Discrete-time $Geo^X/G/1$ queue with unreliable server and multiple adaptive delayed vacations

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Received 7 March 2007; received in revised form 27 August 2007

Abstract

In this paper we consider a discrete-time $Geo^X/G/1$ queue with unreliable server and multiple adaptive delayed vacations policy in which the vacation time, service time, repair time and the delayed time all follow arbitrary discrete distribution. By using a concise decomposition method, the transient and steady-state distributions of the queue length are studied, and the stochastic decomposition property of steady-state queue length has been proved. Several common vacation policies are special cases of the vacation policy presented in this study. The relationship between the generating functions of steady-state queue length at departure epoch and arbitrary epoch is obtained. Finally, we give some numerical examples to illustrate the effect of the parameters on several performance characteristics.

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MSC: primary 60K25; secondary 90B22

Keywords: Discrete-time queueing model; Multiple adaptive delayed vacation; Unreliable; Transient and steady-state distributions; Stochastic decomposition

1. Introduction

Recent years, discrete-time queueing systems have been receiving increasing attention due to their wide applications in design and control of manufacturing and telecommunication systems, and in modeling and analyzing of computer communication networks; for example, the flexible manufacturing system (FMS), the broadband integrated services digital network (B-ISDN) and the asynchronous transfer model (ATM) [4,21,27]. In discrete-time system, time is treated as a discrete variable (slot), and arrivals and departures can only occur at boundary epochs of time slots. To avoid simultaneity of arrival and departure, the systems are classified into the early arrival system and the late arrival system [12]. The discrete-time system has been found to be more appropriate in modeling computer and telecommunication systems because the basic time unit in these systems is a binary code. Moreover, the discrete-time system can be used to approximate the continuous systems.

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doi:10.1016/j.cam.2007.08.019
The concept of vacation policy was introduced into modeling analysis of queueing systems in [15] and two standard vacation policies were defined: single vacation policy and multiple vacation policy. Since then, various vacation policies have been introduced and a considerable amount of work has been done. A literature survey on queueing systems with server vacations can be found in [9]. Choudhury [7] modeled a batch arrival $M^X/G/1$ queueing system with a single vacation. A batch arrival $M^X/G/1$ queueing system with multiple vacations was first studied in [3]. The variations and extensions of these models can be referred to Rosenberg and Yechiali [18], Choudhury [6], Tang and Tang [24], and many others. Zhang and Tian [30] have studied the $Geo/G/1$ queue with multiple adaptive vacations and further in [26] Tian and Zhang considered a $GI/Geo/1$ queue with multiple vacations. Using a matrix-analytic method, a class of discrete-time single-arrival, single-service vacation models have been studied by Alfa. Some recent discrete-time models with batch arrivals and vacations can be referred to Samanta et al. [19], Chang and Choi [5].

Since the server may probably be subjected to lengthy and unpredictable breakdowns while serving a customer, the assumption of an unreliable server is more reasonable. Systems with breakdowns can also be called repairable systems. Gaver [11] firstly proposed a $M/G/1$ queueing system with interrupted service and priorities. Tang [22] and Li et al. [16] investigated the behavior of the $M/G/1$ queueing system with breakdowns from the point of view of both the queueing model and the reliability. Fiems and Bruneel [10] studied the discrete-time $Geo^X/G/1$ queue with multiple vacations governed by a geometrically distributed timer. However, very few papers are known about discrete-time repairable queueing systems.

In this paper, we consider a discrete-time $Geo^X/G/1$ queue with unreliable server and multiple adaptive delayed vacations policy, in which the vacation time, service time, repair time and the delayed time all follow arbitrary discrete distribution. Takagi [20] first proposed the concept of a variant vacation, which is a generalization of the multiple and single vacation policies. In this vacation policy, the maximum number of vacations that the server may take is a random variable. However, the planned vacation process may be interrupted. The server will start a new busy period at the end of a vacation if any arrival occurs in it. Zhang and Tian named this kind of vacation policy for “multiple adaptive vacation policy” in [30]. Ke has analyzed a batch arrival queueing system with balkings and a variant vacation policy [14], Ke has also analyzed a repairable $M/G/1$ queueing model with a variant vacation policy and startup times [13], in which the vacation length is fixed. The repairable $Geo^X/G/1$ queue with multiple adaptive vacations is studied with imbedded Markov chain method in [28].

The multiple adaptive delayed vacations policy is a modification of standard multiple adaptive vacations policy, in which whenever taking a vacation, the server will spend a period of time for vacation preparation. If there is any arrival which occurs in the period of preparation, the server will abort the upcoming vacation and start serving the customers immediately. This vacation policy is more general than most classical vacation policies and the multiple adaptive vacations policy is a special case of it.

The consideration of delay is not only for generalization in mathematics. Firstly, it accommodates the real world more closely. Secondly, it is useful for optimization. Consider the choice that a server station faces in a single-service system when the system becomes empty. It may leave the system to do auxiliary work immediately, or it may stay there for a period of time if new customers are likely to arrive during that time thus he may make more profit. Therefore, it would be practical to consider the multiple adaptive delayed vacations policy for a $Geo^X/G/1$ system with unreliable server. Study about batch arrival queue with delayed vacations and unreliable server in continuous-time cases are referred to Tang and Tang [23], Tang et al. [25]. To the extent of our knowledge, existing work, including those mentioned above, have never investigated this model.

Common procedures for the analysis of queueing models are imbedded Markov chain method [28,30], the supplementary variable method [2,3] introduced in [8], and the theory of matrix-geometric solution [1,29] introduced in [17]. These methods are all based on the assumption of stationary state of system and are concerning system properties on departure epoch $n^+$, while the method in present study analyzes transient state first and then stationary state. We can also get system properties on arbitrary epoch $n$. The supplementary variable method usually becomes too complicated to be solved especially when dealing with models with many variables following general distribution. In this paper, we operate a direct and concise total probability decomposition based on the analysis of the “generalized busy period”. With $z$-transform technology, we derive the probability generating function of both the transient and the steady-state distribution of the queue length, and further the recursion formulae of the queue length distribution in stationary state, which usually cannot be obtained by imbedded Markov chain method. The analysis gives us an intuitive feeling of the structure of the system cycles, and this direct decomposition method operates in continuous situation as well.
The rest of the paper is organized as follows. In the next section, the model of the considered queueing system is described. In Section 3, we study the transient distribution of the queue size in busy period. In Section 4, we study the transient distribution and the steady-state distribution of the queue size at arbitrary epoch. In Section 5, the stochastic decomposition property of steady-state queue length has been proved. In Section 6, we demonstrate that several common queue models are special cases of the queueing model presented in this study. We also give the relationship between the generating functions of steady-state queue length at departure epoch and arbitrary epoch. Finally, we give numerical examples in Section 7.

2. Model formulation

We consider a discrete-time Geo\(^X\)/G/1 system with unreliable server and multiple adaptive delayed vacations policy on first-come, first-serve (FCFS) basis. In discrete time, the time axis is segmented into a sequence of equal time intervals. Furthermore, let the time axis be marked by 0, 1, \ldots, \(n\). The arrivals at different time are independent from each other. Customers arrive in bulk. The number of customers belonging to one arrival batch, denoted by \(D\), is an arbitrary distributed random variable with probability mass function \((p.m.f)\) \(P\{D = j\} = e_j, j = 1, 2, \ldots\) and finite mean value denoted by \(e\). The customers are served one by one. The service order in the same batch is arbitrary, and the service order between different batches is according to the FCFS discipline. For mathematical clarity, we assume that the system is a late arrival system, that is, a potential arrival can only take place in \((n, n)\) and a potential departure can only take place in \((n, n)\). The time between arrivals, denote by \(\tau\), follows a geometric distribution with parameter \(p\), with \(p.m.f\) \(P\{\tau = j\} = p(1 - p)^{j-1}, j = 1, 2, \ldots\). That is to say, a bulk of customers arrive with probability \(p\) in each interval \((n, n)\), \(n = 0, 1, 2, \ldots\), and the probability that no customer arrives is \(1 - p\). The service time, denoted by \(\chi\), has \(p.m.f\) \(P\{\chi = j\} = g_j, j = 1, 2, \ldots\) and the probability generating function \((p.g.f)\) \(G(z)\). The average service time is finite and denoted by \(e\).

The vacation policy of this queueing system is multiple adaptive delayed vacation policy. Whenever the system becomes empty, the server will leave for vacations. The server may take at most \(H\) vacations with random length \(V\). Before each vacation, the server will spend some time for preparation. If a batch of customers arrive in the preparation (delayed) time, the server will abort the upcoming vacation and begin to serve the customers immediately. A vacation will be taken if no customer arrives in delayed time. At each vacation completion instant, the server return to check the system to see if there is any customer waiting. The server will start serving the customer if there is any customer waiting or he will prepare for another vacation if no customer is waiting and the number of vacations taken is less than \(H\). If the total \(H\) vacations have been finished, the server will stay idle and wait for the next arrival. Briefly speaking, there are three cases of starting a new busy period:

Case 1: starting a new busy period in the preparation periods before vacations;
Case 2: starting a new busy period at the completion instant of a vacation;
Case 3: starting a new busy period in idle time.

The \(p.m.f\) and the \(p.g.f\) of \(H\) is \(P\{H = j\} = h_j, j = 1, 2, \ldots\) and \(G_H(z)\), respectively. The delayed time, denoted by \(Y\), has \(p.m.f\) \(P\{Y = j\} = y_j, j = 0, 1, 2, \ldots\) and finite mean and variance. Vacations, denoted by \(V\), are independent and identically distributed \((i.i.d)\) variable with \(P\{V = j\} = v_j, j = 1, 2, \ldots\) and have finite mean and variance. The \(p.g.f\) of \(Y\) and \(V\) are \(y(z)\) and \(v(z)\), respectively. The vacation policy will degenerate into standard multiple adaptive vacations policy when \(P\{Y = 0\} = 1\).

The queueing system will occasionally come into breakdowns. The interval between two breakdowns, that is, the life-span of the system, denoted by \(X\), is an arbitrary distributed discrete random variable with \(p.m.f\) \(P\{X = j\} = R(1 - R)^{j-1}, j = 1, 2, \ldots\). Breakdowns only occur in busy period, and when they take place, the customers being served have to wait. Whenever the server fails, it is immediately repaired at a repair facility, where the repair time, denoted by \(W\), is a generally distributed discrete random variable with finite mean \(\beta\). After repairing, the system renews, and the service will go on. The service time of the interrupted customer will be inherited.

The arriving process, the service process, the maximum vacation number \(H\), the life-span of the system \(X\), repair time \(W\), delayed time \(Y\), vacation time \(V\) and the number of the customers arrived at one time \(D\) are all independent from each other.

It is also assumed that the server will not take a vacation if there is not any customer at \(t = 0\) and the server will stay idle and wait for the first arrival. However, as we shall prove later, the steady-state performances are independent from the initial state.
3. The transient distribution of queue length in generalized busy period

Let \( \tilde{\mathcal{Y}} \) present the “generalized service time”. The “generalized service time” of a customer (the \( n \)th arrival) is the time interval between the start and the end of the service of the \( n \)th arrived customer, which includes the reparation caused by system failure. For \( j \geq 1 \), let

\[
\tilde{g}_j^{[k]} = P\{\tilde{\mathcal{Y}} = j; k \text{ breakdowns occur in } \tilde{\mathcal{Y}}\}, \quad k \geq 0.
\]

Because of the mutual independence of the random variables and the fact that “\( k \) breakdowns occur in \( \tilde{\mathcal{Y}} \)” means that the service time is midst the sum of \( k \) and \( k + 1 \) lifetimes, we have

\[
\tilde{g}_j = P\{\tilde{\mathcal{Y}} = j\} = \sum_{k=0}^{\infty} P\{\tilde{\mathcal{Y}} = j; k \text{ breakdowns occur in } \tilde{\mathcal{Y}}\} = \sum_{k=0}^{\infty} \tilde{g}_j^{[k]} = \sum_{l=1}^{j} \sum_{k=0}^{l-1} P\{\tilde{\mathcal{Y}} = l\} \cdot \sum_{i=1}^{k} P\{X_i = \mathcal{Y} < \sum_{i=1}^{k+1} X_i\} = \sum_{l=1}^{j} \sum_{k=0}^{l-1} P\{\tilde{\mathcal{Y}} = l\} \cdot \sum_{i=1}^{k} P\{X_i = \mathcal{Y} < \sum_{i=1}^{k+1} X_i\} = \sum_{l=1}^{j} \sum_{k=0}^{l-1} P\{\tilde{\mathcal{Y}} = l\} \cdot \sum_{i=1}^{k} P\{X_i = \mathcal{Y} < \sum_{i=1}^{k+1} X_i\} = \sum_{l=1}^{j} \sum_{k=0}^{l-1} P\{\tilde{\mathcal{Y}} = l\} \sum_{i=1}^{k} P\{X_i = \mathcal{Y} < \sum_{i=1}^{k+1} X_i\} \cdot \left(\frac{1}{l}\right) R^k (1 - R)^{l-k},
\]

where \( X_i \) and \( W_i \) denote \( i \)th life-span and \( i \)th repair time, respectively.

Therefore,

\[
\tilde{g}_j = P\{\tilde{\mathcal{Y}} = j\} = \sum_{l=1}^{j} \sum_{k=0}^{l-1} P\{\tilde{\mathcal{Y}} = l\} \sum_{i=1}^{k} P\{X_i = \mathcal{Y} < \sum_{i=1}^{k+1} X_i\} = \sum_{l=1}^{j} \sum_{k=0}^{l-1} P\{\tilde{\mathcal{Y}} = l\} \sum_{i=1}^{k} P\{X_i = \mathcal{Y} < \sum_{i=1}^{k+1} X_i\} \cdot \left(\frac{1}{l}\right) R^k (1 - R)^{l-k},
\]

and the p.g.f of \( \{\tilde{g}_j, j \geq 1\} \) is

\[
\tilde{G}(z) = \sum_{j=1}^{\infty} \tilde{g}_j z^j = G[z R \cdot w(z) + z \tilde{R}], \quad |z| < 1,
\]

and the mean of generalized service time is

\[
E[\tilde{\mathcal{Y}}] = E[\mathcal{Y}](1 + R \cdot E[W]) = \mathcal{X}(1 + R \beta),
\]

where \( w(z) \) presents the p.g.f of the repair time \( W \), \( w(z) = \sum_{k=0}^{\infty} P\{W = k\} z^k; \mathcal{X} = 1 - R \). The results are similar to their counterparts in common-time situation (see Gaver [11] and Tang [22]).

Treating \( \tilde{\mathcal{Y}} \) as its counterpart in common queue model in which the server is always available, the system being discussing is equal to common discrete-time Geo*/G/1 with multiple adaptive delayed vacations. Its input interval follows a geometric distribution with parameter \( p \), and the service times are i.i.d and follow distribution \( \{\tilde{g}_j, j \geq 1\} \).

For mathematical clarity, we note that “generalized busy period” is from the instant when the system terminates idle period and begins to serve customers until the system becomes idle again. Let \( \tilde{b} \) represent the length of generalized busy period beginning with only one customer. Similar to the discussing of the busy period of classic Geo*/G/1 queueing system, we have

**Lemma 1.** Let \( \tilde{b}(z) = \sum_{n=0}^{\infty} P\{\tilde{b} = n\} z^n, |z| < 1 \), then \( \tilde{b}(z) \) subjects to the equation \( \tilde{b}(z) = \tilde{G}[z (\tilde{\rho} + p A(\tilde{b}(z)))] \), mean

\[
E[\tilde{b}] = \begin{cases} \frac{\mathcal{X}(1 + R \beta)}{1 - \tilde{\rho}} & \tilde{\rho} < 1 \\ \infty & \tilde{\rho} \geq 1 \end{cases},
\]

where \( \tilde{\rho} = 1 - \rho \), \( \tilde{\rho} = \rho z e^2 + R \) denotes the traffic intensity, \( A(z) = \sum_{k=1}^{\infty} e_k z^k \) denotes the p.g.f of the distribution \( \{e_k, k \geq 1\} \).
“System idle time”, denoted by $\tilde{\tau}$, is from the instant when the system becomes idle until a new batch of customers arrives. The system idle time of the queueing system is the residual time of arriving interval $\tau$. Because of Markov property, $\tilde{\tau}$ follows the same distribution as $\tau$.

In the remaining part of this section, we will give the distribution of the queue length in generalized busy period beginning with only one customer. Let $Q_j(n)$ represent the probability that the queueing length equals to $j$ at instant $n$ in generalized busy period $\tilde{b}$, which is given by

$$Q_j(n) = P\{\tilde{b} > n \geq 0; N(n) = j\}, \quad j = 1, 2, \ldots,$$

where the instant $n = 0$ is the beginning of the busy period, and the boundary condition is $Q_1(0) = 1, Q_j(0) = 0, j > 1$.

**Theorem 1.** If $|z| < 1$, let $q_j(z) = \sum_{n=0}^{\infty} Q_j(n)z^n$ present the p.g.f of $Q_j(n)$, we have

$$q_1(z) = \frac{\tilde{b}(z)[1 - \tilde{G}(z)\tilde{p}]}{(1 - z\tilde{p})G(z\tilde{p})}, \quad (1)$$

$$q_j(z) = \frac{\tilde{b}(z)}{G(z\tilde{p})} \sum_{i=1}^{j-1} \sum_{m[i]=j-1} e_{m_1} \cdots e_{m_i} \sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} \tilde{g}_k \binom{n}{i} p^i (1 - p)^{n-i} + \frac{1}{G(z\tilde{p})} \sum_{l=0}^{j-1} q_{j-l}(z) \tilde{b}^{l+1}(z) \tilde{G}(z\tilde{p}) \sum_{i=1}^{l} \sum_{m[i]=j} e_{m[i]} e_{m_1} \cdots e_{m_k} z^k \tilde{g}_k \binom{k}{i} p^i (1 - p)^{k-i}, \quad (2)$$

where $m[k] = m_1 + m_2 + \cdots + m_k, k \geq 1$, $\sum_{m[k]=j} = \sum_{m[i]=1}^{i_1} \sum_{m[i_2]=1}^{i_2} \cdots \sum_{m[i_k]=1}^{i_k}$, and $i_1 + i_2 + \cdots + i_k = j$.

**Proof.** The proof is similar to its counterpart proof in continuous-time system [24]. $\square$

4. The transient distribution and the steady-state distribution of queue length

Let $P_{i,j}(n)$ be the condition probability $P_{i,j}(n) = P\{N(n) = j|N(0) = i\}$, and let $p_{i,j}(z) = \sum_{n=0}^{\infty} z^n P_{i,j}(n)$ present the p.g.f of $P_{i,j}(n)$.

**Theorem 2.** If $|z| < 1$, then $p_{0,0}(z)$ and $p_{i,0}(z)$ are given by

$$p_{0,0}(z) = \frac{1}{1 - z\tilde{p}} \left[ 1 + \frac{pz[1 - \theta(z\tilde{p})]A(\tilde{b}(z))}{(1 - z\tilde{p})[1 - t(z)]} \right], \quad (3)$$

$$p_{i,0}(z) = \frac{[1 - \theta(z\tilde{p})]\tilde{b}^i(z)}{(1 - z\tilde{p})[1 - t(z)]}, \quad i \geq 1, \quad (4)$$

where

$$A = z[pA(\tilde{b}(z)) + \tilde{p}]; \quad v(z\tilde{p}) = \sum_{j=1}^{\infty} (z\tilde{p})^j P\{V = j\}; \quad y(z\tilde{p}) = \sum_{j=0}^{\infty} (z\tilde{p})^j P\{Y = j\};$$

$$\theta(z\tilde{p}) = y(z\tilde{p})v(z\tilde{p}); \quad G_H(\theta(z\tilde{p})) = \sum_{j=1}^{\infty} h_j[\theta(z\tilde{p})]^j; \quad A(\tilde{b}(z)) = \sum_{k=1}^{\infty} e_k \tilde{b}^k(z);$$

$$t(z) = \frac{pzA(\tilde{b}(z))}{1 - z\tilde{p}}[1 - y(z\tilde{p}) + y(z\tilde{p})G_H(\theta(z\tilde{p})) - \theta(z\tilde{p})G_H(\theta(z\tilde{p}))]$$

$$+ y(z\tilde{p})[v(A) + G_H(\theta(z\tilde{p}))v(z\tilde{p}) - G_H(\theta(z\tilde{p}))v(A)].$$
Proof. Let \( s_k = \sum_{i=0}^{k} (V_i + Y_i), \) \( l_k = \sum_{i=1}^{k} \tilde{t}_i, \) \( k \geq 1 \) and \( s_0 = 0, l_0 = 0. \) The system is empty at time \( n \) if and only if the system is in idle period at that time, and the condition probability \( P_{0,0}(n) \) can be divided into two parts according to whether the first batch of customers arrived before \( n. \) Considering the condition that system starts from idle state, we have

\[
P_{0,0}(n) = P\{0 \leq n < \tilde{t}_1\} + P\{\tilde{t}_1 \leq n, N(n) = 0\}
\]

\[
= (1 - p)^n + \sum_{k=1}^{n} e_k P\{\tilde{t}_1 + \tilde{b}^{(k)} \leq n < \tilde{t}_1 + \tilde{b}^{(k)} + \tilde{r}_2\}
\]

\[
+ \sum_{k=1}^{n} e_k \sum_{j=1}^{\infty} h_j P\{s_j < \tilde{r}_2; \tilde{t}_1 + \tilde{b}^{(k)} + \tilde{r}_2 \leq n; N(n) = 0\}
\]

\[
+ \sum_{k=1}^{n} e_k \sum_{j=1}^{\infty} h_j \sum_{r=1}^{j} P\{s_{r-1} < \tilde{r}_2 \leq s_r - Y_r; \tilde{t}_1 + \tilde{b}^{(k)} + \tilde{r}_2 \leq n; N(n) = 0\}
\]

\[
+ \sum_{k=1}^{n} e_k \sum_{j=1}^{\infty} h_j \sum_{r=1}^{j} P\{s_{r-1} + Y_r < \tilde{r}_2 \leq s_r; \tilde{t}_1 + \tilde{b}^{(k)} + s_r \leq n; N(n) = 0\}.
\]

The second part of formula (5) is the probability that time \( n \) is among the second “system idle time”, which is equal to

\[
\sum_{k=1}^{n} e_k \sum_{i=k}^{n} (1 - p)^{n-i} P\{\tilde{t}_1 + \tilde{b}^{(k)} = i\}.
\]

The third part of formula (5) is the probability that time \( n \) is after the second “system idle time” and the second “generalized busy period” starts in idle time, which is equal to

\[
\sum_{k=1}^{n} e_k \sum_{j=1}^{\infty} h_j P\{s_j < \tilde{r}_2; \tilde{t}_1 + \tilde{b}^{(k)} + \tilde{r}_2 \leq n; N(n) = 0\}
\]

\[
= \sum_{k=1}^{n} e_k \sum_{j=1}^{\infty} h_j \sum_{m=1}^{\infty} e_m \sum_{i=k+1}^{n} p_{m,0}(n-i-r) P\{s_j < r\} P\{\tilde{r}_2 = r\} P\{\tilde{t}_1 + \tilde{b}^{(k)} = i\}.
\]

The fourth part of formula (5) is the probability that time \( n \) is after the second “system idle time” and the second “generalized busy period” starts in a preparation period, which is equal to

\[
\sum_{k=1}^{n} e_k \sum_{j=1}^{\infty} h_j \sum_{r=1}^{j} P\{s_{r-1} \leq \tilde{r}_2 \leq s_r - Y_r; \tilde{t}_1 + \tilde{b}^{(k)} + \tilde{r}_2 \leq n; N(n) = 0\}
\]

\[
= \sum_{k=1}^{n} e_k \sum_{j=1}^{\infty} h_j \sum_{r=1}^{\infty} e_m \sum_{i=k+1}^{n} \sum_{u=1}^{n-i} p_{m,0}(n-i-u) P\{s_{r-1} < u \leq s_r - Y_r\} P\{\tilde{t}_1 + \tilde{b}^{(k)} = i\} P\{\tilde{r}_2 = u\}
\]

\[
= \sum_{k=1}^{n} e_k \sum_{j=1}^{\infty} h_j \sum_{r=1}^{\infty} e_m \sum_{i=k+1}^{n} \sum_{u=1}^{n-i} p_{m,0}(n-i-u)
\]

\[
\times [ P\{s_{r-1} < u\} - P\{s_{r-1} + Y_r < u\} ] P\{\tilde{t}_1 + \tilde{b}^{(k)} = i\} P\{\tilde{r}_2 = u\}.
\]
The fifth part of formula (5) is the probability that time \( n \) is after the second “system idle time” and the second “generalized busy period” starts at the end of a vacation, which is equal to

\[
\sum_{k=1}^{n} \sum_{j=1}^{\infty} h^{j} \sum_{r=1}^{j} P\{s_{r-1} + Y_r < \tilde{\tau}_2 \leq s_r; \tilde{\tau}_1 + \tilde{\tau}^{(k)} + s_r \leq n; N(n) = 0\}
\]

\[
= \sum_{k=1}^{n} \sum_{j=1}^{\infty} h^{j} \sum_{r=1}^{j} \sum_{v=1}^{\infty} \sum_{m_1=1}^{\infty} e_m \cdots \sum_{m_v}^{\infty} e_{m_v} \sum_{i=k+1}^{n} \sum_{u=1}^{n-i} \sum_{s=1}^{n-i-u} p_{m[v],0}(n-i-u-s)
\times P\{u < \tilde{\tau}_2 \leq u + s; u < \tilde{\tau} + l_{v-1} \leq u + s < \tilde{\tau} + l_v\} P\{V = s\} P\{\tilde{\tau}_1 + \tilde{\tau}^{(k)} = i\} P\{s_{r-1} + Y_r = u\}
\]

\[
= \sum_{k=1}^{n} \sum_{j=1}^{\infty} h^{j} \sum_{r=1}^{j} \sum_{v=1}^{\infty} \sum_{m_1=1}^{\infty} e_m \cdots \sum_{m_v}^{\infty} e_{m_v} \sum_{i=k+1}^{n} \sum_{u=1}^{n-i} \sum_{s=1}^{n-i-u} p_{m[v],0}(n-i-u-s)
\times \left( \frac{s}{v} \right)^{u+s-v} P\{V = s\} P\{\tilde{\tau}_1 + \tilde{\tau}^{(k)} = i\} P\{s_{r-1} + Y_r = u\}.
\] (9)

Substituting (6)–(9) into (5), and getting the p.g.f of both sides of the equation, we have

\[
p_{0,0}(z) = \frac{1}{1-zp} + \frac{pzA(\tilde{\tau}(z))}{(1-zp)^2} + \frac{(pz)^2 G_H(\theta(z)) A(\tilde{\tau}(z))}{(1-zp)^2} \sum_{i=1}^{\infty} e_i p_{i,0}(z)
\]

\[
+ \frac{(pz)^2 [1 - y(z)] [1 - G_H(\theta(z)) A(\tilde{\tau}(z))]}{(1-zp)^2 (1-\theta(z))} \sum_{i=1}^{\infty} e_i p_{i,0}(z)
\]

\[
+ \frac{pzA(\tilde{\tau}(z)) y(z) [1 - G_H(\theta(z))]}{(1-zp)(1-\theta(z))} \sum_{i=1}^{\infty} \sum_{m_1=1}^{\infty} \cdots \sum_{m_i=1}^{\infty} e_m \cdots e_{m_i} p_{m[v],0}(z)
\]

\[
\times \sum_{n=0}^{\infty} z^n \left( \begin{array}{c} n \\ i \end{array} \right) p^i (1-p)^{n-i} P\{V = n\}.
\] (10)

For \( i \geq 1 \), similarly, we have

\[
p_{i,0}(n) = \sum_{k=i}^{n} (1-p)^{n-k} P\{\tilde{\tau}^{(i)} = k\}
\]

\[
+ \sum_{j=1}^{\infty} h_j \sum_{m=1}^{\infty} \sum_{l=1}^{n-k} \sum_{u=0}^{l} p_{m,0}(n-k-l) P\{s_j = u\} P\{\tilde{\tau} = l\} P\{\tilde{\tau}^{(i)} = k\}
\]

\[
+ \sum_{j=1}^{\infty} h_j \sum_{r=1}^{j} \sum_{v=1}^{\infty} \sum_{m_1=1}^{\infty} \cdots \sum_{m_v=1}^{\infty} e_{m_1} \cdots e_{m_v} \sum_{i=k+1}^{n} \sum_{u=1}^{n-i} \sum_{s=1}^{n-i-u} p_{m[v],0}(n-k-u-s) \left( \frac{u+s}{v} \right)^{u+s-v} P\{V = s\} P\{s_{r-1} + Y_r = u\} P\{\tilde{\tau}^{(i)} = k\}
\]

\[
+ \sum_{j=1}^{\infty} h_j \sum_{r=1}^{j} \sum_{m=1}^{\infty} \sum_{l=1}^{n-k} \sum_{u=0}^{l} p_{m,0}(n-k-l)
\]

\[
\times \left( \sum_{w=1}^{u} P\{s_{j-1} = w\} - \sum_{w=1}^{j} P\{s_{j-1} + Y_j = w\} \right) P\{\tilde{\tau} = l\} P\{\tilde{\tau}^{(i)} = k\}.
\] (11)
Getting the p.g.f of both sides of the equation, we have

\[
P_{t,0}(z) = \frac{\tilde{b}^i(z)}{1 - z\tilde{p}} + \frac{pz\tilde{b}^i(z)}{(1 - z\tilde{p})(1 - \theta(z\tilde{p}))} \sum_{i=1}^{\infty} e_i P_{i,0}(z) + \frac{pz\tilde{b}^i(z)[1 - y(z\tilde{p})][1 - G_H(\theta(z\tilde{p}))]}{(1 - z\tilde{p})(1 - \theta(z\tilde{p}))} \sum_{i=1}^{\infty} e_i P_{i,0}(z) + \frac{\tilde{b}^i(z)y(z\tilde{p})[1 - G_H(\theta(z\tilde{p}))]}{1 - \theta(z\tilde{p})} \sum_{i=1}^{\infty} \sum_{m_1=1}^{\infty} \cdots \sum_{m_i=1}^{\infty} e_{m_1} \cdots e_{m_i} P_{m[i],0}(z)
\times \sum_{n=0}^{\infty} z^n \binom{n}{i} p^i (1 - p)^{n-i} P\{V = n\}. \tag{12}
\]

(10) and (6) give the relationship between \(p_{0,0}(z)\) and \(p_{1,0}(z)\):

\[
P_{t,0}(z) = \left(1 - z\tilde{p}\right)\tilde{b}^i(z) \left\{ P_{0,0}(z) - \frac{1}{1 - z\tilde{p}} \right\}. \tag{13}
\]

Substituting (13) into (10) gives (3), and furthermore, we get (4). \(\square\)

**Theorem 3.** If \(|z| < 1\), then \(p_{0,j}(z)\) and \(p_{i,j}(z)\) are given by

\[
p_{0,j} = \frac{pz}{(1 - z\tilde{p})(1 - t(z))} \left\{ \frac{A(\tilde{b}(z))(1 - \theta(z\tilde{p}))}{\tilde{b}(z)} q_j(z) + y(z\tilde{p})A(\tilde{b}(z))[1 - G_H(\theta(z\tilde{p}))]
\times [\delta_{2,j}(z) + \delta_{3,j}(z)] + [1 - y(z\tilde{p})v(A) + y(z\tilde{p})G_H(\theta(z\tilde{p}))[v(A) - v(z\tilde{p})]]\delta_{1,j}(z) \right\}. \tag{14}
\]

\[
p_{i,j}(z) = \sum_{k=1}^{i} q_{j-i+k}(z)\tilde{b}^{i-1}(z) + \frac{\tilde{b}^i(z)}{1 - t(z)} \left\{ \frac{t(z) - \theta(z\tilde{p})}{\tilde{b}(z)} q_j(z) + y(z\tilde{p})[1 - G_H(\theta(z\tilde{p}))]
\times [\delta_{2,j}(z) + \delta_{3,j}(z)] + \frac{pz[1 - y(z\tilde{p}) + G_H(\theta(z\tilde{p}))][y(z\tilde{p}) - \theta(z\tilde{p})]}{(1 - z\tilde{p})}\delta_{1,j}(z) \right\}. \tag{15}
\]

where

\[
\delta_{1,j}(z) = \sum_{k=1}^{j-1} q_{j-k}(z)\tilde{b}^{k+1}(z) \left[ A(\tilde{b}(z)) - \sum_{i=1}^{k} e_i \tilde{b}^i(z) \right],
\]

\[
\delta_{2,j}(z) = \sum_{k=1}^{j-1} q_{j-k}(z)\tilde{b}^{k+1}(z) \left\{ v(A) - v(z\tilde{p}) - \sum_{m[i]=i}^{k} \tilde{b}^m[i](z)e_{m_1} \cdots e_{m_i} \sum_{n=0}^{\infty} z^n \binom{n}{i} p^i (1 - p)^{n-i} P\{V = n\} \right\},
\]

\[
\delta_{3,j}(z) = \sum_{i=1}^{j} \sum_{m[i]=j} e_{m_1} \cdots e_{m_i} \sum_{n=0}^{\infty} z^n \sum_{m=i+1}^{\infty} \binom{n}{i} p^i (1 - p)^{n-i} P\{V = m\}.
\]
Proof. Note that \( P(\tilde{\mathbf{b}}^{(i)} > n \geq 0; N(n) = j) = \sum_{k=1}^{j} Q_{j-k+i}(z) \ast \tilde{B}^{(k-1)}(z) \), similar to Theorem 2, we have

\[
P_{0,j}(n) = \sum_{k=1}^{\infty} e_k P(\tilde{\tau}_1 \leq n < \tilde{\tau}_1 + \tilde{b}^{(k)}; N(n) = j) + \sum_{k=1}^{\infty} e_k P(\tilde{\tau}_1 + \tilde{b}^{(k)} + \tilde{\tau}_2 \leq n; N(n) = j)
\]

\[
= \sum_{k=1}^{\infty} e_k \sum_{i=0}^{\infty} \sum_{r=1}^{n} \sum_{l=i}^{n-r} Q_{j-k+i}(n-r-l) P(\tilde{\tau}_1 + \tilde{b}_2 + \ldots + \tilde{b}_{l-1} = l) P(\tilde{\tau}_1 = r)
\]

\[
+ \sum_{k=1}^{\infty} e_k \sum_{i=0}^{\infty} \sum_{r=1}^{n} \sum_{l=i}^{n-r} \sum_{v=1}^{m} \sum_{m=1}^{n-i} \sum_{i=1}^{k+1} \sum_{r=1}^{n-v} \sum_{w=1}^{v} P_{m,v,j}(n-i-r) P(s_j = w) P(\tilde{\tau}_2 = r) P(\tilde{\tau}_1 + \tilde{b}^{(k)} = i)
\]

\[
+ \sum_{k=1}^{\infty} e_k \sum_{i=0}^{\infty} \sum_{r=1}^{n} \sum_{l=i}^{n-r} \sum_{v=1}^{m} \sum_{m=1}^{n-i} \sum_{i=1}^{k+1} \sum_{r=1}^{n-v} \sum_{w=1}^{v} P_{m,v,j}(n-i-w) \times (P(s_{r-1} < w) - P(s_{r-1} + Y_r < w)) P(\tilde{\tau}_2 = w) P(\tilde{\tau}_1 + \tilde{b}^{(k)} = i)
\]

\[
+ \sum_{k=1}^{\infty} e_k \sum_{i=0}^{\infty} \sum_{r=1}^{n} \sum_{l=i}^{n-r} \sum_{v=1}^{m} \sum_{m=1}^{n-i} \sum_{i=1}^{k+1} \sum_{r=1}^{n-v} \sum_{w=1}^{v} P_{m,v,j}(n-i-w) u \times (\frac{w}{\varepsilon}) p^e p^{w+u-\varepsilon} P(V = u) P(s_{r-1} + Y_r = w) P(\tilde{\tau}_1 + \tilde{b}^{(k)} = i)
\]

\[
= \sum_{k=1}^{\infty} e_k \sum_{i=0}^{\infty} \sum_{r=1}^{n} \sum_{l=i}^{n-r} Q_{j-k+i}(n-r-l) P(\tilde{\tau}_1 + \tilde{b}_2 + \ldots + \tilde{b}_{l-1} = l)
\]

\[
+ \sum_{k=1}^{\infty} h_{v} \sum_{i=0}^{\infty} \sum_{r=1}^{n} \sum_{l=i}^{n-r} \sum_{v=1}^{m} \sum_{m=1}^{n-i} \sum_{i=1}^{k+1} \sum_{r=1}^{n-v} \sum_{w=1}^{v} P_{m,v,j}(n-k-r) P(s_j = w) P(\tilde{\tau} = r) P(\tilde{\mathbf{b}}^{(i)} = k)
\]

\[
+ (P(s_{r-1} < w) - P(s_{r-1} + Y_r < w)) P(\tilde{\tau} = w) P(\tilde{\mathbf{b}}^{(i)} = k)
\]

\[
+ \sum_{k=1}^{\infty} h_{v} \sum_{i=0}^{\infty} \sum_{r=1}^{n} \sum_{l=i}^{n-r} \sum_{v=1}^{m} \sum_{m=1}^{n-i} \sum_{i=1}^{k+1} \sum_{r=1}^{n-v} \sum_{w=1}^{v} P_{m,v,j}(n-k-w) \times \left( \frac{w}{\varepsilon} \right) p^e p^{n-i-w-\varepsilon} p^w P(V = u) P(s_{r-1} + Y_r = w) P(\tilde{\mathbf{b}}^{(i)} = k)
\]

\[
+ \sum_{k=1}^{\infty} h_{v} \sum_{i=0}^{\infty} \sum_{r=1}^{n} \sum_{l=i}^{n-r} \sum_{v=1}^{m} \sum_{m=1}^{n-i} \sum_{i=1}^{k+1} \sum_{r=1}^{n-v} \sum_{w=1}^{v} P_{m,v,j}(n-k-w) \times \left( \frac{w}{\varepsilon} \right) p^e p^{n-i-w-\varepsilon} p^w P(V = u) P(s_{r-1} + Y_r = w) P(\tilde{\mathbf{b}}^{(i)} = k).
\]
Getting the p.g.f of both sides of Eqs. (16) and (17), we have

\[ p_{0,j}(z) = \frac{pz}{1-zp} \sum_{k=1}^{\infty} e_k \sum_{l=1}^{k} q_{j-k+l}(z) \tilde{b}^{l-1}(z) + \left( \frac{pz}{1-zp} \right)^2 A(\tilde{b}(z)) G_{\bar{\theta}(z \bar{p})} \sum_{i=1}^{\infty} e_i p_{i,j}(z) \]

\[ + \frac{p^2 z^2 A(\tilde{b}(z))[1-y(z \bar{p})][1-G_{\bar{\theta}(z \bar{p})}]}{(1-zp)(1-\theta(z \bar{p}))} \sum_{i=1}^{\infty} e_i p_{i,j}(z) \]

\[ + \frac{pz A(\tilde{b}(z)) y(z \bar{p})[1-G_{\bar{\theta}(z \bar{p})}]}{(1-zp)(1-\theta(z \bar{p}))} \sum_{l=1}^{\infty} \sum_{m[l]=j} P(D[l]=l) p_{lj}(z) \sum_{n=0}^{\infty} z^n \left( \begin{array}{c} n \vspace{1mm} \hline i \end{array} \right) p^i (1-p)^{n-i} P(V = n) \]

\[ + \frac{pz A(\tilde{b}(z)) y(z \bar{p})[1-G_{\bar{\theta}(z \bar{p})}]}{(1-zp)(1-\theta(z \bar{p}))} \sum_{j} \sum_{l=1}^{\infty} e_{m_1} \cdots e_{m_j} \sum_{n=0}^{\infty} z^n \left( \begin{array}{c} n \vspace{1mm} \hline e \end{array} \right) p^e (1-p)^{n-e} P(V = m) \]

\times \sum_{m=n+1}^{\infty} P(V = m). \tag{18} \]

\[ p_{i,j}(z) = \sum_{k=1}^{i} q_{j-i+k}(z) \tilde{b}^{i-1}(z) + \frac{pz \tilde{b}^{i}(z) G_{\bar{\theta}(z \bar{p})}}{1-zp} \sum_{k=1}^{\infty} e_k p_{k,j}(z) \]

\[ + \frac{pz \tilde{b}^{i}(z) [1-y(z \bar{p})][1-G_{\bar{\theta}(z \bar{p})}]}{(1-zp)(1-\theta(z \bar{p}))} \sum_{k=1}^{\infty} e_k p_{k,j}(z) \]

\[ + \tilde{b}^{i}(z) \frac{y(z \bar{p})[1-G_{\bar{\theta}(z \bar{p})}]}{1-\theta(z \bar{p})} \sum_{e=1}^{\infty} \sum_{l=e}^{\infty} P(D[l]=l) p_{lj}(z) \sum_{n=0}^{\infty} z^n \left( \begin{array}{c} n \vspace{1mm} \hline e \end{array} \right) p^e (1-p)^{n-e} P(V = n) \]

\[ + \tilde{b}^{i}(z) \frac{y(z \bar{p})[1-G_{\bar{\theta}(z \bar{p})}]}{1-\theta(z \bar{p})} \sum_{e=1}^{\infty} \sum_{m[e]=j} e_{m_1} \cdots e_{m_e} \sum_{n=0}^{\infty} z^n \left( \begin{array}{c} n \vspace{1mm} \hline e \end{array} \right) p^e (1-p)^{n-e} P(V = m) \]

\times \sum_{m=n+1}^{\infty} \left( \begin{array}{c} n \vspace{1mm} \hline e \end{array} \right) p^e (1-p)^{n-e} P(V = m). \tag{19} \]

With (18) and (19), we can get the relationship between \( p_{0,j}(z) \) and \( p_{i,j}(z) \):

\[ p_{i,j}(z) = \sum_{k=1}^{i} q_{j-i+k}(z) \tilde{b}^{i-1}(z) \]

\[ + \frac{(1-zp) \tilde{b}^{i}(z)}{zp A(\tilde{b}(z))} \left[ p_{0,j}(z) - \frac{zp}{1-zp} \sum_{k=1}^{n} e_k \sum_{l=1}^{k} q_{j-k+l}(z) \tilde{b}^{l-1}(z) \right], \quad i \geq 1. \tag{20} \]

Substitute (20) into (18) and note that when \( j \leq 0 \), \( q_j(z) = 0 \). After some algebraic simplification,

\[ p_{0,j}(z) = t(z) p_{0,j}(z) + f(z) [1-y(z \bar{p})v(A) + y(z \bar{p})v(z \bar{p})] \sum_{k=1}^{\infty} e_k \sum_{l=1}^{k} q_{j-k+l}(z) \tilde{b}^{l-1}(z) \]

\[ + f(z) y(z \bar{p}) A(\tilde{b}(z)) \sum_{l=1}^{\infty} \sum_{m[l]=j} P(D[l]=l) \sum_{k=1}^{\infty} z^n \left( \begin{array}{c} n \vspace{1mm} \hline e \end{array} \right) p^e (1-p)^{n-e} P(V = n) \]

\[ + f(z) y(z \bar{p}) A(\tilde{b}(z)) \delta_{3,j}(z). \tag{21} \]
Note that
\[
\sum_{k=1}^{\infty} e_k \sum_{l=1}^{k} q_{j-k+l}(z) b^{l-1}(z)
\]
\[
= \frac{A(\hat{b}(z))}{b(z)} q_j(z) + \delta_{1,j}(z) \sum_{l=1}^{\infty} \sum_{i=l}^{\infty} P[D[i] = l] \sum_{k=1}^{l} q_{j-l+k}(z) b^{k-1}(z) \sum_{n=0}^{\infty} z^n \left( \frac{n}{e} \right) p^n (1 - p)^{n-i} P[V = n]
\]
\[
= \sum_{k=1}^{\infty} \hat{b}^{k-1}(z) \sum_{l=k}^{\infty} q_{j-l+k}(z) \sum_{i=1}^{l} P[D[i] = l] \sum_{n=0}^{\infty} z^n \left( \frac{n}{e} \right) p^n (1 - p)^{n-i} P[V = n]
\]
\[
= q_j(z) \frac{b(z)}{b(z)} [v(A) - v(z \hat{b})] + \delta_{3,j}(z),
\]
we can get (14). Combining (20) gives (15). □

**Theorem 4.** Let \( p_j = \lim_{n \to \infty} P_{i,j}(n) \), then

1. when \( \hat{p} \geq 1 \), \( p_j = 0 \), \( j \geq 0 \);
2. when \( \hat{p} < 1 \),

\[
p_0 = (1 - \hat{p}) \frac{1 - \theta(\hat{p})}{1 - y(\hat{p}) + y(\hat{p})[G_H(\theta(\hat{p}))[1 - v(\hat{p})] + (1 - G_H(\theta(\hat{p})))pE[V]],}
\]

\[
p_j = \frac{p(1 - \hat{p})}{[1 - y(\hat{p}) + y(\hat{p})[G_H(\theta(\hat{p}))[1 - v(\hat{p})] + (1 - G_H(\theta(\hat{p})))pE[V]]}
\]

\[\times [1 - \theta(\hat{p})] A_{1,j} + [1 - y(\hat{p}) + y(\hat{p})\theta(\hat{p})(1 - v(\hat{p}))] A_{1,j} + y(\hat{p})[1 - \theta(\hat{p})](A_{2,j} + A_{3,j})\] \quad (23)

where \( E[V] \) is the mean value of a vacation

\[
A_{1,j} = \sum_{i=1}^{j-1} \Omega_{j-i} \left[ 1 - \sum_{k=1}^{i} e_k \right],
\]

\[
A_{2,j} = \sum_{i=1}^{j} \sum_{m[i]=j}^{i} e_m \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} P[V = m] \left( \frac{n}{i} \right) p^i (1 - p)^{n-i},
\]

\[
A_{3,j} = \sum_{i=1}^{j-1} \Omega_{j-i} \left[ 1 - v(\hat{p}) - \sum_{k=1}^{i} \sum_{m[k]=k}^{i} e_m \sum_{m=0}^{\infty} \left( \frac{m}{k} \right) p^k (1 - p)^{m-k} P[V = m] \right], \quad j \geq 1;
\]

\[
\Omega_1 = \frac{1 - \hat{G}(\hat{p})}{p\hat{G}(\hat{p})},
\]

\[
\Omega_j = \frac{1}{\hat{G}(\hat{p})} \sum_{i=1}^{j-1} \sum_{m[i]=j-1}^{i} e_m \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \hat{g}_k \left( \frac{n}{i} \right) p^i (1 - p)^{n-i}
\]

\[
+ \frac{1}{\hat{G}(\hat{p})} \sum_{j=1}^{j-1} \Omega_{j-i} \left[ 1 - \hat{G}(\hat{p}) - \sum_{i=1}^{l} \sum_{m[i]=i}^{l} e_m \sum_{k=i}^{\infty} \hat{g}_k \left( \frac{k}{j} \right) p^j (1 - p)^{k-j} \right], \quad j > 1;
\]

and when \( j \leq 0 \), \( \sum_{k=1}^{j} = 0 \).

**Proof.** In discrete-time situation we have \( p_j = \lim_{n \to \infty} P_{i,j}(n) = \lim_{z \to 1}(1 - z) p_{i,j}(z) \). Combining Lemma 1 and Theorems 1–3, and using L’Hospital’s rule, Theorem 4 can be obtained. □
5. The stochastic decomposition

**Theorem 5.** The total steady-state queue length $L$ can be decomposed into the sum of two independent random variables: $L = L_0 + L_1$. $L_0$ is the steady-state queue length of unreliable Geo$^X/G/1$ queueing system without vacation and $L_1$ is the number of additional customers due to the adaptive multiple delayed vacations. Let $\pi(z) = \sum_{j=0}^{\infty} z^j p_j$ present the p.g.f of the steady-state queue length, then

$$\pi(z) = \frac{(1 - \hat{p})(1 - z)\hat{g}(\zeta)}{[\hat{g}(\zeta) - z]} \cdot \Phi(z),$$

where $\zeta = \hat{p} + pA(z)$,

$$\Phi(z) = \frac{[1 - \theta(\hat{p})G_H(\theta(\hat{p}))][1 - A(z)] + y(\hat{p})[1 - G_H(\theta(\hat{p}))][A(z) - v(\zeta)]}{[1 - A(z)][1 - y(\hat{p}) + y(\hat{p})G_H(\theta(\hat{p}))][1 - v(\hat{p})] + (1 - G_H(\theta(\hat{p})))pE[V]].$$

**Proof.** Using p.g.f’s definition and $P(z) = \sum_{j=0}^{\infty} z^j p_j = p_0 + \sum_{j=1}^{\infty} z^j p_j$, and combining Theorem 4, we can obtain Theorem 5. Some important equations that should be paid attention to are given below:

$$\Psi(z) = \sum_{j=1}^{\infty} z^j \Omega_j = \frac{z(1 - z)[1 - \hat{g}(\zeta)]}{p[1 - A(z)][\hat{g}(\zeta) - z]},$$

$$\sum_{j=1}^{\infty} z^j A_{1,j} = \frac{z - A(z)}{1 - z} \Psi(z),$$

$$\sum_{j=1}^{\infty} z^j A_{2,j} = \sum_{i=1}^{\infty} z^j \sum_{i=1}^{\infty} e_{m_1} \ldots e_{m_i} \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} P[V = m] \binom{n}{i} p^i (1 - p)^{n-i}$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} z^j P[D_1 + \cdots + D_i = j] \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} P[V = m] \binom{n}{i} p^i (1 - p)^{n-i}$$

$$= \sum_{i=1}^{\infty} \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} P[V = m] \binom{n}{i} (pA(z))^i (1 - p)^{n-i}$$

$$= \frac{1 - v(\zeta)}{p[1 - A(z)] - \frac{1 - v(\hat{p})}{p}},$$

$$\sum_{j=1}^{\infty} z^j A_{3,j}$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{j-1} \Omega_{j-i} \left\{ 1 - v(\hat{p}) - \sum_{k=1}^{i} \sum_{m[k]=k} e_{m_{1}} \ldots e_{m_{k}} \sum_{m=0}^{\infty} \binom{m}{k} p^k (1 - p)^{m-k} P[V = m] \right\}$$

$$= z \frac{[1 - v(\hat{p})]}{1 - z} \Psi(z) - \Psi(z) \sum_{i=1}^{\infty} z^i \sum_{k=1}^{i} \sum_{l=k}^{i} P[D_1 + \cdots + D_k = l] \sum_{m=0}^{\infty} \binom{m}{k} p^k (1 - p)^{m-k} P[V = m]$$
\[
\psi(z) = \frac{z[1 - v(\tilde{p})]}{1 - z} \Psi(z) - \Psi(z) \sum_{i=1}^{\infty} \sum_{l=1}^{i} \sum_{k=1}^{l} P[D_1 + \cdots + D_k = l] \sum_{m=0}^{\infty} \binom{m}{k} p^k (1 - p)^{m-k} P[V = m]
\]

\[
= \frac{z[1 - v(\tilde{p})]}{1 - z} \Psi(z) - \Psi(z) \sum_{i=1}^{\infty} \sum_{l=1}^{i} \sum_{k=1}^{l} P[D_1 + \cdots + D_k = l] \sum_{m=0}^{\infty} \binom{m}{k} p^k (1 - p)^{m-k} P[V = m]
\]

\[
= \frac{z[1 - v(\tilde{p})]}{1 - z} \Psi(z) - \Psi(z) \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \left( \binom{m}{k} (pA(z))^k (1 - p)^{m-k} P[V = m] \right)
\]

\[
= \Psi(z) \left( \frac{z[1 - v(\tilde{p})]}{1 - z} - \frac{v(\xi) - v(\tilde{p})}{1 - z} \right). \square
\]

**Theorem 6.** The mean queue length of the discrete-time Geo^X / G / 1 queue with unreliable server and adaptive multiple vacations is

\[
E[L] = \tilde{\rho} + \frac{e^3 p^2[E[\tilde{X}^2] - E[\tilde{X}]] + (E[D^2] - e)}{2e(1 - \tilde{\rho})} + \frac{\eta(\tilde{\rho})[1 - G_H(0(\tilde{\rho}))] p^2 e(E[V^2] - E[V])}{2\eta},
\]

where \( \eta = 1 - y(\tilde{\rho}) + y(\tilde{\rho})G_H(0(\tilde{\rho})) - \theta(\tilde{\rho})G_H(0(\tilde{\rho})) + y(\tilde{\rho}) p E[V] \).

**Proof.** We can easily obtain (24) by using \( E[L] = \frac{d}{dz} [\pi(z)] \big|_{z=1} \). \square

### 6. Several special cases

Some common models are special cases of the model discussed in this paper.

**Corollary 1.** If \( H = \infty \), \( G_H(z) = 0 \), then

\[
\pi(z) = \frac{(1 - \tilde{\rho})(1 - z) \tilde{g}(\xi)}{[\tilde{g}(\xi) - z]} \frac{1 - A(z)}{[1 - A(z)][1 - y(\tilde{\rho}) + y(\tilde{\rho}) p E[V]]}.
\]

This is the p.g.f of the discrete-time Geo^X / G / 1 queue with unreliable server and multiple delayed vacations.

**Corollary 2.** If \( H = 1 \), \( G_H(z) = z \), then

\[
\pi(z) = \frac{(1 - \tilde{\rho})(1 - z) \tilde{g}(\xi)}{[\tilde{g}(\xi) - z]} \frac{(1 + y(\tilde{\rho}) v(\tilde{\rho})) [1 - A(z)] + y(\tilde{\rho}) [A(z) - v(\xi)]}{[1 - A(z)][1 - y(\tilde{\rho}) + y(\tilde{\rho}) [p E[V] + v(\tilde{\rho})]]}.
\]

This is the p.g.f of the discrete-time Geo^X / G / 1 queue with unreliable server and single delayed vacation.

**Corollary 3.** If \( Y = \infty \), \( y(z) = 0 \), then

\[
\pi(z) = \frac{(1 - \tilde{\rho})(1 - z) \tilde{g}(\xi)}{[\tilde{g}(\xi) - z]}.
\]

This is the discrete-time unreliable Geo^X / G / 1 queue without vacations. It is significant that the expression has no relationship with random variables \( H, Y \) and \( V \). If the system never fails i.e., the lifetime of the system follows \( P[X = \infty] = 1 \), we could obtain the p.g.f of standard Geo^X / G / 1 queue:

\[
\pi(z) = \frac{(1 - \rho)(1 - z) \tilde{g}(\xi)}{[\tilde{g}(\xi) - z]}.
\]

**Corollary 4.** If \( Y = 0 \), \( y(z) = 1 \), then

\[
\pi(z) = \frac{(1 - \tilde{\rho})(1 - z) \tilde{g}(\xi)}{[\tilde{g}(\xi) - z]} \cdot \Phi(z),
\]
where
\[
\phi(z) = \frac{[1 - v(\bar{p})G_H(v(\bar{p}))][1 - A(z)] + y(\bar{p})[1 - G_H(v(\bar{p}))][A(z) - v(\bar{p})]}{[1 - A(z)][1 - y(\bar{p}) + y(\bar{p})][G_H(v(\bar{p}))][1 - v(\bar{p})] + (1 - G_H(v(\bar{p})))pE[V]]}.
\]

This is the p.g.f of the discrete-time Geo\(^X\) / G / 1 queue with unreliable server and multiple adaptive vacations, but without delay.

We would like to emphasize that in papers about discrete-time queueing system with imbedded Markov chain method the steady-state results concern the departure epoch \(t = n^+\), while in this study we get the result at arbitrary epoch \(t = n\). The existence of diversity is natural since the two techniques and the epoch being concerned are totally different. According to these results, the relationship between the p.g.f of the steady-state queue length at point \(t = n\) and \(t = n^+\) could be obtained. Xu and Zhu [28] obtained the p.g.f of this model when \(Y = 0\) at point \(t = n^+\), so we have

**Corollary 5.** Let \(\Pi^+(z)\) present the p.g.f of the steady-state queue length of this model when \(Y = 0\) at point \(t = n^+\), and \(\Pi(z)\) present the p.g.f of the steady-state queue length at point \(t = n\), respectively, then

\[
\Pi(z) = \Pi^+(z)\frac{e(1 - z)}{1 - A(z)}.
\]

The relationship still holds for queueing system without vacation and breakdowns.

7. Numerical results

In this section, we present some numerical results to illustrate the effect of variant parameters on the main performance measures of our system.

We consider a service system with batch arriving tasks (named task A) whose arrivals follow geometric distribution with parameter \(p\). Each service of task A needs \(\chi\) length of time. The batch size distribution is taken as \(P\{D = k\} = e_k = 0.1, k = 1, 2, \ldots, 10\). System fails occasionally. When system becomes idle, the service station will start to perform \(H\) auxiliary tasks (named task B) whose length is \(V\). But before each task B, the service station will stay in idle for a period of time \(Y\). Tasks A and B make different profits. It is obvious that both the variation of controllable variable \(Y\) and \(H\) play essential role in the optimization of the system.

An important feature of this work is the recursion formulae provided by Theorem 4. Utilizing the results obtained in Eq. (22) and (23), we present stationary distribution of queue length in Table 1. Here we consider that \(\chi\), lifetime of the system \(X\) and repair time \(W\) follow geometric distribution, respectively. We take \(p = 0.05\). Four cases \((EX = 100, EW = 5); EX = 100, EW = 10; EX = 200, EW = 5; EX = 200, EW = 10\) are considered to illustrate the effect of variant lifetime and repair time. The mean service time is 2. \(Y\) and \(V\) follow geometric distribution with

<table>
<thead>
<tr>
<th></th>
<th>((EX = 100, EW = 5))</th>
<th>((EX = 100, EW = 10))</th>
<th>((EX = 200, EW = 5))</th>
<th>((EX = 200, EW = 10))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(p_0)</td>
<td>0.3882</td>
<td>0.3630</td>
<td>0.4009</td>
<td>0.3882</td>
</tr>
<tr>
<td>(p_1)</td>
<td>0.0438</td>
<td>0.0420</td>
<td>0.0443</td>
<td>0.0435</td>
</tr>
<tr>
<td>(p_2)</td>
<td>0.0442</td>
<td>0.0425</td>
<td>0.0446</td>
<td>0.0438</td>
</tr>
<tr>
<td>(p_3)</td>
<td>0.0441</td>
<td>0.0426</td>
<td>0.0444</td>
<td>0.0438</td>
</tr>
<tr>
<td>(p_4)</td>
<td>0.0435</td>
<td>0.0423</td>
<td>0.0437</td>
<td>0.0432</td>
</tr>
<tr>
<td>(p_5)</td>
<td>0.0424</td>
<td>0.0414</td>
<td>0.0425</td>
<td>0.0421</td>
</tr>
<tr>
<td>(p_6)</td>
<td>0.0407</td>
<td>0.0400</td>
<td>0.0407</td>
<td>0.0404</td>
</tr>
<tr>
<td>(p_7)</td>
<td>0.0384</td>
<td>0.0379</td>
<td>0.0382</td>
<td>0.0381</td>
</tr>
<tr>
<td>(p_8)</td>
<td>0.0354</td>
<td>0.0352</td>
<td>0.0351</td>
<td>0.0351</td>
</tr>
<tr>
<td>(p_9)</td>
<td>0.0316</td>
<td>0.0318</td>
<td>0.0312</td>
<td>0.0314</td>
</tr>
<tr>
<td>(p_{10})</td>
<td>0.0270</td>
<td>0.0276</td>
<td>0.0265</td>
<td>0.0269</td>
</tr>
<tr>
<td>(\sum_{j=0}^{10} p_j)</td>
<td>0.7793</td>
<td>0.7463</td>
<td>0.7921</td>
<td>0.7765</td>
</tr>
</tbody>
</table>

Table 1
The steady distribution of the queue length for different system lifetime and repair time
parameters $\zeta = 0.5$ and $v = 0.2$, respectively. $H$ follows geometric distribution with parameter 0.2. Table 1 shows that the steady distribution probabilities $p_i$ are larger for higher values of $EX$ when the value of $i$ is small (here $i \leq 10$). Similar behavior is observed for lower values of $EW$.

In Fig. 1(a) and (b), we illustrate the effect of variant $H$. Here $\chi$, $X$, $W$, $Y$ and $V$ all follow geometric distribution with means $E[\chi] = 2$, $E[X] = 100$, $E[W] = 5$, $E[Y] = 2$ and $E[V] = 20$, respectively. In Fig. 1 (a), system idle probability $p_0$ is plotted versus arrival rate $p$. We have presented five curves corresponding to constant $H = 1, 2, 3, 5, \infty$, respectively. In Fig. 1(b), mean queue length $E[L]$ is plotted versus $p$. As we expected, the graphs show that as $H$ increases $p_0$ decreases and $E[L]$ increases. The figures are useful for optimization; for example, they show that the results for $H = 5$ and $\infty$ are very close. Considering the truth that having $H$ value greater than 5 will not degrade the performance significantly, it may be more beneficial to choose $H$ as infinite rather than taking $H$ value between 5 and infinite, since the server will handle task B without checking $H$ value and degrading performance.

Fig. 2(a) and (b) illustrate the effect of variant $Y$ and $V$. Here $\chi$, $X$ and $W$ have the same distribution and mean as in Fig. 1, while $Y$ and $V$ are governed by a geometric distribution with parameter $\zeta$ and $v$, respectively. We assume
that $H$ follows geometric distribution with mean 5. In Fig. 2 (a) $p_0$ is plotted versus $\zeta$. Three curves are presented corresponding to $\alpha = 0.02$, 0.1 and 0.5, respectively. It is clear that $p_0$ increases as the delayed time increases. Fig. 2 (a) illustrates that $p_0$ is larger for shorter vacation time. However, when the delayed time is infinite, three curves converge to the same point, which has been proved by Corollary 3. The same discussing holds for Fig. 2(b), which illustrates the behavior of $E[L]$ as function of $\zeta$.

8. Conclusion

A discrete-time Geo$^X$/G/1 queue with unreliable server and multiple adaptive delayed vacations policy has been studied. The transient and steady-state distributions of the queue length are obtained, and the stochastic decomposition property of steady-state queue length has been proven. Several common vacation policies are special cases of the vacation policy presented in this study. The relationship between the generating functions of steady-state queue length at departure epoch and arbitrary epoch is obtained. Some numerical examples are presented to illustrate the effect of variant parameters on several performance characteristics.

Since the system is unreliable, the study of reliability problems is significant. So the results are planned to be reported later in some journals in the field of reliability theory.

Acknowledgment

The authors wish to thank the anonymous referees for their valuable comments and suggestions for the improvement of this paper and the supports from Natural Science Foundation of the Education Department of Sichuan Province ([2006]A067) and the Talent Introduction Foundation of Sichuan Normal University of China.

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