* \), we have
\[ f(x) \geq f(x^*) + \langle \nabla f(x^*), x - x^* \rangle, \]
\[ g(x) \geq g(x^*) + \langle \nabla g(x^*), x - x^* \rangle. \]

Hence,
\[ \lambda f(x) \geq \lambda f(x^*) + \langle \lambda \nabla f(x^*), x - x^* \rangle \]
and
\[ f(x) + g(x) \geq f(x^*) + g(x^*) + \langle \nabla f(x^*) + \nabla g(x^*), x - x^* \rangle, \]
for any \( x \in C \).

According to the uniqueness of \( \nabla f(x^*) \) and \( \nabla g(x^*) \) and Lemma 3.1, we conclude what we want to prove is correct.

**Example 3.1.** A convex and concave fuzzy mapping
\[ f(x) = (a, x) + b \]
is a differential at \( \bar{x} \in R^n_+ \), and \( \nabla f(\bar{x}) = a \), where \( a \in F_0(R^n), \bar{x} \in R^n_+, b \in R \).

In fact, assume that there exists \( \xi \in F_0(R^n) \), such that
\[ f(x) \geq f(\bar{x}) + (\xi, x - \bar{x}), \]
for any \( x \in R^n_+ \), that is,
\[ \langle a, x \rangle + b \geq \langle a, \bar{x} \rangle + b + (\xi, x - \bar{x}), \]
hence,
\[ \langle a, x \rangle \geq \langle a, \bar{x} \rangle + (\xi, x - \bar{x}). \]
Let $a = (a_1, \ldots, a_n)^T$, $x = (x_1, \ldots, x_n)^T$, $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T$, and $\xi = (\xi_1, \ldots, \xi_n)^T$, where $a_i = \{(a_i(r)_*, a_i(r)^*, r) : r \in [0, 1]\}$ and $\xi_i = \{(\xi_i(r)_*, \xi_i(r)^*, r) : r \in [0, 1]\}$ are all fuzzy numbers. Hence, the above inequality can be written as

$$\sum_{i=1}^{n} x_i a_i(r)_* \geq \sum_{i=1}^{n} \bar{x}_i a_i(r)_* + \sum_{i=1}^{n} \xi_i(r)(x_i - \bar{x}_i)$$

and

$$\sum_{i=1}^{n} x_i a_i(r)^* \geq \sum_{i=1}^{n} \bar{x}_i a_i(r)^* + \sum_{i=1}^{n} \xi_i(r)^*(x_i - \bar{x}_i),$$

i.e.,

$$\sum_{i=1}^{n} (a_i(r)_* - \xi_i(r)_*)(x_i - \bar{x}_i) \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} (a_i(r)^* - \xi_i(r)^*)(x_i - \bar{x}_i) \geq 0.$$

The above formulas are true for any $x \in \mathbb{R}^n$, this implies that

$$\xi_i(r)_* = a_i(r)_* \quad \text{and} \quad \xi_i(r)^* = a_i(r)^*,$$

for any $r \in [0, 1]$. This leads to the conclusion that $\xi = a$ by Theorem 2.1. This shows that $\xi$ is unique, and $\xi = \nabla f(\bar{x}) = a$. This completes the proof.

**EXAMPLE 3.2.** The convex fuzzy mapping $h(x) = x^T Q x (x \in \mathbb{R}^n_+)$ is differential at $\bar{x} \in \text{Int} \mathbb{R}^n_+$. In fact, assume that there exists $\xi \in F_0(R^n)$, such that

$$h(x) \geq h(\bar{x}) + \langle \xi, x - \bar{x} \rangle,$$

for any $x \in \mathbb{R}^n_+$, that is,

$$x^T Q x \geq \bar{x}^T Q \bar{x} + \langle \xi, x - \bar{x} \rangle,$$

hence,

$$\left(\{ (x^T Q(r)_* x, x^T Q(r)^* x, r) : r \in [0, 1]\}\right)_{n \times n} \geq \left(\{ (\bar{x}^T Q(r)_* \bar{x}, \bar{x}^T Q(r)^* \bar{x}, r) : r \in [0, 1]\}\right)_{n \times n}$$

$$+ \left\{ \left( \sum_{i=1}^{n} \xi_i(r)_*(x_i - \bar{x}_i), \sum_{i=1}^{n} \xi_i(r)^*(x_i - \bar{x}_i), r \right) : r \in [0, 1] \right\},$$

where $x = (x_1, \ldots, x_n)^T$, $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)^T$, and $\xi = (\xi_1, \ldots, \xi_n)^T$.

Hence, the above inequality can be written as

$$x^T Q(r)_* x \geq \bar{x}^T Q(r)_* \bar{x} + \sum_{i=1}^{n} \xi_i(r)_*(x_i - \bar{x}_i),$$

$$x^T Q(r)^* x \geq \bar{x}^T Q(r)^* \bar{x} + \sum_{i=1}^{n} \xi_i(r)^*(x_i - \bar{x}_i),$$

for any $r \in [0, 1]$, where $Q(r)_* = (q_{ij}(r)_*)_{n \times n}$, $Q(r)^* = (q_{ij}(r)^*)_{n \times n}$.

Let

$$F(x, \xi)(r)_* = x^T Q(r)_* x - \bar{x}^T Q(r)_* \bar{x} - \xi(r)_*^T (x - \bar{x}),$$

$$F(x, \xi)(r)^* = x^T Q(r)^* x - \bar{x}^T Q(r)^* \bar{x} - \xi(r)^*_T (x - \bar{x}).$$

Then we must have

$$F(x, \xi)(r)_* \geq 0, \quad F(x, \xi)(r)^* \geq 0 \quad (\forall x \in \mathbb{R}^n_+).$$
Let
\[ F(x, \xi)(r) = 2Q(r)x - \xi(r) = 0, \]
\[ F_x(x, \xi)(r) = x - \bar{x} = 0. \]

It is easy to know that \( F(x, \xi)(r) \) achieves the minimum zero as \( \xi(r) = 2Q(r)x \) and \( x = \bar{x} \).

Similarly, we can prove that \( F(x, \xi)(r)^* \) achieve the minimum zero as \( \xi(r)^* = 2Q(r)^*\bar{x} \) and \( x = \bar{x} \).

Hence,
\[ \xi = \{(\xi(r), \xi(r)^*, r) : r \in [0, 1]\} \]
\[ = \{(2Q(r)x, 2Q(r)^*\bar{x}, r) : r \in [0, 1]\} \]
\[ = \{(2Q(r)x, 2Q(r)^*\bar{x}, r) : r \in [0, 1]\} \]
\[ = 2\{(Q(r)x, Q(r)^*, r) : r \in [0, 1]\} \bar{x} = 2Q\bar{x}. \]

This is leads to the conclusion that \( \xi \) is unique, and \( \nabla h(\bar{x}) = \xi = 2Q\bar{x} \). This completes the proof.

We now focus on the problems of minimizing and maximizing a convex fuzzy mapping over a convex set and develop the necessary and/or sufficient conditions for optimality.

Let \( f \) be a fuzzy mapping, and consider the problem to minimize \( f(x) \) subject to \( x \in S \). A point \( x \in S \) is called a feasible solution to the problem. If \( \bar{x} \in S \) and \( f(\bar{x}) \leq f(x) \) for each \( x \in S \), then \( \bar{x} \) is called an optimal solution, or a global minimum point. If \( \bar{x} \in S \) and there exists a neighborhood \( N(\bar{x}) \) around \( \bar{x} \) such that \( f(\bar{x}) \leq f(x) \) for each \( x \in S \cap N(\bar{x}) \), then \( \bar{x} \) is called a local optimal solution, or a local minimum point.

A global maximum point and a local maximum point for a fuzzy mapping can be defined similarly.

**Theorem 3.3.** Let \( f : E \to F_0 \) be a convex fuzzy mapping, then \( \bar{x} \) is a global minimum point if and only if \( 0 \in \partial f(\bar{x}) \).

**Proof.** By the definition of subgradient, \( 0 \in \partial f(\bar{x}) \) if and only if
\[ f(x) \geq f(\bar{x}) + (0, x - \bar{x}) = f(\bar{x}), \]
for any \( x \in E \).

**Corollary.** Let \( f : E \to F_0 \) be a differential convex fuzzy mapping, then \( \bar{x} \) is a global minimum point if and only if \( \nabla f(\bar{x}) = 0 \).

**Theorem 3.4.** Let \( f : C \to F_0 \) be a convex fuzzy mapping, and let \( C \) be a nonempty convex set in \( E \). Consider the problem to maximize \( f(x) \) subject to \( x \in C \). If \( \bar{x} \in C \) is a local optimal solution, then \( \langle \xi, x - \bar{x} \rangle \leq 0 \) for each \( x \in C \), where \( \xi \in \partial f(\bar{x}) \).

**Proof.** Suppose that \( \bar{x} \in C \) is a local optimal solution. Then, a neighborhood \( N(\bar{x}) \) exists such that \( f(\bar{x}) \geq f(x) \) for each \( x \in C \cap N(\bar{x}) \). Let \( x \in C \), and note that \( \bar{x} + \lambda(x - \bar{x}) \in C \cap N(\bar{x}) \) for \( \lambda > 0 \) and sufficiently small. Hence,
\[ f(\bar{x} + \lambda(x - \bar{x})) \leq f(\bar{x}). \]

By \( \xi \in \partial f(\bar{x}) \) and the convexity of \( f \), we have
\[ f(\bar{x} + \lambda(x - \bar{x})) \geq f(\bar{x}) + \langle \xi, \bar{x} + \lambda(x - \bar{x}) - \bar{x} \rangle = f(\bar{x}) + \lambda\langle \xi, x - \bar{x} \rangle. \]

The above inequality, together with \( f(\bar{x} + \lambda(x - \bar{x})) \leq f(\bar{x}) \), implies that \( \lambda\langle \xi, x - \bar{x} \rangle \leq 0 \), and, dividing by \( \lambda > 0 \), the result follows.
COROLLARY. In addition to the assumptions of the theorem, suppose that $f$ is differentiable. If $\bar{x} \in C$ is a local optimal solution, then $\langle \nabla f(\bar{x}), x - \bar{x} \rangle \leq 0$ for all $x \in C$.

Let $f : S \to F_0$, $g_i : S \to F_0 (i = 1, \ldots, m)$ be fuzzy mappings, $S \subseteq E$. The following problem:

$$\begin{align*}
\min & \quad f(x), \\
\text{s.t.} & \quad g_i(x) \leq 0, \quad x \in S
\end{align*}$$

is called a fuzzy programming, denoted by (FP). If $S \subseteq E$ is a convex set, $f$ and $g_i$ $(i = 1, \ldots, m)$ are all convex fuzzy mappings, then problem (FP) is called a convex fuzzy programming, denoted by (FCP).

THEOREM 3.5. Let $S \subseteq E$ be a convex set, $f$ and $g_i$ $(i = 1, \ldots, m)$ are all convex fuzzy mappings. Consider convex fuzzy programming

$$\begin{align*}
\min & \quad f(x), \\
\text{s.t.} & \quad g_i(x) \leq 0, \quad x \in S.
\end{align*}$$

Then, we have the following.

1. The feasible set is convex.
2. Suppose that $\bar{x} \in S$ is a local optimal solution to the problem, then $\bar{x}$ is a global optimal solution. If $f$ is strictly convex, then $\bar{x}$ is the unique global optimal solution.

PROOF.

1. By Theorem 2.3, we know that $S_0(g_i) = \{x \in S : g_i(x) \leq 0\}$ $(i = 1, \ldots, m)$ are convex sets. The feasible set for (FCP) is $T = S \cap (\bigcap_{i=1}^{m} S_0(g_i))$, as the intersection of convex sets is convex.
2. Since $\bar{x}$ is a local optimal solution, there exists a neighborhood $N(\bar{x})$ around $\bar{x}$ such that $f(\bar{x}) \leq f(x)$, for each $x \in S \cap N(\bar{x})$. (*)

By contradiction, suppose that $\bar{x}$ is not a global optimal solution so that $f(x^*) < f(\bar{x})$ for some $x^* \in T$. By the convexity of $f$, the following is true for each $\lambda \in (0, 1)$:

$$f(\lambda x^* + (1 - \lambda)\bar{x}) \leq \lambda f(x^*) + (1 - \lambda)f(\bar{x}) \leq \lambda f(\bar{x}) + (1 - \lambda)f(\bar{x}) = f(\bar{x}).$$

But for $\lambda > 0$ and sufficiently small, $\lambda x^* + (1 - \lambda)\bar{x} \in S \cap N(\bar{x})$. Hence, the above inequality contradicts with (*), this leads to the conclusion that $\bar{x}$ is a global optimal solution.

Suppose that $\bar{x}$ is not the unique global optimal solution so that there exists $\hat{x} \in T$, $\hat{x} \neq \bar{x}$, such that $f(\hat{x}) = f(\bar{x})$. By strict convexity,

$$f\left(\frac{1}{2} \bar{x} + \frac{1}{2} \hat{x}\right) < \frac{1}{2} f(\bar{x}) + \frac{1}{2} f(\bar{x}) = f(\bar{x}).$$

By the convexity of $T$, $(1/2)\bar{x} + (1/2)\hat{x} \in T$, and the above inequality violates global optimality of $\bar{x}$. Hence, $\bar{x}$ is the unique global minimum.

Taking $g_i(x) = 0$ $(i = 1, \ldots, m)$, Theorem 5.5 in [7] is obtained.

4. SADDLE-POINTS AND LAGRANGIAN DUALITY

Duality theory plays a central role in mathematical programming. This theory is closely related to the theory of so-called minimax problems and saddle-points. (See [26–28].)

In this section, we focus on the duality theory for fuzzy programming.

Let $f, g, F$ be fuzzy mappings defined on $X \subseteq R^n$, $Y \subseteq R^m$, and $X \times Y \subseteq R^n \times R^m$, respectively. We assume that the global minima and maxima which we address below do exist, and is a fuzzy number, in all cases.
LEMMA 4.1. Suppose that \( f(x) \leq g(y) \) for all \((x, y) \in X \times Y\). Then,

\[
\max_{x \in X} f(x) \leq \min_{y \in Y} g(y).
\]

PROOF. By Theorem 2.1, \( f(x), g(y) \) can be written as

\[
f(x) = \{(f(x)_*(r), f(x)^*(r), r) : r \in [0, 1]\},
g(y) = \{(g(y)_*(r), g(y)^*(r), r) : r \in [0, 1]\}.
\]

By \( f(x) \leq g(y) \), we have

\[
f(x)_*(r) \leq g(y)_*(r), \quad f(x)^*(r) \leq g(y)^*(r),
\]

for each \( r \in [0, 1] \) and all \((x, y) \in X \times Y\), one has that

\[
\max_{x \in X} f(x)_*(r) \leq \min_{y \in Y} g(y)_*(r), \quad \max_{x \in X} f(x)^*(r) \leq \min_{y \in Y} g(y)^*(r).
\]

By Lemma 2.2, we have

\[
\max_{x \in X} f(x) = \max_{x \in X} \{(f(x)_*(r), f(x)^*(r), r) : r \in [0, 1]\}\]
\[
= \{(\max_{x \in X} f(x)_*(r), \max_{x \in X} f(x)^*(r), r) : r \in [0, 1]\}\]
\[
\leq \{(\min_{y \in Y} g(y)_*(r), \min_{y \in Y} g(y)^*(r), r) : r \in [0, 1]\}\]
\[
= \min_{y \in Y} g(y),
\]

i.e.,

\[
\max_{x \in X} f(x) \leq \min_{y \in Y} g(y).
\]

Under certain conditions, the above inequality can be satisfied as equality

\[
\max_{x \in X} f(x) = \min_{y \in Y} g(y).
\]

Each result of this kind is called a duality theorem.

LEMMA 4.2.

\[
\max_{y \in Y} \min_{x \in X} F(x, y) \leq \min_{y \in Y} \max_{x \in X} F(x, y).
\]

PROOF. By Theorem 2.1, \( F(x, y) \) can be written as

\[
F(x, y) = \{(F(x, y)_*(r), F(x, y)^*(r), r) : r \in [0, 1]\}.
\]

By the operations of fuzzy numbers, we have

\[
\min_{x \in X} F(x, y) = \min_{x \in X} \{(F(x, y)_*(r), F(x, y)^*(r), r) : r \in [0, 1]\}\]
\[
= \left\{ \left( \min_{x \in X} F(x, y)_*(r), \min_{x \in X} F(x, y)^*(r), r \right) : r \in [0, 1] \right\}\]
\[
\leq \{(F(x, y)_*(r), F(x, y)^*(r), r) : r \in [0, 1]\} = F(x, y),
\]

for any \( y \in Y\).
Duality Theory

Similarly,

\[
F(x, y) = \{(F(x, y)_*(r), F(x, y)^*(r), r) : r \in [0, 1]\}
\]

\[
\leq \left\{ \left( \max_{y \in Y} F(x, y)_*(r), \max_{y \in Y} F(x, y)^*(r), r \right) : r \in [0, 1] \right\}
\]

\[
\leq \max_{y \in Y} \left\{(F(x, y)_*(r), F(x, y)^*(r), r) : r \in [0, 1]\right\}
\]

\[
\leq \max_{y \in Y} F(x, y),
\]

for any \(x \in X\).

Hence,

\[
\min_{x \in X} F(x, y) \leq \max_{y \in Y} F(x, y),
\]

for any \((x, y) \in X \times Y\). Note that \(\min_{x \in X} F(x, y)\), \(\max_{y \in Y} F(x, y)\) are fuzzy mappings in \(y\) and \(x\), respectively. By Lemma 4.1, one has that

\[
\max_{y \in Y} \min_{x \in X} F(x, y) \leq \min_{x \in X} \max_{y \in Y} F(x, y).
\]

In addition, under certain conditions, we can prove that

\[
\max_{y \in Y} \min_{x \in X} F(x, y) = \min_{x \in X} \max_{y \in Y} F(x, y).
\]

Each result of this type is called a minimax theorem.

**DEFINITION 4.1.** The point \((\bar{x}, \bar{y}) \in X \times Y\) is called a saddle-point of \(F : X \times Y \rightarrow F_0\) if

\[
F(\bar{x}, y) \leq F(\bar{x}, \bar{y}) \leq F(x, \bar{y}),
\]

for every \((x, y) \in X \times Y\).

**THEOREM 4.1.** The point \((\bar{x}, \bar{y})\) is a saddle-point of \(F : X \times Y \rightarrow F_0\) if and only if

\[
F(\bar{x}, \bar{y}) = \max_{y \in Y} \min_{x \in X} F(x, y) = \min_{x \in X} \max_{y \in Y} F(x, y).
\]

**PROOF.** Let \((\bar{x}, \bar{y})\) be a saddle-point of \(F\), then

\[
F(\bar{x}, y) \leq F(\bar{x}, \bar{y}) \leq F(x, \bar{y}), \quad \text{for every } (x, y) \in X \times Y.
\]

One has that

\[
\max_{y \in Y} F(\bar{x}, y) \leq F(\bar{x}, \bar{y}) \leq \min_{x \in X} F(x, \bar{y}),
\]

hence,

\[
\min_{x \in X} \max_{y \in Y} F(x, y) \leq F(\bar{x}, \bar{y}) \leq \max_{y \in Y} \min_{x \in X} F(x, y).
\]

According to Lemma 4.2, we have

\[
\max_{y \in Y} \min_{x \in X} F(x, y) = \min_{x \in X} \max_{y \in Y} F(x, y).
\]

That is,

\[
F(\bar{x}, \bar{y}) = \max_{y \in Y} \min_{x \in X} F(x, y) = \min_{x \in X} \max_{y \in Y} F(x, y).
\]

Conversely, if

\[
F(\bar{x}, \bar{y}) = \max_{y \in Y} \min_{x \in X} F(x, y) = \min_{x \in X} \max_{y \in Y} F(x, y),
\]
then we have
\[
\min_{x \in X} F(x, y) = F(\bar{x}, \bar{y}) = \max_{y \in Y} F(\bar{x}, y).
\]
Therefore, \( F(\bar{x}, y) \leq F(\bar{x}, \bar{y}) \leq F(x, y) \) for any \((x, y) \in X \times Y\), i.e., \((\bar{x}, \bar{y})\) is a saddle-point.

Consider the following fuzzy programming problem (FP), which is called the primal problem.

\[
\begin{align*}
\min & \quad f(x), \\
\text{s.t.} & \quad g_i(x) \leq 0, \quad x \in S,
\end{align*}
\]

where \( f : S \to F_0 \), \( g_i : S \to F_0 (i = 1, \ldots, m) \) are all fuzzy mappings, and \( S \subseteq E \).

The Lagrangian of (FP) is
\[
L(x, \lambda_1, \ldots, \lambda_m) = f(x) + \lambda_1 g_1(x) + \cdots + \lambda_m g_m(x),
\]
\[
\lambda_i \geq 0 \quad (i = 1, \ldots, m).
\]

It is easy to show that the Lagrangian \( L(x, \lambda_1, \ldots, \lambda_m) \) is a fuzzy mapping from \( S \times R^m \) to \( F_0 \).

It can be written as
\[
L(x, \lambda) = f(x) + \lambda \circ g(x),
\]
where \( \lambda = (\lambda_1, \ldots, \lambda_m)^T \), \( g(x) = (g_1(x), \ldots, g_m(x))^T \).

The Lagrangian dual problem (DFP) is presented below.

\[
\max_{\lambda \geq 0} d(\lambda),
\]

where \( d(\lambda) = \min_{x \in S} L(x, \lambda) \).

The objective function \( d(\lambda) \) of (DFP) is often called dual fuzzy mapping.

**Lemma 4.3.** If \( d(\lambda) = \min_{x \in S} L(x, \lambda) \) exists, then it is a concave fuzzy mapping.

**Proof.** By Lemma 2.3, we have
\[
d [(1 - \alpha)\lambda^1 + \alpha \lambda^2] = \min_{x \in S} \{ f(x) + [(1 - \alpha)\lambda^1 + \alpha \lambda^2] \circ g(x) \}
\]
\[
= \min_{x \in S} \{(1 - \alpha) [f(x) + \lambda^1 \circ g(x)] + \alpha [f(x) + \lambda^2 \circ g(x)] \}
\]
\[
\geq (1 - \alpha) \min_{x \in S} [f(x) + \lambda^1 \circ g(x)] + \alpha \min_{x \in S} [f(x) + \lambda^2 \circ g(x)]
\]
\[
= (1 - \alpha) d (\lambda^1) + \alpha d (\lambda^2),
\]
for any \( \lambda^1, \lambda^2 \in R_+^m \) and \( \alpha \in [0, 1] \), \( d(\lambda) \) is a concave fuzzy mapping.

**Theorem 4.2.** **Weak Duality Theorem.** Let \( x \) be a feasible solution to problem (FP), and let \( \lambda \) be a feasible solution to problem (DFP). Then, \( f(x) \geq d(\lambda) \).

**Proof.** By the definition of \( d(\lambda) \), and since \( \lambda \circ g(x) \leq 0 \), we have
\[
d(\lambda) = \min_{x \in S} L(x, \lambda) \leq L(x, \lambda) = f(x) + \lambda \circ g(x) \leq f(x).
\]

**Corollary 1.** \( \min_{x \in X} f(x) \geq \max_{\lambda \geq 0} d(\lambda) \). Where \( X = \{ x \in S : g_i(x) \leq 0, i = 1, \ldots, m \} \).

The above result is an immediate consequence of Theorem 4.2 and Lemma 4.1.

**Corollary 2.** If \( f(\bar{x}) = d(\bar{\lambda}) \), then \( \bar{x} \) and \( \bar{\lambda} \) solve the (FP) and (DFP), respectively.

**Proof.** By Theorem 4.2, we have
\[
f(x) \geq d (\bar{\lambda}),
\]
for every \( x \in X = \{ x \in S : g_i(x) \leq 0, i = 1, \ldots, m \} \). Hence, \( f(x) \geq f(\bar{x}) \).
Applying Theorem 4.2 again, one has that \( f(\bar{x}) \geq d(\lambda) \) for every \( \lambda \in \mathbb{R}_+^m \). Hence, \( d(\bar{\lambda}) \geq d(\lambda) \). This leads to the conclusion that \( \bar{x} \) and \( \bar{\lambda} \) solve the (FP) and (DFP), respectively.

If \( f(\bar{x}) \geq d(\bar{\lambda}) \), then the difference \( f(\bar{x}) - d(\bar{\lambda}) \) is called the "duality gap".

Since
\[
\sup_{\lambda \geq 0} \{ f(x) + \lambda \circ g(x) \} = \begin{cases} f(x), & x \in X, \\ +\infty, & x \notin X, \end{cases}
\]
where \( X = \{ x \in S : g_i(x) \leq 0, \ i = 1, \ldots, m \} \).

Problem (FP) can be restated in the form
\[
\min_{x \in X} \max_{\lambda \geq 0} L(x, \lambda).
\]

The dual problem (DFP) of (FP) can be written as
\[
\max_{\lambda \geq 0} d(\lambda) = \max_{\lambda \geq 0} \min_{x \in X} L(x, \lambda).
\]

Hence, if \( f(\bar{x}) = d(\bar{\lambda}) \), then
\[
L(\bar{x}, \bar{\lambda}) = \max_{\lambda \geq 0} \min_{x \in X} L(x, \lambda) = \min_{x \in X} \max_{\lambda \geq 0} L(x, \lambda),
\]
i.e., the minimax theorem of fuzzy mapping \( L(x, \lambda) \) holds.

By Theorem 4.1, \( (\bar{x}, \bar{\lambda}) \) is a saddle-point to \( L(x, \lambda) \) and set \( X \) and \( \mathbb{R}_+^m \), that is,
\[
L(\bar{x}, \lambda) \leq L(\bar{x}, \bar{\lambda}) \leq L(x, \lambda),
\]
for any \( (x, \lambda) \in X \times \mathbb{R}_+^m \).

The following result is then an immediate consequence of the above discussion.

**Theorem 4.3.** A point \( (\bar{x}, \bar{\lambda}) \in X \times \mathbb{R}_+^m \) is a saddle-point of the fuzzy Lagrangian \( L(x, \lambda) \) if and only if \( \bar{x} \) is a global minimum point of the primal problem (FP), \( \bar{\lambda} \) is a global maximum point of the dual (DFP), and the optimal values \( f(\bar{x}) \) of (FP) and \( d(\bar{\lambda}) \) of (DFP) coincide.

The next theorem provide a set of necessary and sufficient conditions for \( (\bar{x}, \bar{\lambda}) \in X \times \mathbb{R}_+^m \) to be a saddle-point of \( L(x, \lambda) \), and hence, for the equivalent duality theorem above.

**Theorem 4.4.** A point \( (\bar{x}, \bar{\lambda}) \in X \times \mathbb{R}_+^m \) is a saddle-point of the Lagrangian \( L(x, \lambda) \) if and only if the following conditions hold.

1. \( L(\bar{x}, \lambda) = \min_{x \in X} L(x, \lambda) \).
2. \( g_i(\bar{x}) \leq 0 \) (\( i = 1, \ldots, m \)).
3. \( \lambda \circ g(\bar{x}) = 0 \).

**Proof.** Condition (1) is equivalent to the inequality
\[
L(\bar{x}, \lambda) \leq L(x, \lambda),
\]
for every \( x \in X \).

When (2) and (3) hold, then
\[
L(\bar{x}, \lambda) = f(\bar{x}) + \lambda \circ g(\bar{x}) \leq f(\bar{x}) = f(\bar{x}) + \bar{\lambda} \circ g(\bar{x}) = L(\bar{x}, \bar{\lambda}),
\]
for every \( \lambda \geq 0 \).

Hence,
\[
L(\bar{x}, \lambda) \leq L(\bar{x}, \bar{\lambda}) \leq L(x, \lambda),
\]
for every \( (x, \lambda) \in X \times \mathbb{R}_+^m \).
This shows that a point \((\bar{x}, \bar{\lambda}) \in X \times \mathbb{R}^m\) is a saddle-point of the Lagrangian \(L(x, \lambda)\).

Conversely, if \((\bar{x}, \bar{\lambda}) \in X \times \mathbb{R}^m\) is a saddle-point of the Lagrangian \(L(x, \lambda)\). That is,

\[
L(\bar{x}, \lambda) \leq L(\bar{x}, \bar{\lambda}) \leq L(x, \lambda),
\]

for every \((x, \lambda) \in X \times \mathbb{R}^m\), i.e.,

\[
f(\bar{x}) + \lambda \circ g(\bar{x}) \leq f(\bar{x}) + \bar{\lambda} \circ g(\bar{x}) \leq f(x) + \bar{\lambda} \circ g(x).
\]

According to the inequality at the left and (1) in Lemma 2.1, we have

\[
\lambda \circ g(\bar{x}) \leq \bar{\lambda} \circ g(\bar{x}),
\]

i.e.,

\[
\sum_{i=1}^{m} \lambda_i g_i(\bar{x}) \leq \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x}).
\]

By Theorem 2.1, every \(g_i(x)\) can be written as

\[
g_i(x) = \{(g_i(x)_*(r), g_i(x)^*(r), r) : r \in [0, 1]\} \quad (i = 1, \ldots, m).
\]

By the order of fuzzy numbers, one has that

\[
\sum_{i=1}^{m} \lambda_i g_i(\bar{x})_*(r) \leq \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x})_*(r), \quad \sum_{i=1}^{m} \lambda_i g_i(\bar{x})^*(r) \leq \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x})^*(r),
\]

for each \(r \in [0, 1]\).

Clearly, this implies that

\[
g_i(\bar{x})_*(r) \leq 0, \quad g_i(\bar{x})^*(r) \leq 0 \quad (i = 1, \ldots, m),
\]

for every \(r \in [0, 1]\), or else (*) can be violated by appropriately making a component of \(\lambda\) infinitely large in magnitude.

Now by taking \(\lambda_i = 0\) \((i = 1, \ldots, m)\) in (*), we obtain that

\[
\sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x})_*(r) \geq 0, \quad \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x})^*(r) \geq 0.
\]

Noting that \(\bar{\lambda}_i \geq 0\) \((i = 1, \ldots, m)\) and \(g_i(\bar{x}) \leq \bar{0}\) \((i = 1, \ldots, m)\), we have that

\[
\bar{\lambda}_i g_i(\bar{x})_*(r) \leq 0, \quad \bar{\lambda}_i g_i(\bar{x})^*(r) \leq 0 \quad (i = 1, \ldots, m),
\]

for every \(r \in [0, 1]\), we must have

\[
\sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x})_*(r) = 0, \quad \sum_{i=1}^{m} \bar{\lambda}_i g_i(\bar{x})^*(r) = 0,
\]

i.e., \(\bar{\lambda} \circ g(\bar{x}) = \bar{0}\).

Hence, conditions (1)–(3) hold.
5. KKT CONDITIONS FOR (FCP) AND ITS APPLICATIONS TO (FLP) AND (FQP)

Consider the following fuzzy programming problem, which we call the convex fuzzy programming (FCP):

\[
\begin{align*}
\text{min} & \quad f(x), \\
\text{s.t.} & \quad g_i(x) \leq \bar{0}, \quad x \in S,
\end{align*}
\]

where \( f : S \to F_0, g_i : S \to F_0 \) \((i = 1, \ldots, m)\) are all convex fuzzy mappings, and \( S \subseteq E \).

The Lagrangian of (FCP) is

\[
L(x, \lambda) = f(x) + \lambda_1 g_1(x) + \cdots + \lambda_m g_m(x) = f(x) + \lambda \circ g(x),
\]

where \( \lambda = (\lambda_1, \ldots, \lambda_m)^\top, \lambda_i \geq 0 (i = 1, \ldots, m), g(x) = (g_1(x), \ldots, g_m(x))^\top \).

By Theorem 2.5, Lagrangian \( L(\cdot, \lambda) \) is a convex fuzzy mapping from \( S \) to \( F_0 \) for each \( \lambda \geq 0 \).

The Lagrangian dual problem (DFP) of (FCP) is presented below, and denoted by (DFCP).

\[
\max_{\lambda \geq 0} d(\lambda),
\]

where \( d(\lambda) = \min_{x \in S} L(x, \lambda) \).

We discuss the KKT optimality conditions for problem (FCP).

**Theorem 5.1.** Let \( X = \{ x \in S : g_i(x) \leq \bar{0}, i = 1, \ldots, m \}, \bar{x} \in X \). Suppose that \( f, g_i \) \((i = 1, \ldots, m)\) is differential at \( \bar{x} \) and satisfies the following conditions (called KKT conditions for (FCP)), that is, there exist \( \lambda \geq 0 \) such that

\[
\nabla f(\bar{x}) + \nabla g(\bar{x})^\top \lambda = \bar{0}, \quad \bar{\lambda} \circ g(\bar{x}) = \bar{0}. (*).
\]

Then, \((\bar{x}, \bar{\lambda}) \in S \times R_m^n \) is a saddle-point for the Lagrangian \( L(x, \lambda) \). Conversely, suppose that \((\bar{x}, \bar{\lambda}) \in (f S) \times R_m^n \) is a saddle-point for the Lagrangian \( L(x, \lambda) \). Then, \( \bar{x} \) is an optimal solution to problem (FCP), and \((\bar{x}, \bar{\lambda})\) satisfies the KKT conditions specified by (*).

**Proof.** Suppose that \( (\bar{x}, \bar{\lambda}) \) with \( \bar{x} \in X \) and \( \bar{\lambda} \geq 0 \), satisfies the KKT conditions specified by (*).

By convexity of \( f : S \to F_0 \) and \( g_i : S \to F_0 \) \((i = 1, \ldots, m)\), and \( f, g_i \) \((i \in I)\) are differential at \( \bar{x} \). According to Definition 3.1, we get

\[
\begin{align*}
f(x) & \geq f(\bar{x}) + \langle \nabla f(\bar{x}), x - \bar{x} \rangle, \\
g_i(x) & \geq g_i(\bar{x}) + \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle,
\end{align*}
\]

for all \( x \in S \). By Lemma 2.1 and the condition \( \nabla f(\bar{x}) + \nabla g(\bar{x})^\top \bar{\lambda} = \bar{0} \), we have

\[
\begin{align*}
f(x) + \sum_{i=1}^m \bar{\lambda}_i g_i(x) & \geq f(\bar{x}) + \langle \nabla f(\bar{x}), x - \bar{x} \rangle \\
& \quad + \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \langle \nabla g_i(\bar{x}), x - \bar{x} \rangle \\
& = f(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x}) + \langle \nabla f(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i \nabla g_i(\bar{x}), x - \bar{x} \rangle \\
& = f(\bar{x}) + \sum_{i=1}^m \bar{\lambda}_i g_i(\bar{x})
\end{align*}
\]

It follows that \( L(x, \bar{\lambda}) \geq L(\bar{x}, \bar{\lambda}) \) for all \( x \in S \), i.e., \( L(\bar{x}, \bar{\lambda}) = \min_{x \in S} L(x, \bar{\lambda}) \).
Since $\bar{x} \in X$, we have $g_i(\bar{x}) \leq 0$ $(i = 1, \ldots, m)$, and noting that $\bar{\lambda} \circ g(\bar{x}) = 0$, by Theorem 4.4, $(\bar{x}, \bar{\lambda})$ is a saddle-point for the Lagrangian $L(x, \lambda)$.

To prove the converse, suppose that $(\bar{x}, \bar{\lambda}) \in \bigcap S \times R_+^m$ is a saddle-point solution. Since $L(\bar{x}, \lambda) \leq L(\bar{x}, \bar{\lambda})$ for all $\lambda \geq 0$, we have

$$f(\bar{x}) + \lambda \circ g(\bar{x}) \leq f(\bar{x}) + \bar{\lambda} \circ g(\bar{x}).$$

By Lemma 2.1, we get $\lambda \circ g(\bar{x}) \leq \bar{\lambda} \circ g(\bar{x})$. Using proof as in Theorem 4.4, that $g(\bar{x}) \leq 0$, $\bar{\lambda} \circ g(\bar{x}) = 0$, This shows that $\bar{x}$ is feasible to problem (FCP).

Since $L(\bar{x}, \lambda) \leq L(x, \lambda)$ for all $x \in S$, then $\bar{x}$ solves the problem to minimize $L(x, \lambda)$ subject to $x \in S$. Since $\bar{x} \in \bigcap S$, then $\nabla_x L(\bar{x}, \lambda) = 0$. By Theorem 3.2, we have

$$\nabla_x L(\bar{x}, \lambda) = \nabla f(\bar{x}) + \nabla g(\bar{x})^T \lambda.$$

It follows that

$$\nabla f(\bar{x}) + \nabla g(\bar{x})^T \lambda = 0.$$

By Theorem 4.3, $\bar{x}$ is a global minimum point of the primal problem (FP), $\bar{\lambda}$ is a global maximum point of the dual (DFP), and the optimal values $f(\bar{x})$ of (FCP) and $d(\bar{\lambda})$ of (DFCP) coincide.

Consider the "perturbed" primal problem

$$\min f(x), \quad \text{s.t.} \quad g_i(x) \leq z_i, \quad x \in S, \quad i = 1, \ldots, m. \tag{z FCP}$$

Let $Z = \{z \in R^m : g(x) \leq z \text{ for some } x \in S\}$ denote the set of all vectors $z$ for which the perturbed problem (zFCP) has nonempty feasible domain and let $v(z) = \min \{f(x) : x \in S, g(x) \leq z\}$ be the optimal objective function value of the perturbed problem (zFCP).

**Lemma 5.1.** $Z$ is a convex subset in $R^m$ and $v(z)$ is a convex and nonincreasing fuzzy mapping on $Z$.

**Proof.** Suppose that $z^1, z^2 \in Z$ and $\lambda \in [0, 1]$, then there exist $x^1, x^2 \in S$ such that $g(x^1) \leq z^1$ and $g(x^2) \leq z^2$. By the convexity of $S$, and $g_i(x)$ $(i = 1, \ldots, m)$ is a convex fuzzy mapping on $S$ in $g(x) = (g_1(x), \ldots, g_m(x))^T$. It follows that $(1 - \lambda)x^1 + \lambda x^2 \in S$ and

$$g (\lambda (1 - \lambda)x^1 + \lambda x^2) = (g_1 ((1 - \lambda)x^1 + \lambda x^2), \ldots, g_m ((1 - \lambda)x^1 + \lambda x^2))^T$$

$$\leq ((1 - \lambda)g_1 (x^1) + \lambda g_1 (x^2), \ldots, (1 - \lambda)g_m (x^1) + \lambda g_m (x^2))^T$$

$$= (1 - \lambda)g_1 (x^1, \ldots, g_m (x^1)) + \lambda g_1 (x^2, \ldots, g_m (x^2))$$

$$= (1 - \lambda)g (x^1) + \lambda (g (x^2))$$

$$\leq (1 - \lambda)z^1 + \lambda z^2.$$

By the definition of $Z$, we have $(1 - \lambda)z^1 + \lambda z^2 \in Z$, that is, $Z$ is convex.

Since $g_i$ $(i = 1, \ldots, m)$ is convex on $S$, it follows that

$$g ((1 - \lambda)x^1 + \lambda x^2) \leq (1 - \lambda)g (x^1) + \lambda g (x^2) \leq (1 - \lambda)z^1 + \lambda z^2,$$

for $x^1, x^2 \in S$ with $g(x^1) \leq z^1, g(x^2) \leq z^2$, and $\lambda \in [0, 1]$.

By

$$\{f(x) : x = (1 - \lambda)x^1 + \lambda x^2, x^1 \in S, x^2 \in S, g(x^1) \leq z^1, g(x^2) \leq z^2\}$$

$$\subseteq \{f(x) : x = (1 - \lambda)x^1 + \lambda x^2, x^1 \in S, x^2 \in S, g(x) \leq (1 - \lambda)z^1 + \lambda z^2\}.$$
We have
\[
v((1-\lambda)z^1 + \lambda z^2) = \min \{f(x) : x \in S, \ g(x) \leq (1-\lambda)z^1 + \lambda z^2\}
\leq \min \{f(x) : x = (1-\lambda)x^1 + \lambda x^2, \ x^1 \in S, \ x^2 \in S, \ g(x^1) \leq z^1, \ g(x^2) \leq z^2\}
= \min \{f((1-\lambda)x^1 + \lambda x^2) : x^1 \in S, \ x^2 \in S, \ g(x^1) \leq z^1, \ g(x^2) \leq z^2\}
\leq (1-\lambda)\min \{f(x^1) : x^1 \in S, \ g(x^1) \leq z^1\} + \lambda \min \{f(x^2) : x^2 \in S, \ g(x^2) \leq z^2\}
= (1-\lambda)v(z^1) + \lambda v(z^2),
\]
for every \(z^1, z^2 \in Z\) and \(\lambda \in [0, 1]\). One has that \(v(z)\) is a convex fuzzy mapping on \(Z\).

If \(z^1 \leq z^2, z^1, z^2 \in Z\), then, by
\[
\{f(x) : x \in S, \ g(x) \leq z^1\} \subseteq \{f(x) : x \in S, \ g(x) \leq z^2\},
\]
we have
\[
v(z^1) = \min \{f(x) : x \in S, \ g(x) \leq z^1\} \geq \min \{f(x) : x \in S, \ g(x) \leq z^2\} = v(z^2),
\]
that is, \(v(z)\) is nonincreasing on \(Z\).

**Theorem 5.2.** Let \(S \subseteq E\) be nonempty, compact, and convex. If convex fuzzy programming problem
\[
\begin{align*}
&\min f(x), \\
&\text{s.t.} \quad g_i(x) \leq 0, \quad x \in S, \quad i = 1, \ldots, m
\end{align*}
\]
has an optimal solution \(\bar{x}\), then there exists \(\bar{\lambda} \in R^m_+\) such that \((\bar{x}, \bar{\lambda})\) is a saddle-point of the Lagrangian \(L(x, \lambda)\) and \(-\bar{\lambda} \in \partial v(0)\).

**Proof.** Suppose that \((\bar{x}, \bar{\lambda})\) is a saddle-point of the Lagrangian \(L(x, \lambda)\), then by Theorem 4.3, we get
\[
v(0) = \min \{f(x) : x \in S, \ g(x) \leq 0\} = f(\bar{x}) = d(\bar{\lambda})
= \min_{x \in S} L(x, \bar{\lambda}) = \min_{x \in S} \{f(x) + \bar{\lambda} \circ g(x)\}
= \min_{x \in S} \{f(x) + \bar{\lambda} \circ (g(x) - z) + \langle \bar{\lambda}, z \rangle\}
= \min_{x \in S} \{f(x) + \bar{\lambda} \circ (g(x) - z)\} + \langle \bar{\lambda}, z \rangle
\leq v(z) + \langle \bar{\lambda}, z \rangle.
\]

It follows that \(v(z) \geq v(0) + \langle -\bar{\lambda}, z - 0 \rangle\) for every \(z \in Z\), that is, \(-\bar{\lambda} \in \partial v(0)\).

We now apply the above results to fuzzy linear programming and fuzzy quadratic programming. Consider the fuzzy linear programming problem
\[
\begin{align*}
&\min c^T x, \\
&\text{s.t.} \quad Ax \leq b, \quad x \in R^n_+,
\end{align*}
\]
where \(b \in R^m, c, \) and \(A\) are \(n\)-dimensional fuzzy vector and \(m \times n\) fuzzy matrix, respectively.

Since \(x \geq 0, f(x) = c^T x\) is convex and concave fuzzy mapping and \(g(x) = Ax - b\) is an \(m\)-dimensional fuzzy vector. Hence, the fuzzy programming (FLP) is a convex fuzzy programming, whose Lagrangian is given by
\[
L(x, \lambda) = c^T x + \langle \lambda, Ax - b \rangle,
\]
where \(\lambda = (\lambda_1, \ldots, \lambda_m)^T, \lambda_i \geq 0 (i = 1, \ldots, m)\).
It is easy to show that the Lagrangian $L(x, \lambda)$ is a fuzzy mapping from $R^n_+ \times R^m_+$ to $F_0$, and $L(\cdot, \lambda)$ is convex and concave fuzzy mapping. The Lagrangian dual problem of (FLP) is presented below, and denoted by (DFLP).

$$\max_{\lambda \geq 0} d(\lambda),$$  \hspace{1cm} (DFLP)

where $d(\lambda) = \min_{x \in S} L(x, \lambda)$.

Since

$$d(\lambda) = \min_{x \geq 0} L(x, \lambda) = \min_{x \geq 0} \{c^T x + \langle \lambda, Ax - b \rangle\}$$

$$= \min_{x \geq 0} \{c^T x + \langle \lambda, Ax \rangle + \langle \lambda, -b \rangle\}$$

$$= \min_{x \geq 0} \{c^T x + A^T \lambda, x \} + \langle \lambda, -b \rangle\}.$$

Hence, $d(\lambda) = -\langle \lambda, b \rangle$ if $c^T + A^T \lambda \geq 0$; $d(\lambda) = -\infty$ otherwise. Therefore, the dual problem (DFLP) is also equivalent to the problem

$$\max -\langle \lambda, b \rangle,$$

s.t. $c^T + A^T \lambda \geq 0, \lambda \in R^m_+.$

By Theorem 4.2 and its corollaries, we have the following results.

**THEOREM 5.3.** WEAK DUALITY THEOREM. Let $x$ be a feasible solution to the primal problem (FLP), and let $\lambda$ be a feasible solution to dual problem (DFLP). Then $\langle c, x \rangle \geq -\langle \lambda, b \rangle$.

**COROLLARY.** If $\langle c, \bar{x} \rangle = -\langle \bar{\lambda}, b \rangle$, where $\bar{x}$ and $\bar{\lambda}$ are feasible solutions to problems (FLP) and (DFLP), respectively, then $\bar{x}$ and $\bar{\lambda}$ solve the (FLP) and (DFLP).

**THEOREM 5.4.** Let $X = \{x \in R^n_+: Ax \leq b\}$, then $(\bar{x}, \bar{\lambda}) \in X \times R^m_+$ is a saddle-point of fuzzy mapping $L(x, \lambda)$ if and only if $\bar{x}$ and $\bar{\lambda}$ are optimal solutions to the primal problem (FLP) and the dual problem (DFLP), and the optimal values coincide.

**THEOREM 5.5.** Let $X = \{x \in R^n_+: Ax \leq b\}$, a point $(\bar{x}, \bar{\lambda}) \in X \times R^m_+$ is a saddle-point of the Lagrangian $L(x, \lambda)$ if and only if the following conditions hold.

1. $c^T + A^T \lambda = 0.$
2. $\langle \bar{\lambda}, A\bar{x} - b \rangle = 0.$

**PROOF.** By Theorem 3.3, $L(\bar{x}, \bar{\lambda}) = \min_{x \in X} L(x, \bar{\lambda})$ if and only if $0 \in \partial L(\bar{x}, \bar{\lambda})(\partial L(\bar{x}, \bar{\lambda})$ represents the subdifferential of convex fuzzy $L(x, \bar{\lambda})$ at $\bar{x}$).

Since

$$L(x, \bar{\lambda}) = c^T x + \langle \bar{\lambda}, Ax - b \rangle = \langle c^T + A^T \bar{\lambda}, x \rangle + \langle \bar{\lambda}, -b \rangle.$$

It leads to the conclusion that $L(x, \bar{\lambda})$ is differential at $\bar{x}$ by Example 3.1, hence, $\partial L(\bar{x}, \bar{\lambda})$ is a singleton, that is, $\nabla_x L(x, \bar{\lambda}) = 0$. One has that

$$\nabla_x L(x, \bar{\lambda}) = c^T + A^T \bar{\lambda}.$$

It follows that $c^T + A^T \bar{\lambda} = 0$. By Theorem 4.4, this completes the proof.

Furthermore, from the KKT conditions, we have the complementary condition

$$\langle \bar{\lambda}, A\bar{x} - b \rangle = 0.$$
Consider the fuzzy quadratic programming problem, denoted by (FQP)

\[ \min \quad c^T x + \frac{1}{2} x^T Q x, \]
\[ \text{s.t.} \quad Ax \leq b, \quad x \in \mathbb{R}^n, \]

where the \( n \times n \) matrix \( Q \) is symmetric positive definite, \( b \in \mathbb{R}^m \), and \( A \) are \( n \)-dimensional fuzzy vector and an \( m \times n \) fuzzy matrix, respectively.

By Example 3.2 and Theorem 2.5, \( f(x) = c^T x + (1/2)x^T Q x \) \( x \in \mathbb{R}^n_+ \) is a convex fuzzy mapping, and \( g(x) = Ax - b \) \( x \in \mathbb{R}^n_+ \) is convex and concave. From this point of view, (FQP) is a special case of convex fuzzy programming.

The corresponding Lagrangian of (FQP) is

\[ L(x, \lambda) = c^T x + \frac{1}{2} x^T Q x + \langle \lambda, Ax - b \rangle, \]

where \( \lambda = (\lambda_1, \ldots, \lambda_m)^T \), \( \lambda_i \geq 0 \) \( (i = 1, \ldots, m) \).

Lagrangian \( L(x, \lambda) \) is a fuzzy mapping from \( \mathbb{R}^n_+ \times \mathbb{R}^m_+ \) to \( F_0 \), and \( L(\cdot, \lambda) \) is convex.

The dual problem of (FQP) is given by

\[ \max_{\lambda \geq 0} d(\lambda), \quad \text{(DFQP)} \]

where \( d(\lambda) = \min_{x \geq 0} L(x, \lambda) \).

\[ d(\lambda) A = \min_{x \geq 0} L(x, \lambda) = \min_{x \geq 0} \left\{ c^T x + \frac{1}{2} x^T Q x + \langle \lambda, Ax - b \rangle \right\} \]
\[ = \min_{x \geq 0} \left\{ (c^T + A^T \lambda, x) + \frac{1}{2} x^T Q x + \langle \lambda, -b \rangle \right\}. \]

By Theorem 5.1 and Example 3.2, and applying Theorem 3.2, it follows that KKT conditions of (FQP) are

\[ c^T + Q \bar{x} + A^T \bar{\lambda} = 0 \quad \text{and} \quad \langle \bar{\lambda}, A \bar{x} - b \rangle = 0. \]

6. CONCLUSIONS

Motivated by earlier research works [14–16, 22, 24, 25, 29–31], the duality theory of fuzzy programming problems with fuzzy coefficients are formulated by using the fuzzy scalar product proposed in [16] and subdifferential of convex fuzzy mapping proposed in this paper. The corresponding theorems for fuzzy programming problems are derived. Although various investigators in the literature [22, 24, 25, 29–31] have discussed the fuzzy programming models, they merely considered the properties of the global optimal solutions in the case where fuzzy mappings in the models are various generalized convexity. A solution concept of fuzzy optimization problems in [14–16] is essentially similar to the notion of Pareto solution in multiobjective programming problems. Hence, the fuzzy programming problems discussed in this paper is different from the works in [14–16]. However, owing to the above excellent works, the duality theory for a class of fuzzy programming problems in this paper has similar structures to the traditional mathematical programming problems. We will try to explore fuzzy convex analysis and fuzzy optimization and relationships between them in the future.

REFERENCES