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Topology design of tensegrity structures via mixed integer programming

Shintaro Ehara [1], Yoshihiro Kanno *

Department of Mathematical Informatics, University of Tokyo, Tokyo 113-8656, Japan

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ABSTRACT

This paper presents a numerical method for finding a tensegrity structure based on the ground structure method. We first solve a mixed integer programming (MIP) problem which maximizes the number of struts over the self-equilibrium condition of axial forces and the discontinuity condition of struts. Subsequently we solve the minimization problem of the number of cables in order to remove redundant self-equilibrium modes, which is also formulated as an MIP. It is regarded to be advantageous that our method does not require any connectivity information of cables and struts to be known in advance, while the obtained tensegrity structure is guaranteed to satisfy the discontinuity condition of struts rigorously.

1. Introduction

Tensegrity structure is a class of tension structures, which consists of pin-jointed members transmitting only axial forces. According to the definition given by Fuller (1975), a tensegrity structure is a prestressed pin-jointed structure consisting of continuous tensile members (cables) and discontinuous compressive members (struts). Since such a structure does not necessarily have enough stiffness from the practical point of view, the concept of tensegrity has been generalized extensively; see, e.g., Motro (2003), Wang (2004), and the references therein.

Light-weight characteristic of tensegrity and tensegric structure is recognized as a significant advantage for space structures over the conventional structural systems in civil engineering (Fu, 2005). In aerospace engineering tensegrity structures have been adopted as deployable structures (Sultan and Skelton, 2003; Furuya, 1992). Recently tensegrity structures have also received increasing attention in various research fields including biomechanics (Baudriller et al., 2006), cellular biology (Ingber, 1998), and discrete mathematics (Jórdan et al., 2009).

In this paper we propose an optimization-based approach to find a tensegrity structure which rigorously satisfies the discontinuity condition of struts. It should be emphasized that our method does not require any information of the connectivity of cables and struts to be known in advance, which is regarded as a major contribution of this paper.

There have been many studies on form-finding of tensegrity structures; see, e.g., Zhang and Ohsaki (2006), Masic et al. (2005), Juan and Mirats Tur (2008), Tibert and Pellegrino (2003), Zhang et al. (2006), Gomez Estrada et al. (2006). However, as input data, those methods require to specify the connectivity of members as well as the labeling indicating whether each member is to be a cable or strut. Based on the group representation theory, a systematic approach was presented to enumerate topologies, i.e. connectivities and labelings of members, of tensegrity structures which share a common group-theoretic symmetry property (Connelly and Terrell, 1995; Connelly and Back, 1998). Particularly, for tensegrity structures with a rotational symmetry property, form-finding methods utilizing such a symmetry property have been proposed (Masic et al., 2005; Sultan et al., 2002), and the stability conditions of such tensegrities were also investigated by using the group representation theory (Zhang et al., 2009). However, these methods based on the group theory assume that the group symmetry underlying a family of tensegrities is known in advance, i.e. it is necessary to specify a symmetry property of tensegrity structures before the form-finding process. Thus, it still remains as a challenging problem to find a new pair of the connectivity and the labeling of members of a tensegrity structure satisfying its definition rigorously.

We propose a numerical algorithm for finding a tensegrity structure based on the ground structure method, which has been widely used for topology optimization of discrete structures. Given a pin-jointed structure with the specified locations of nodes and sufficiently many candidate members, our problem is to find a labeling of members which indicates whether each member is to be a cable,

* Corresponding author. Address: Department of Mathematical Informatics, Graduate School of Information Science and Technology, University of Tokyo, Bunkyo, Tokyo 113-8656, Japan. Tel.: +81 3 5841 6906; fax: +81 3 5841 6886.
E-mail address: kanno@mist.i.u-tokyo.ac.jp (Y. Kanno).
[1] Present address: Bank of Tokyo-Mitsubishi UFJ, Ltd., 2-7-1, Marunouchi, Chiyoda, Tokyo 100-8388, Japan.

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a strut, or removed, so that the resulting structure becomes a tensegrity structure. It is shown that the discontinuity condition of struts can be written as a system of linear inequalities in terms of the axial forces and some additional binary, i.e. 0–1, variables.

Our approach consists of two parts; at the first step we find a self-equilibrium mode of axial forces satisfying the discontinuous condition of struts, while at the second step we remove redundant cables from the structure obtained at the first step. At each step we solve a *mixed integer programming* (MIP) problem. See e.g., Wolsey (1998) for basics of MIP. As a variant of the problem to be solved at the first step, we also present another MIP for finding a tensegrity unit, or module (Murakami and Nishimura, 2001), which can be connected one-by-one to obtain a larger tensegrity structure. Note that we restrict ourselves to finding a self-equilibrated configuration, and consider neither the stability nor the stiffness of tensegrity structures in this paper. It is emphasized that our approach does not require any labeling of members or any underlying group symmetry property to be known in advance. To the authors' knowledge, no efficient algorithm has been proposed for design of tensegrity structures which does not require any information of the connectivity of struts and cables, even for the specified locations of nodes which can exist. Most of existing methods require to specify the sets of struts and cables, or to specify at least an underlying group symmetry property, and then aim at finding the locations of nodes.

This paper is organized as follows. Section 2 introduces the discontinuity condition of struts as well as the minimal tensegrity for the given set of struts and candidates of cables. In Sections 3 and 4, we present an algorithm in which we solve two MIPs sequentially; an MIP for finding a self-equilibrium mode satisfying the discontinuous condition of struts is formulated in Section 3 to obtain a feasible set of struts, and the problem of minimizing the number of cables for the given set of struts is formulated as another MIP in Section 4. Section 5 discusses a simple scheme for designing a tensegrity module. Numerical results are presented in Section 6; tensegrity structures with non-symmetric and symmetric configurations are presented in Sections 6.1 and 6.2, respectively, and examples of tensegrity modules in Sections 6.3 and 6.4. Some conclusions are drawn in Section 7.

A few words regarding our notation: all vectors are assumed to be column vectors. The $(m+n)$-dimensional column vector $(u^{\mathrm{T}}, v^{\mathrm{T}})^{\mathrm{T}}$ consisting of $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ is often written simply as (u, v). The cardinality of the set S is denoted by $|S|$. For example, if $S = \{1, \dots, m\}$, then $|S| = m$.

2. Self-equilibrated configuration and tensegrity conditions

Consider a pin-jointed structure without any support. We say that the structure with prestresses, i.e.with a nonzero axial forces vector, is at the state of self-equilibrium if it satisfies the static equilibrium condition when no external load is applied.

Suppose that the locations of nodes of a structure are specified in the three-dimensional space. Let V and E denote the set of nodes and the set of members, respectively, where $|V| = n$ and $|E| = m$. For convenience, let $E = \{1, \dots, m\}$. We denote by $q = (q_i) \in \mathbb{R}^m$ the vector of axial forces of members. The self-equilibrium condition in terms of the axial forces is written as

$$Hq = 0.$$

Here, $H \in \mathbb{R}^{3n \times m}$ denotes the equilibrium matrix, which is a constant matrix. The degree of static indeterminacy is defined by

$$s = \dim(\mathrm{Ker}\, H) = m - \mathrm{rank}\, H. \tag{1}$$

Note that a cable and a strut transmit only compressive and tensile forces, respectively, and hence we have $q_i > 0$ for a cable and $q_i < 0$ for a strut.

Although there exist various definitions of tensegrity structures (Motro, 2003), we employ the one which consists of the self-equilibrium condition and the discontinuity condition of struts. More precisely, we consider a free-standing pin-jointed prestressed structure, any two struts of which do not share a common node. Let $E(n_j) \subset E$ denote the set of indices of the members which are connected to the node $n_j \in V$. The definition above is formally stated as follows.

Definition 2.1. A structure is said to be a tensegrity satisfying the discontinuity condition of struts if there exists a vector $\hat{q} \in \mathbb{R}^m$ satisfying

$$\hat{q} \neq 0 : H\hat{q} = 0, \tag{2}$$

$$|\{i \in E(n_j) | \hat{q}_i < 0\}| \leqslant 1, \quad \forall n_j \in V. \tag{3}$$

Note that we remove the member i if $\hat{q}_i = 0$ in the conditions of Definition 2.1. Since Definition 2.1 requires only the existence of self-equilibrium mode of axial forces which satisfies the discontinuity condition of struts, the self-equilibrium mode of a tensegrity structure defined by Definition 2.1 is not unique in general. From such a tensegrity structure it might be possible to remove some cables without changing the locations of struts. Suppose that at least one strut has vanishing axial force if we remove any cable from a tensegrity structure. Then the tensegrity has no redundant cables, it is regarded as the *minimal* one for the given set of struts and set of cable candidates. Such a minimal tensegrity structure has no redundant self-equilibrium mode, while it still has the self-equilibrium mode satisfying the discontinuity condition of struts.Such a minimal tensegrity structure is formally defined below.

For $\hat{q}$ satisfying the conditions in Definition 2.1, define $E_{\mathrm{cable}}, E_{\mathrm{strut}} \subset E$ by

$$E_{\mathrm{cable}} = \{i \in E | \hat{q}_i > 0\}, \tag{4}$$

$$E_{\mathrm{strut}} = \{i \in E | \hat{q}_i < 0\}, \tag{5}$$

i.e. E_{cable} and E_{strut} are the sets of member indices of cables and struts, respectively. The following definition captures the essence of a tensegrity structure which is minimal in the sense that it includes no redundant cables for the given set of struts and set of cable candidates.

Definition 2.2. A tensegrity structure satisfying the discontinuity condition of struts is said to be a minimal tensegrity if it satisfies

$$\left\{ q \,\middle|\, \begin{array}{l} Hq = 0 \\ q_{i'} = 0 \\ q_i \geqslant 0 \ (\forall i \in E_{\mathrm{cable}} \setminus \{i'\}) \\ q_j < 0 \ (\forall j \in E_{\mathrm{strut}}) \end{array} \right\} = \emptyset, \quad \forall i' \in E_{\mathrm{cable}}.$$

Remark 2.3. The degree of static indeterminacy of a minimal tensegrity structure is not necessarily equal to one.

In the following sections we propose an algorithm for finding a tensegrity from a given ground structure. Our algorithm consists of two parts corresponding to Definitions 2.1 and 2.2. The former (Section 3) corresponds to finding a self-equilibrium mode satisfying the discontinuous condition of struts and the latter (Section 4) to removing redundant cables for the set of struts which is obtained at the former step.

3. Topology with discontinuous struts

We present here an MIP problem for finding a tensegrity structure satisfying the conditions in Definition 2.1.

Based on the conventional ground structure method, consider a pin-jointed structure with fixed locations of nodes and sufficiently many members that can exist. We solve an MIP presented below in order to find a self-equilibrium mode $\hat{q}$. By using $\hat{q}$ we assign members as follows.

$$\begin{cases} \hat{q}_i > 0 & \Rightarrow \text{ the member } i \text{ is to be a cable;} \\ \hat{q}_i < 0 & \Rightarrow \text{ the member } i \text{ is to be a strut;} \\ \hat{q}_i = 0 & \Rightarrow \text{ the member } i \text{ is to be removed.} \end{cases} \tag{6}$$

We first reformulate (3) in Definition 2.1 into a tractable form. We introduce a binary variable, $x_i \in \{0,1\}$, associated with each member in order to indicate whether the member i is a strut or not. Let M and ε be positive constants, where M is sufficiently large, i.e. $0 < \varepsilon \ll M$. Consider the linear inequalities

$$-Mx_i \leqslant q_i \leqslant M(1-x_i) - \varepsilon \tag{7}$$

for each $i \in E$. Since $x_i \in \{0,1\}$, we see that (7) is equivalent to

$$\begin{cases} q_i < 0 & \Longleftrightarrow \quad x_i = 1, \\ q_i \geqslant 0 & \Longleftrightarrow \quad x_i = 0, \end{cases} \tag{8}$$

i.e. $x_i = 1$ if and only if the member i is supposed to be a strut. Hence, the number of struts connected to the node n_j is equal to $\sum_{i\in E(n_j)} x_i$. Since at most one strut can be connected to each node, we have

$$\sum_{i\in E(n_j)} x_i \leqslant 1, \quad \forall n_j \in V. \tag{9}$$

Note that the total number of struts is given by $\sum_{i\in E} x_i$. Since it is natural to attempt to choose as many struts as possible from the given ground structure, we consider the maximization problem of the number of struts. From (7) and (9), the maximization problem of the number of struts of a tensegrity structure satisfying the discontinuity condition of struts is formulated as

$$(\text{MIP-1}): \max_{q,x} \quad \sum_{i\in E} x_i \tag{10a}$$

$$\text{s.t.} \quad Hq = 0, \tag{10b}$$

$$-Mx_i \leqslant q_i \leqslant M(1-x_i) - \varepsilon, \quad \forall i \in E, \tag{10c}$$

$$\sum_{i\in E(n_j)} x_i \leqslant 1, \qquad \forall n_j \in V, \tag{10d}$$

$$x \in \{0,1\}^{|E|}. \tag{10e}$$

Note that the problem (10) is a 0–1 mixed integer programming problem. We denote by $(\hat{q}, \hat{x})$ the optimal solution of (10). The characteristics of members are determined from $\hat{q}$ according to (6). We next verify that the obtained structure satisfies the conditions in Definition 2.1.

We have already shown that (3) in Definition 2.1 is equivalent to (10c)–(10e). The equality condition in (2) is explicitly included in (MIP-1) as (10b). Hence, it suffices to consider the condition $\hat{q} \neq 0$ in (2).

If $\hat{x} \neq 0$, then (8) implies $\hat{q} \neq 0$, which is as expected. Hence, suppose $\hat{x} = 0$ in the following. Note that the trivial solution, $q = x = 0$, is feasible for (MIP-1). Since (MIP-1) maximizes the number of struts, $\hat{q} = 0$ implies that it is impossible to construct a tensegrity structure from the given ground structure. On the other hand, if $\hat{q} \neq 0$, then the optimal solution includes no struts but some cables. Since we consider a free-standing ground structure, a structure consisting only of cables cannot satisfy the self-equilibrium condition. Hence, $\hat{q} \neq 0$ with $\hat{x} = 0$ is not feasible for (MIP-1).

The discussion above is summarized as follows.

- $\hat{x} \neq 0 \Rightarrow$ a tensegrity structure with discontinuity condition of struts is obtained from $\hat{q}$ according to (6);
- $\hat{x} = 0 \Rightarrow$ it is impossible to find a tensegrity structure from the given ground structure.

Note again that in this paper we restrict ourselves to finding a self-equilibrated configuration, and the issues of stability and the stiffness of tensegrity structures are not dealt with.

Remark 3.1. In this paper we focus on tensegrity structures satisfying the conditions in Definition 2.1, i.e. the number of struts connected to each node is at most one. However, the presented formulation can be immediately applied to a class of tensegric structures with more relaxed connectivity condition of struts. For example, suppose a tensegric structure such that each node is connected to at most p struts, where p is a given natural number. Such a structure can be found by replacing (10d) in (MIP-1) with $\sum_{i\in E(n_j)} x_i \leqslant p \ (\forall n_j \in V)$.

4. Topology with minimal cables

In this section we present an MIP for finding the minimal tensegrity (Definition 2.2) from the given tensegrity satisfying the discontinuity condition of struts (Definition 2.1).

We have shown in Section 3 that a tensegrity structure satisfying the conditions of Definition 2.1 can be obtained by solving (MIP-1) in (10). By using the optimal solution of (MIP-1), a feasible set of struts, E_{cable}, for the given ground structure is specified by (5). As the second stage of tensegrity design we attempt to remove cables as many as possible in order to obtain the minimal tensegrity for the specified E_{cable} and set of cable candidates.

Consider a ground structure, which has the same nodes, V, as those used in (MIP-1), and includes any member in E_{strut}. Let $\overline{E}_{\text{cable}}$ be the set of candidates of cables. We choose $\overline{E}_{\text{cable}}$ so that $\overline{E}_{\text{cable}} \supseteq E_{\text{cable}}$ and $\overline{E}_{\text{cable}} \cup E_{\text{strut}} = \emptyset$, where $\overline{E}_{\text{cable}}$ is defined by (4) from the optimal solution of (MIP-1). Then the minimization problem of the number of cables, (MIP-2) presented below, is guaranteed to be feasible, because the optimal solution of (MIP-1) becomes a trivial feasible solution of (MIP-2). Note again that the ground structure for (MIP-2), defined with the set of nodes V and the set of members $\overline{E}_{\text{cable}} \cup E_{\text{strut}}$, is not necessarily same as that for (MIP-1). It should be clear that, throughout this section, the equilibrium matrix H and axial forces vector q are defined for the ground structure for (MIP-2), i.e. $H \in \mathbb{R}^{3n\times|\overline{E}_{\text{cable}}\cup E_{\text{strut}}|}$ and $q \in \mathbb{R}^{|\overline{E}_{\text{cable}}\cup E_{\text{strut}}|}$.

Associated with each $i \in \overline{E}_{\text{cable}}$, we introduce a binary variable $y_i \in \{0,1\}$ which indicates whether the cable i can be removed or not. Consider the linear inequalities

$$0 \leqslant q_i \leqslant My_i, \tag{11}$$

where M is a sufficiently large positive constant. Since $y_i \in \{0,1\}$, we see that (11) is equivalent to

$$q_i \geqslant 0, \tag{12}$$

$$\begin{cases} q_i > 0 & \Rightarrow \quad y_i = 1, \\ q_i = 0 & \Leftarrow \quad y_i = 0. \end{cases} \tag{13}$$

We remove the member i if $y_i = 0$. From (13) it follows that by minimizing y_i over (11), y_i becomes equal to 1 if and only if $q_i > 0$. Consequently, the minimization problem of the number of cables with the specified E_{strut} is formulated as

$$(\text{MIP-2}): \min_{q,y} \quad \sum_{i\in\overline{E}_{\text{cable}}} y_i \tag{14a}$$

$$\text{s.t.}\quad Hq = 0, \tag{14b}$$

$$q_i \leqslant -\varepsilon, \qquad \forall i \in E_{\text{strut}}, \tag{14c}$$

$$0 \leqslant q_i \leqslant My_i, \quad \forall i \in \overline{E}_{\text{cable}}, \tag{14d}$$

$$y \in \{0,1\}^{|\overline{E}_{\text{cable}}|}, \tag{14e}$$

where $0 < \varepsilon \ll M$. Note that (14e) is a 0–1 mixed integer programming problem.

Let (q^*, y^*) denote the optimal solution of the problem (14e). Observe that in (14e) we attempt to minimize the sum of y_i, from which and (13) we obtain

$$\begin{cases} q_i^* > 0 & \iff \quad y_i^* = 1, \\ q_i^* = 0 & \iff \quad y_i^* = 0, \end{cases}$$

at the optimal solution. Hence, the optimal value of the problem (14e) is equal to the number of remaining cables. By removing cables corresponding to $y_i^* = 0$, we can obtain the minimal tensegrity which does not include any redundant cables.

Remark 4.1. We have presented a method in which we solve two MIPs sequentially in order to obtain a tensegrity structure satisfying the conditions in Definitions 2.1 and 2.2. It is possible to formulate a single MIP for finding such a tensegrity directly from the given ground structure, i.e. the minimization problem of the number of cables with the specified lower-bound of the number of struts can be formulated as an MIP. However, from our preliminary numerical experiments it was observed that it is very difficult to solve this single MIP by existing well-developed software for MIPs, e.g. CPLEX (ILOG, 2008). It is conjectured that the lower bound constraint on the number of struts cannot be dealt with efficiently, because the locations of struts are not known in advance. This motivates us to propose a two-phase algorithm. Note that at the second step we specify the locations of struts, which are found at the first step. This may explain the reason why MIP at the second step of our method can be solved more efficiently for moderately large structures compared with the single MIP formulation.

5. Topology of tensegrity module

We here consider a problem of finding a tensegrity unit, or module, which can be connected one-by-one to obtain a new tensegrity structure with a larger size.

We start with a sufficient condition for tensegrity modules. Suppose that a tensegrity structure, which is referred to as the tensegrity (T), satisfying the conditions in Definitions 2.1 and 2.2. Consider a set of two struts, which is called the linking part (A), as shown in Fig. 1. The two struts consisting of the linking part (A) are referred to as the struts (A-1) and (A-2). Suppose that there exists another set of two struts, (B-1) and (B-2), where the position relationship between (B-1) and (B-2) is the same as that between (A-1) and (A-2). We call the set of (B-1) and (B-2) the linking part (B) as shown in Fig. 1. Then the tensegrity (T) can be connected to another (T) by superimposing the linking part (A) of the former (T) on the linking part (B) of the latter (T). Here, we replace each duplicate strut in the superimposed linking part with a single strut. It is easy to see that the obtained tensegrity structure, (T2) illustrated in Fig. 1, also satisfies the conditions in Definitions 2.1 and 2.2, because we have not added any additional cables and struts in order to construct (T2). Thus a new tensegrity with a larger size can be obtained from the two tensegrity modules (T). Similarly, (T2) can be connected with (T) again.

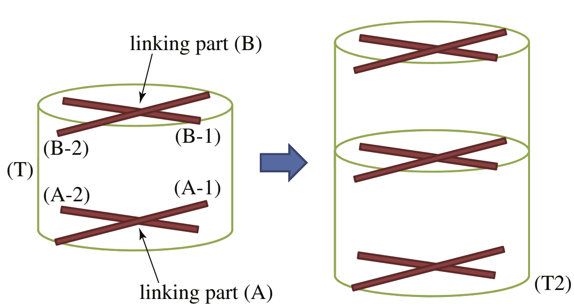


Fig. 1. Schematic representation of a tensegrity module. Thick lines indicate struts.

In the procedure of the ground structure method it is easy to prepare two candidates of linking parts such that the position relationship of members is parallel to each other. Let E_{connect} denote the set of struts of candidates of linking parts. We solve the following MIP, instead of (MIP-1), in order to find a tensegrity unit satisfying the discontinuity condition of struts:

$$\begin{aligned} (\text{MIP-1}_{\text{unit}}) : \max_{q,x} \quad & \sum_{i \in E} x_i \\ \text{s.t.} \quad & Hq = 0, \\ & -Mx_i \leqslant q_i \leqslant M(1 - x_i) - \varepsilon, \quad \forall i \in E, \\ & \sum_{i \in E(n_j)} x_i \leqslant 1, \qquad \forall n_j \in V, \\ & x_i = 1, \qquad \forall i \in E_{\text{connect}}, \\ & x \in \{0,1\}^{|E|}, \end{aligned}$$

where $0 < \varepsilon \ll M$. In a manner similar to Section 4, a tensegrity unit without redundant cables can be obtained by solving (MIP-2) for E_{strut} defined from the optimal solution of $(\text{MIP-1}_{\text{unit}})$.

6. Numerical examples

Various tensegrity structures are found by using the proposed method based on MIPs. Computation has been carried out on Quad-Core Xeon E5450 (3GHz) with 16GB RAM. We solve MIPs by using CPLEX Ver.11.2 (ILOG, 2008) with the default settings.

6.1. Tensegrity structures with randomly generated nodal coordinates

Consider a ground structure shown in Fig. 2a, which consists of 12 nodes and 66 members ($|V| = 12$ and $|E| = 66$). Note that there exists a member which connects any two node, i.e. the topology of the ground structure is the perfect graph with the 12 nodes. The coordinates of all the nodes are randomly generated on the surface of a sphere.

We solve (MIP-1) in (10) to find a tensegrity satisfying the discontinuity condition of struts. The optimal solution obtained is shown in Fig. 2b, which consists of 6 struts and 30 cables. The degree of static indeterminacy is 6.

We next solve (MIP-2) in (14e) in order to find the tensegrity with the minimum number of cables. The ground structure for (MIP-2) is defined by Fig. 2b, i.e. $\overline{E}_{\text{cable}} := E_{\text{cable}}$ in (14e). The optimal structure obtained is illustrated in Fig. 2c, which consists of 6 struts and 25 cables, and the degree of static indeterminacy of which is equal to 1.

We solve similar examples for ground structures with various numbers of nodes. In each case the ground structure for (MIP-1) is defined as the perfect graph, while the ground structure for (MIP-2) is defined by the optimal solution of (MIP-1). The results are listed in Table 1. Here, n and $|E|$ denote the numbers of nodes and members of the ground structure for (MIP-1), respectively, $|E_{\text{strut}}|$ and $|E_{\text{cable}}|$ are the numbers of struts and cables of the

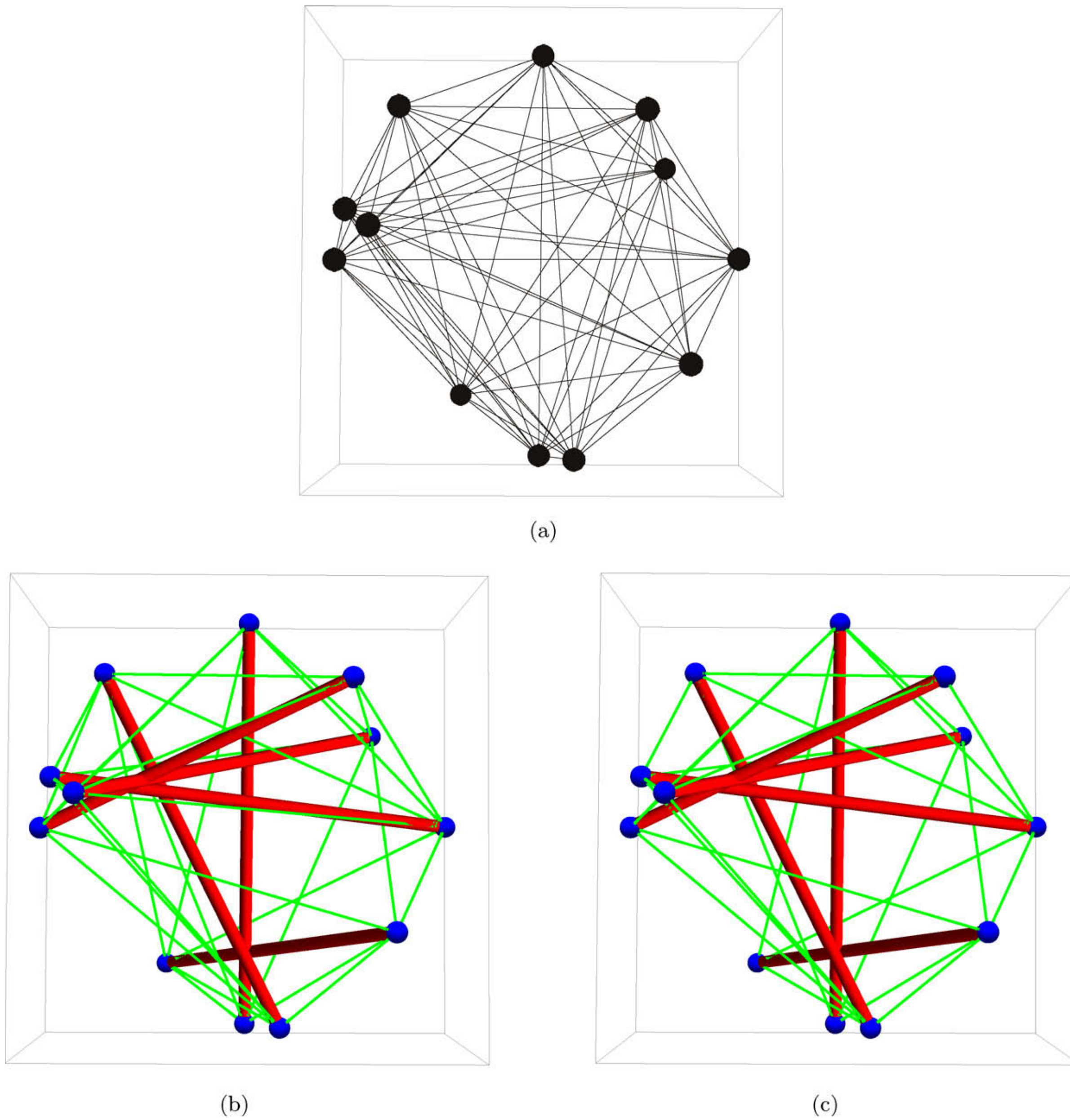


Fig. 2. A 12-node example. Thick and thin lines indicate struts and cables, respectively. (a) The ground structure with 12 nodes and 66 members; (b) The optimal solution of (MIP-1); (c) The optimal solution of (MIP-2).

Table 1
Computational results of (MIP-1) and (MIP-2) for ground structures with randomly generated nodal coordinates.

n	(MIP-1)					(MIP-2)			
	$\|E\|$	CPU (s)	$\|E_{\mathrm{strut}}\|$	$\|E_{\mathrm{cable}}\|$	s	$\|\overline{E}_{\mathrm{cable}}\|$	CPU (s)	Cable	s
12	66	0.26	6	30	6	30	0.34	25	1
20	190	0.03	10	56	12	56	0.22	45	1
40	780	0.39	20	124	30	124	0.57	97	3
60	1770	3.30	30	193	49	–	–	–	–

optimal solution of (MIP-1), respectively, s is the degree of static indeterminacy, $|\overline{E}_{\mathrm{cable}}|$ and 'cable' are the numbers of cables of the ground structure and optimal solution of (MIP-2), respectively.

It is observed in Table 1 that (MIP-1) is solved efficiently even if we increase n and $|E|$. For $n = 60$, (MIP-2) cannot be solved by using CPLEX within 86,400 s. As n increases, s of the solution of (MIP-1) increases, and hence the number of redundant cables increases. Consequently, the number of cables which are removed in (MIP-2) also increases, which makes it difficult to solve (MIP-2) for $n = 60$.

6.2. Tensegrity structures with symmetric configurations

We next consider an example shown in Fig. 3, where the ground structure has some symmetric properties. It is often that the symmetry of a tensegrity configuration causes the rank-deficiency of the equilibrium matrix. Hence, the number of cables of the minimum tensegrity depends on its topology, and the conventional Maxwell counting rule does not necessarily hold (Connelly et al., 2009; Calladine, 1978).

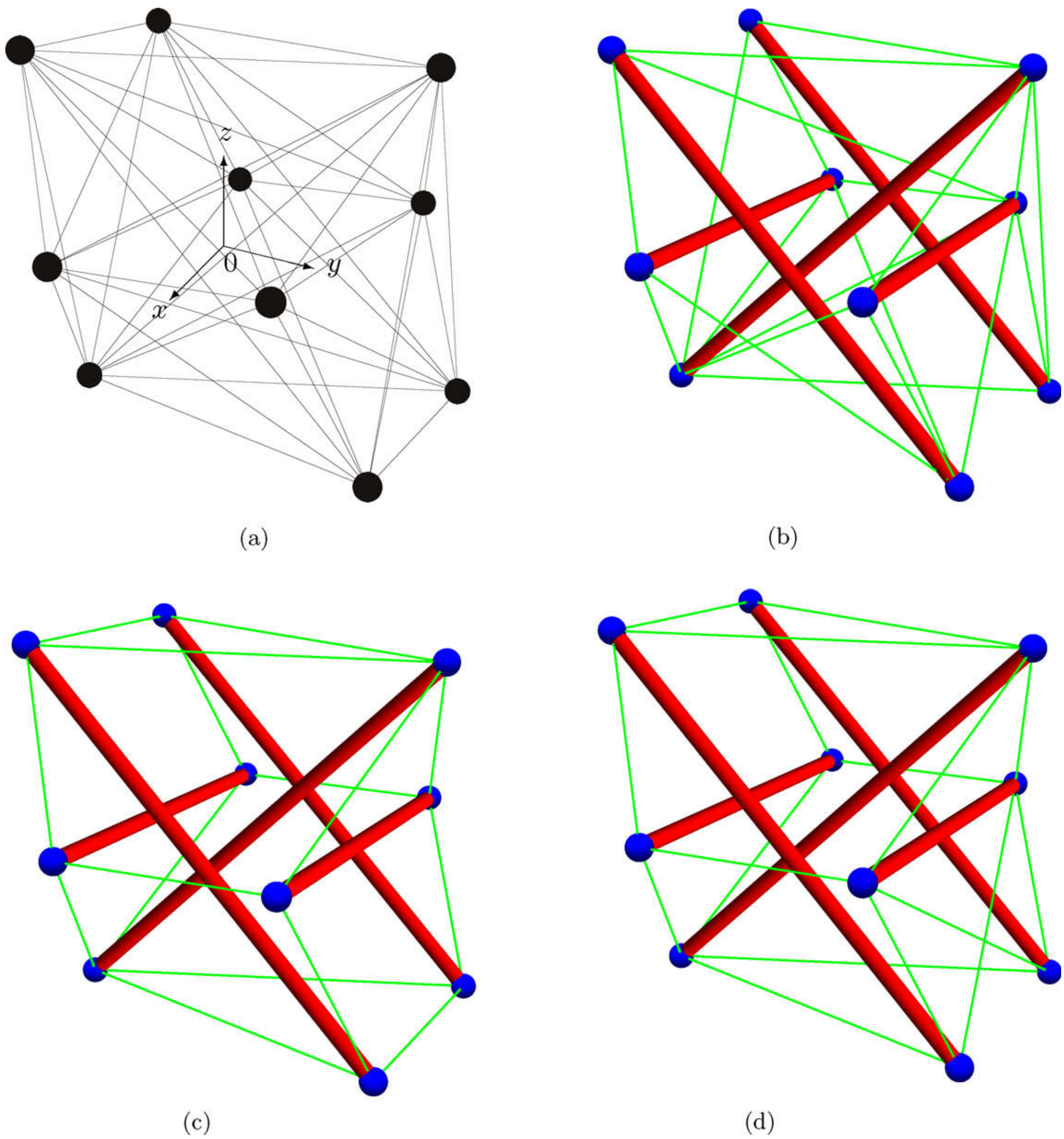


Fig. 3. A 10-node symmetric example. (a) The ground structure; (b) The optimal solution of (MIP-1); (c) The optimal solution of (MIP-2); (d) A minimal tensegrity including 17 cables.

Consider a ground structure illustrated in Fig. 3a, where $n = 10$ and $|E| = 41$. The ground structure consists of three layers, where the top and bottom layers are in triangular shapes and the middle one in a rectangular shape. Note that the configuration of this structure is symmetric by the reflection with respect to the *yz*-plane and the rotation around the *x*-axis with the angle π. The optimal solution of (MIP-1) is shown in Fig. 3b, which consists of 5 struts and 18 cables. For finding the tensegrity with the minimum number of cables, we next solve (MIP-2), where the ground structure for (MIP-2) is given by Fig. 3a, i.e. $\overline{E}_{\mathrm{cable}} := E \setminus E_{\mathrm{strut}}$ in (14e). Both of (MIP-1) and (MIP-2) are solved within one second by CPLEX (ILOG, 2008). The optimal solution of (MIP-2) is illustrated in Fig. 3c, which has 5 struts and 16 cables. We see that the tensegrity in Fig. 3c satisfies $21 = 5 + 16 < 3|V| - 6 = 24$, which implies that the conventional Maxwell rule does not hold. Thus this tensegrity is kinematically and statically indeterminate. However, this tensegrity is stabilized by introducing prestresses, i.e. it is prestress stable (Guest, 2006; Connelly and Whiteley, 1996), because the stress matrix of this tensegrity is positive definite (after removing the freedom of rigid-body motions). The prestress stability property is also verified from an actual model that we created. Finally we solve (MIP-2) for a slightly modified ground structure, where the lower-right member is removed. The optimal solution is shown in Fig. 3d, which has 17 cables. Thus the minimum number of cables depends on the topology of tensegrity.

6.3. A tower-type tensegrity-module

In this and the following section we present examples of tensegrity modules obtained by using the method presented in Section 5.

Consider the perfect graph with 18 nodes as a ground structure, where the locations of nodes are shown in Fig. 4a. The four struts (two on the bottom layer and two on the top layer) parallel to the *xy*-plane are specified as the elements of E_{cable}. Fig. 4a

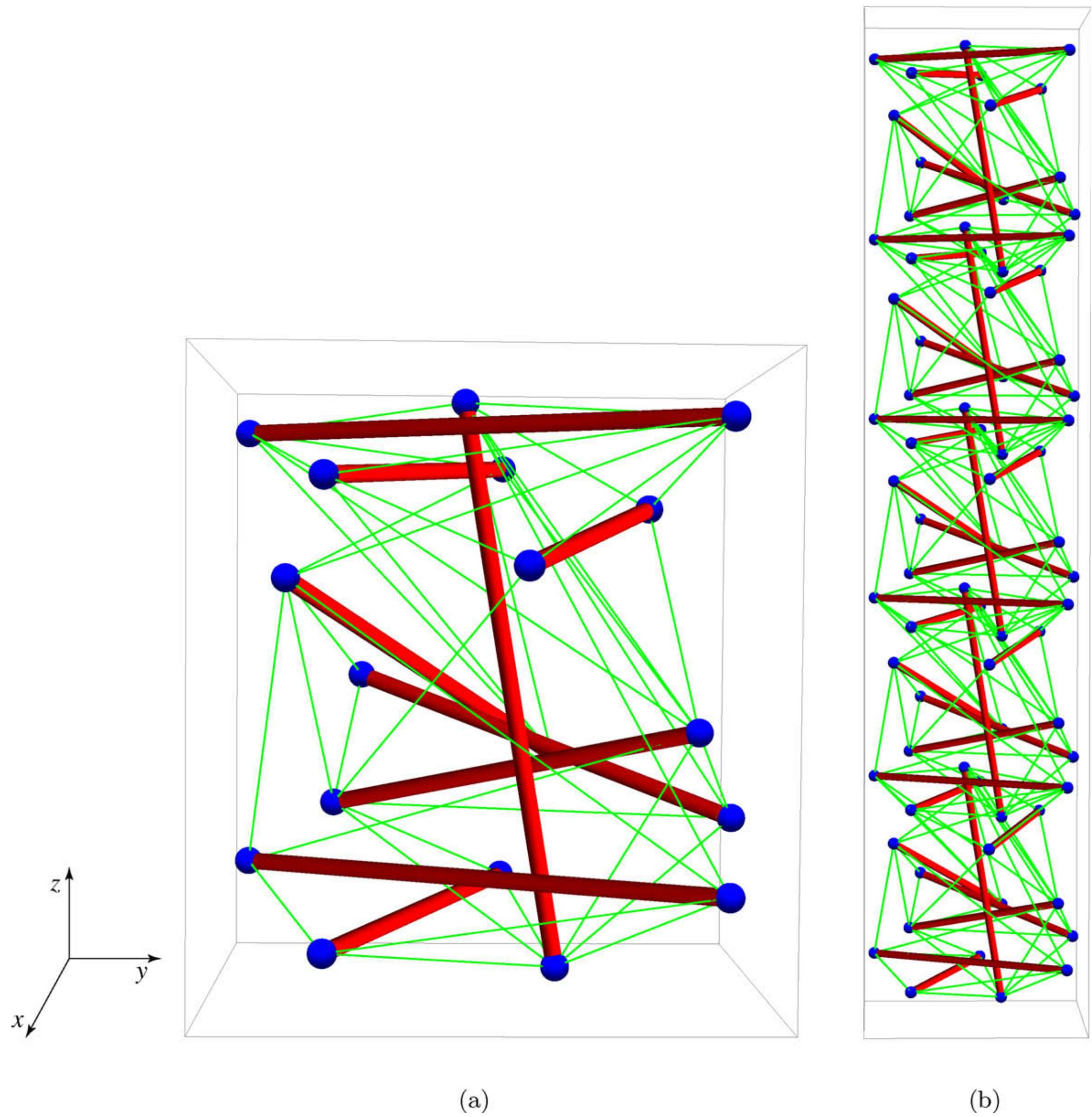


Fig. 4. A tower-type example of tensegrity module. (a) The tensegrity module with parallel linking parts; (b) The tensegrity consisting of 5 modules.

illustrates the tensegrity module with 9 struts, which is obtained by solving (MIP-1_{unit}) and (MIP-2). Each problem can be solved within a few seconds. The structure in Fig. 4a is kinematically determinate, and its degree of static indeterminacy is 1. By connecting five units we obtain the tensegrity structure illustrated in Fig. 4b, the degrees of static indeterminacy of which is 5.

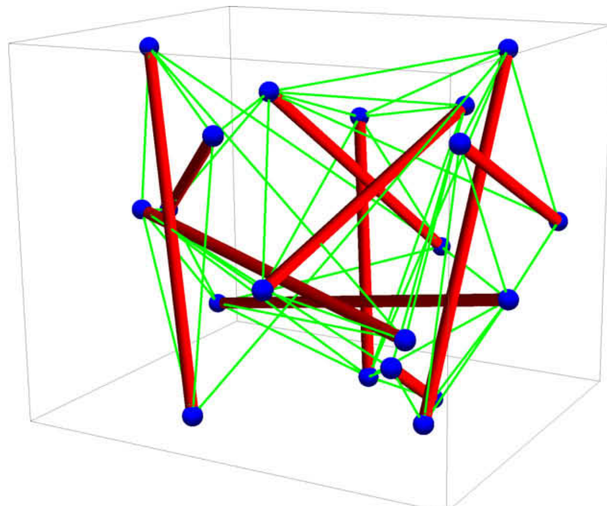
6.4. A ring-type tensegrity-module

Consider a ground structure defined as 18-node perfect graph in a manner similar to Section 6.3. In contrast to Fig. 4, the nodal coordinates of two linking parts are not parallel but are transformed to each other by rotation with the angle $\pi/4$. The obtained tensegrity module is illustrated in Fig. 5a. By connecting eight modules we can obtain the ring-type tensegrity shown in Fig. 5b. The degrees of static indeterminacy of tensegrity structures shown in Figs. 5a and 5b are 1 and 8, respectively. Note that these structures are kinematically determinate.

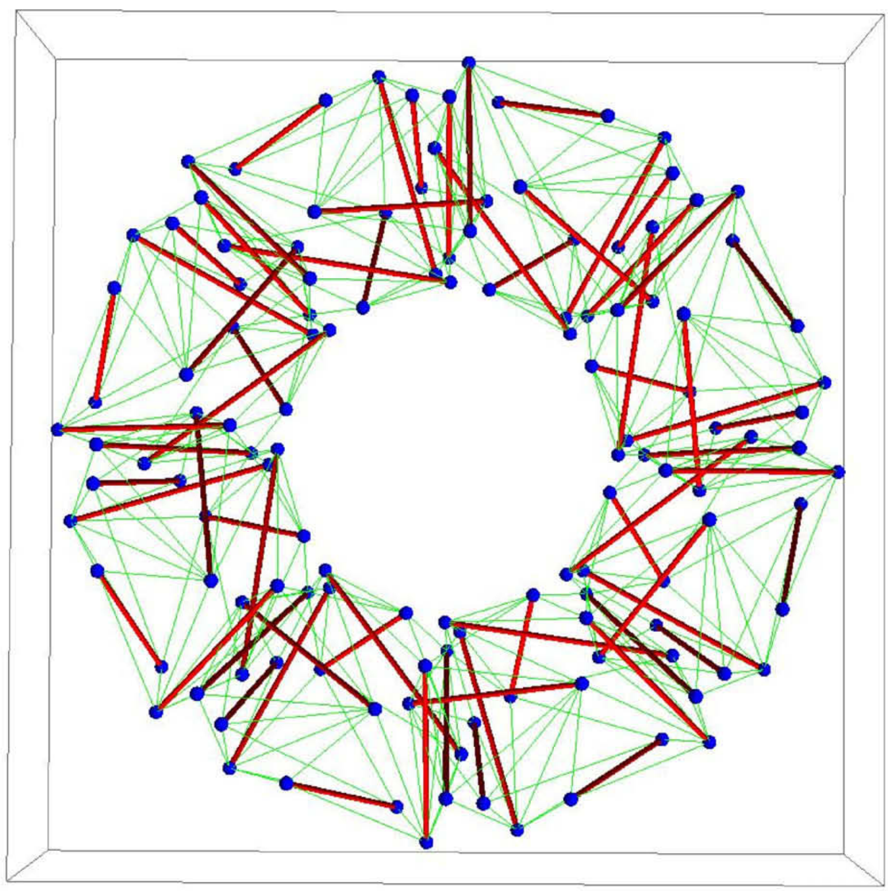
7. Conclusions

In this paper we have presented a numerical method for finding a tensegrity structure based on the ground structure method. In our method we solve the two MIPs (mixed integer programming problems) sequentially in order to find a tensegrity structure which satisfies the self-equilibrium condition as well as the discontinuity condition of struts.

At the first step we solve an MIP which maximizes the number of struts over the self-equilibrium condition and the discontinuity condition of struts. Note that it is very difficult to deal with the discontinuity condition of struts rigorously by existing methods for design of tensegrities. We have shown that this condition can be written as a system of linear inequalities in terms of the axial forces and some additional binary variables. Since the optimal solution obtained at the first step has some self-equilibrium modes in general, we solve an MIP which minimizes the number of cables as the second step. We have also presented a simple scheme to design a tensegrity unit, or module, which can be connected one-by-one. It has been shown



(a)



(b)

Fig. 5. A ring-type example of tensegrity module. (a) The tensegrity module with linking parts which are transformed to each other by the rotation with the angle $\pi/4$; (b) The tensegrity consisting of 8 modules.

that such a module can be found by adding some linear equality constraints to the MIP to be solved at the first step.

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