Regular packings of regular graphs

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Abstract

A graph $H$ is $G$-decomposable if it contains subgraphs $G_1,\ldots,G_h$, $h \geq 2$, isomorphic to $G$ whose sets of edges partition $E(H)$. Wilson (Proceedings of the Fifth British Combinatorial Conference, University of Aberdeen, Aberdeen, 1975, pp. 647–659; Congr. Numer. XV, Utilitas, Math., Winnipeg, Manitoba, 1976) proved that, given a nonempty graph $G$, the complete graph $K_N$ is $G$-decomposable for $N$ large enough, provided some natural divisibility conditions hold. Fink and Ruiz (Czechoslovak Math. J. 36 (111) (1986) 172) proved that a noncomplete $G$-decomposable graph $H$ exists even within the class of circulant graphs. The order $N_0(G)$ of the smallest $G$-decomposable regular graph is known only for particular classes of graphs or for graphs with small maximum degree. We give some tools to study the problem of determining $N_0(G)$ when $G$ is a connected regular graph. These tools are applied to obtain upper and lower bounds of $N_0(G)$ for regular graphs of degree $r \geq |V(G)|/2$. Families of extremal graphs which attain the bounds are also given. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

A set of graphs $G_1,\ldots,G_h$ of order $n$, $h \geq 2$, can be packed into a graph $H$ if $H$ contains a set of edge-disjoint subgraphs $G'_1,\ldots,G'_h$, where each $G'_i$ is isomorphic to $G_i$, $1 \leq i \leq h$. When the edge sets of $G'_1,\ldots,G'_h$ partition the edge set of $H$ then the graph $H$ is decomposable into the graphs $G_1,\ldots,G_h$ and we write $H = G_1 \oplus \cdots \oplus G_h$. In particular, when $G_1 \simeq \cdots \simeq G_h \simeq G$, the graph $H$ is said to be $G$-decomposable. Given a graph $G$, a natural question to ask is if there is a $G$-decomposable graph $H$. The answer is trivially positive if no additional conditions are imposed on $H$, and thus it is usual to require $H$ to be regular and connected. Wilson [7] proves that, for $N$ sufficiently large, all complete graphs of order

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\( N \) are \( G \)-decomposable, provided that some natural divisibility conditions hold. Fink and Ruiz [5] prove that the graph \( H \) can be even required to be a noncomplete circulant graph. None of these results provides a regular graph of minimum size which is \( G \)-decomposable.

Let \( N_0(G) \) denote the order of the smallest \( G \)-decomposable regular graph. In this paper, we study this parameter when \( G \) is a connected regular graph.

The value of \( N_0(G) \) is known for some classes of graphs. Fink [4] shows that \( N_0(K_n) = n(n+1)/2 \). Davitt et al. [3] compute the value of this parameter for complete bipartite graphs, \( N_0(K_{m,m}) = 3m \), cycles \( N_0(C_n) = n > 4 \), \( n \)-cubes, \( n \geq 3 \), \( N_0(Q_n) = 2^n \), and complete \( t \)-partite graphs \( N_0(K_{k[t]}) = kt(t+1)/2 \).

Some of these results can be obtained from a theorem of Sauer and Spencer [6] which says that \( N_0(G) = |V(G)| = n \) for graphs with maximum degree \( \Delta(G) < \sqrt{n/2} \). A graph \( G \) of order \( n \) is self-packable if two copies of \( G \) can be packed into \( K_n \). Thus, a regular graph of degree at most \( \sqrt{n/2} \) is self-packable. A weaker version of a conjecture of Bollobás and Eldridge says that a regular graph of degree at most \( \sqrt{n+1} - 1 \) is still self-packable, see [1, Chapter 8]. Note that there are regular self-packable graphs of order \( n > 5 \) and degree up to \( [n-2/2] \) (see a family of examples in Section 3).

Two graphs \( G_1 \) and \( G_2 \) of order \( n \) are packable if \( K_n \) contains edge-disjoint copies isomorphic to \( G_1 \) and \( G_2 \).

Denote by \( \mathcal{G}_{n,r} \) the class of regular graphs of order \( n \) and degree \( r \) and let

\[
L(n,r) = \min \{ N_0(G) : G \in \mathcal{G}_{n,r} \} \quad \text{and} \quad U(n,r) = \max \{ N_0(G) : G \in \mathcal{G}_{n,r} \}.
\]

We study \( L(n,r) \) and \( U(n,r) \) for any positive integers \( n \) and \( r \) such that \( n/2 \leq r < n \). In particular, we prove the following:

1. \( L(n,n-i) = \frac{n}{2}(\frac{n}{2} + 1) = U(n,n-i), \quad 1 \leq i \leq 2 \),
2. \( \frac{n^2}{8} \leq U(n,r) \leq \frac{n}{2}([\frac{n}{2}] + 2), \quad n/2 \leq r \leq n - 3 \),
3. \( \frac{n}{2}k < L(n,r) \leq \frac{n}{2}(k+2), \quad (\frac{2k-3}{2k})n \leq r \leq \frac{(2k-1)n-2}{2k}, \quad k \geq 2 \).

The proofs of the above results are mainly based on what we call the packable partition number. This parameter is introduced in Section 2, where the main tools for our proofs are presented. In Section 3, we obtain the upper and lower bounds, \( U(n,r) \) and \( L(n,r) \), when \( G \) is a dense regular graph and give the extremal graphs for the corresponding values of \( N_0(G) \). Final remarks conclude the paper.

2. The packable partition number

Throughout this section, \( G \) denotes a regular connected simple graph. For a subset \( X \) of vertices of \( G \), we denote by \( G[X] \) the subgraph of \( G \) induced by the vertices in \( X \).
We say that a partition \( P = \{V_1, \ldots, V_h\} \) of \( V(G) \) is a \textit{packable partition} if one of the following conditions holds:

(i) \( h \) is an odd number and \( G[V_i] \) is self-packable for \( 1 \leq i \leq h \).

(ii) \( h \) is an even number and \( G[V_{2i}] \) and \( G[V_{2i-1}] \) are packable, \( 1 \leq i \leq h/2 \).

The \textit{packable partition number} of \( G \) is the minimum number of parts in a packable partition of \( G \), and it is denoted by

\[
\theta(G) = \min\{|P| : P \text{ is a packable partition of } G\}.
\]

Note that a partition into an odd number of stable sets is a packable partition. Hence, we have

\[
\theta(G) \leq \chi(G) + 1,
\]

where \( \chi(G) \) denotes the chromatic number of the graph \( G \). An example where equality holds is shown in Section 3 (Lemma 3.3).

The following theorems are the main tools to obtain the upper and lower bounds for the order \( N_0(G) \) of the smallest \( G \)-decomposable regular graphs.

\textbf{Theorem 1.} For a graph \( G \) with packing partition number \( \theta(G) \),

\[
N_0(G) \leq \frac{n}{2}(\theta(G) + 1).
\]

\textbf{Proof.} Let \( P = \{V_1, \ldots, V_h\} \) be a packable partition of \( V(G) \). Set \( n_i = |V_i| \), \( 1 \leq i \leq h \) and let \( X \) be a set of size \( N = n/2(h + 1) \). In what follows we construct a \( G \)-decomposable graph \( H \) with vertex set \( X \). We consider two cases.

\textit{Case 1:} \( h \) is an odd integer. Let \( M = (m_{ij}) \) be the square matrix of order \( h + 1 \) with entries in \( \{\ast, 0, 1, \ldots, h-1\} \) defined in the following way,

\[
m_{ij} = \begin{cases} 
\ast & i = j, \\
i + j \mod h & 0 \leq i, j < h, \ i \neq j, \\
2(i + j) \mod h & i = h \text{ or } j = h.
\end{cases}
\]

(1)

Clearly \( M \) is an equipotent symmetric latin square of order \( h + 1 \) when \( h \) is odd.

Consider a partition of \( X \) into subsets \( X_{ij} \) with \( 0 \leq i < j \leq h \) such that \( |X_{ij}| = n_{M(i,j)} \), where we set \( n_{\ast} = 0 \). Since \( \sum_{i<j} n_{M(i,j)} = n(h+1)/2 \), for each \( j \), such a partition exists.

Define \( X_{ij} = X_{ji} \) for \( 0 \leq j < i \leq h \). Let \( X_0 = \bigcup_{j=1}^h X_{0j} \). Note that, \( |X_0| = \sum_{i=0}^h n_{M(i,0)} = 0 + n_1 + \cdots + n_{h-1} + n_0 = n \). Similarly, for \( k \geq 1 \), let \( X_k = (\bigcup_{i=0}^{k-1} X_{ik}) \cup (\bigcup_{j=k+1}^h X_{kj}) \).

By the symmetry of \( M \), \( |X_i| = \sum_{i=0}^h n_{M(i,k)} = n \). By construction, each of the subsets \( X_0, \ldots, X_h \) of \( X \) of cardinality \( n \) is a packable partition. Hence, we put edges colored \( i \) and edges colored \( j \) in \( X_{ij} \) such that the subgraph induced by
Then, for each pair $0 \leq i < j \leq h$, we get a graph with edges colored by integers $0, \ldots, h$, where $X_{ij} \neq X_{j'i'}$, then the pair $\{x, y\}$ belongs to at most one of the sets $X_0, \ldots, X_h$. Therefore, we can add on the set $X_i$ edges colored $i$ such that the graph induced by $X_i$ is isomorphic to $G$. In this way, we end up by constructing a graph $H$ with vertex set $X$ and edge set in which all the edges colored $0, \ldots, h$, are $G$-decomposable.

**Case 2:** $h$ is an even integer. Let $M=(m_{ij})-0 \leq i \leq j \leq h$ be the circulant matrix of order $h+1$,

$$m_{ij}=(j-i) \mod (h+1).$$

For each pair $0 \leq i, j \leq h$ with $i \neq j$, let

$$z_{ij} = \begin{cases} m_{ij} & \text{if } m_{ij} \text{ is odd,} \\ h+1-m_{ij} & \text{if } m_{ij} \text{ is even.} \end{cases}$$

Let $A = \{(2k-1,2k), (2k,2k-1), 1 \leq k \leq h/2\}$ and define $f:A \to [1,h]$ as

$$f(\alpha, \beta) = \begin{cases} \alpha & \alpha \text{ is odd,} \\ h+1-\beta & \alpha \text{ is even.} \end{cases}$$

Note that $f$ is a bijection and $f(\alpha, \beta) + f(\beta, \alpha) \equiv 0 \mod (h+1)$. Consider a partition $X_{ij}$, $0 \leq i < j \leq h$ of $X$ such that $|X_{ij}|=n_{x_{ij}}$ and define $X_{ij}=X_{ji}$ for $0 \leq j < i \leq h$. Note that $n_{x_{ij}}=n_{x_{ji}}$ and $\sum_{i \neq j} |X_{ij}| = (h+1) \sum_{k=1}^{h} n_{k} = 2|X|$, so that the $X_{ij}$'s are well defined. Let $X_i = \bigcup_{j=1}^{h} X_{ik}$ for $0 \leq i \leq h$. Then $|X_i| = \sum_{j \neq i} n_{x_{ij}} = \sum_{k=1}^{h} n_{k} = n$. Moreover, every three of the $X_i$'s have empty intersection.

For a fixed pair of different subscripts $0 \leq i < j \leq h$, let $(\alpha, \beta) \in A$ such that $f(\alpha, \beta) = m_{ij}$. Place edges colored $i$ in $X_{ij}$ such that the resulting induced subgraph is isomorphic to $G[V_{\alpha}]$ and edges colored $j$ such that they induce a subgraph isomorphic to $G[V_{\beta}]$. Moreover, since $G[V_{\alpha}]$ and $G[V_{\beta}]$ are packable, we require the two induced subgraphs to be edge disjoint. By repeating this procedure for each pair of subscripts, we construct a graph $H_1$ such that $H_1[X_{ij}] \cong G[V_{\alpha}] \oplus G[V_{\beta}]$ with $f^{-1}(m_{ij})$. We complete the construction of $H$ by adding, for each $0 \leq i \leq h$, edges colored $i$ such that the subgraph induced by these edges on $X_i$ is isomorphic to $G$. Since those edges joint vertices in different parts of the partition $\{X_{ij}, 0 \leq i < j \leq h\}$ and no two of them are shared by the same $X_i$, we do not create multiple edges. Therefore, $H[X_i] = G_i \cong G$ and $H = G_0 \oplus \cdots \oplus G_h$. This completes the proof. 

Theorem 2 will be used to obtain lower bounds for $N_0(G)$.

**Theorem 2.** Let $G$ be a graph of order $n$. Let $t$ be the largest integer such that $G$ contains a pair of induced subgraphs of order $t$ which are packable. Set $\hat{\Theta}(G) = \lfloor n/t \rfloor$. Then

$$\frac{n}{2}(\hat{\Theta}(G)+1) \leq N_0(G).$$
Proof. Let $H$ be a $G$-decomposable graph, say $H = G_0 \oplus G_1 \oplus \cdots \oplus G_h$. Let $X = V(G_i) \cap V(G_j)$ for different subscripts $0 \leq i, j \leq h$. Then, clearly $G_i[X]$ and $G_j[X]$ are isomorphic to a pair of packable induced subgraphs of $G$ of cardinality $|X|$. Hence, $|X| \leq t$.

Let $n = |V(G_0)| = t\hat{\theta} + \mu$ with $0 \leq \mu < t$ and $\hat{\theta} = \hat{\theta}(G)$. Note that, since each vertex of $H$ is in at least two different copies of $G$, and two of them intersect in at most $t$ vertices, we have $h \geq \hat{\theta} + 1$ if $\mu > 0$ and $h \geq \hat{\theta}$ if $\mu = 0$. Therefore,

$$|V(H)| \geq |V(G_0)| + |V(G_1) \setminus V(G_0)| + \cdots + \left|V(G_h) \setminus \bigcup_{i<h} V(G_i)\right|$$

$$\geq n + (n - t) + (n - 2t) + \cdots + (n - \hat{\theta}t) = (n + \mu) \frac{\hat{\theta} + 1}{2},$$

as claimed. □

As an example of application of the above Theorems, we obtain the following corollary:

Corollary 3 (Fink [4]). $N_0(K_n) = n(n + 1)/2$.

Proof. The partition of $V(K_n)$ into singletons is the only packable partition and thus

$$0(K_n) = n = \hat{\theta}(K_n).$$

Notice that, if $G$ is a self-packable graph, we have the other extremal situation.

$$0(G) = 1 = \hat{\theta}(G).$$

3. Bounds of $N_0(G)$ for dense regular graphs

Recall that $L(n,r) = \min\{N_0(G): G \in \mathcal{G}_{n,r}\}$, and $U(n,r) = \max\{N_0(G): G \in \mathcal{G}_{n,r}\}$, where $\mathcal{G}_{n,r}$ denotes the class of regular graphs of degree $r$ and order $n$.

In this section, we study the values of $L(n,r)$ and $U(n,r)$ when $n/2 \leq r < n$.

Note that when $G$ is a connected regular graph, every vertex of a $G$-decomposable regular graph $H$ belongs to the same number of copies of $G$ in a $G$-decomposition.

Lemma 4. Let $G$ be a regular graph of degree $r$ and order $n$. Let $H$ be a regular $G$-decomposable graph of degree $r_H$ and order $n_H$. Then

$$n_H r_H = nr_h,$$

where $h$ is the number of copies of $G$ in a $G$-decomposition of $H$.

Proof. Let $H = G_1 \oplus \cdots \oplus G_h$ be a $G$-decomposition of $H$. Let $B$ be the bipartite graph with stable sets $V_1 = V(H)$ and $V_2 = \{1, 2, \ldots, h\}$ and where a vertex $x$ in $V_1$
is adjacent to \( i \in V_2 \) if \( x \) belongs to the set \( V(G_i) \). Each vertex \( i \in V_2 \) is adjacent to \( n \) vertices in \( V_1 \) and, since \( G \) is regular of degree \( r \), each vertex in \( V_1 \) is adjacent to \( rH/r \) vertices in \( V_2 \). By double counting the edges of graph \( B \) we obtain the result.

We first consider the graphs with \( r = n - 2 \).

**Proposition 5.** If \( n \) is an even number, then

\[
L(n, n - 2) = \frac{n}{2} \left( \frac{n}{2} + 1 \right) = U(n, n - 2).
\]

**Proof.** The complement \( \tilde{G} \) of \( G \) is just a perfect matching. The partition of \( V(G) \) into sets of size two, each one consisting of the vertices of an edge in \( \tilde{G} \) is clearly a packable partition. Therefore, \( \theta(G) \leq n/2 \). On the other hand, every set of \( s \geq 3 \) vertices in \( G \) induces a subgraph with at least \( s(s - 2)/2 \) edges, which is not packable with any other induced graph of the same number of vertices. Therefore, \( \hat{\theta} \geq n/2 \). The result follows now from Theorem 1 and Theorem 2. \( \square \)

**Lemma 6.** Let \( G \) be a graph of order \( n \) such that its complement \( \tilde{G} \) has a 2-factor, then,

\[
N_0(G) \leq \frac{n}{2} \left( \left\lfloor \frac{n}{3} \right\rfloor + 2 \right).
\]

The bound is tight.

**Proof.** Let us construct a packable partition \( P \) of \( G \) with parts of size at least 3. If a subset \( X \) of vertices belongs to a cycle of length \( l \), \( 3 \leq l \leq 5 \) in \( \tilde{G} \), then \( G[X] \) is self-packable. We thus put \( X \) in \( P \). Let \( C_k \) be a cycle, in \( \tilde{G} \), of length \( l_k = 3\alpha_k + 4\beta_k \geq 6 \), with \( \alpha_k, \beta_k \) nonnegative integers. We partition the cycle \( C_k \) into \( \alpha_k \) paths of length 2 and \( \beta_k \) paths of length 3. Let \( X \) be the set of vertices of one of these paths. Then, \( G[X] \) is clearly self packable and we add \( X \) in \( P \). We thus obtain a packable partition whose smaller part has at least three vertices. In order to get a packable partition, since the parts of \( P \) may have different sizes, to apply Theorem 1 we might have to split one of them into two parts to have an odd number of them. Therefore, \( \hat{\theta}(G) \leq \lfloor n/3 \rfloor + 1 \). Hence,

\[
N_0(G) \leq \frac{n}{2} \left( \left\lfloor \frac{n}{3} \right\rfloor + 2 \right).
\]

The next example shows that the upper bound is tight.

Let \( G \) be a graph whose complement \( \tilde{G} \) is the disjoint union of \( kC_3 \cup C_4 \) with \( k \) odd. Then, by (3) we have

\[
N_0(G) \leq (3k + 4)(k + 3)/2.
\]

Let \( H \) be a \( G \)-decomposable graph, \( H = G_1 \oplus \cdots \oplus G_h \) where \( G_i \cong G \), \( 1 \leq i \leq h \). Let \( \alpha = r_H/r \) and \( X_{ij} = V(G_i) \cap V(G_j) \), \( 1 \leq i \leq j \leq h \). By the choice of \( G \) we have
$|X_{ij}| \leq 4$ and $|X_{ij}| = 4$ implies
\[ G_i[X_{ij}] \simeq G_j[X_{ij}] \simeq \tilde{C}_4. \quad (4) \]
That is, if two copies intersect in $k \geq 4$ points, they must be precisely the four points whose complement is $C_4$.

Let $C = \{(x, i) : x \in V(G_i) \cap V(G_j), 2 \leq i \leq h\}$. We clearly have $|C| = n(x - 1)$. On the other hand, by (4) all copies of $G$ intersect $V(G_1)$ in at most 3 points except possibly a single one with intersection four. Therefore, $|C| \leq 3(h - 1) + 1 = 3h - 2$. Hence,
\[ h \geq \frac{n(x - 1) + 2}{3}, \]
which implies
\[ n_H = \frac{nh}{x} \geq \frac{n(n - 1) + 2}{x}. \]
This is an increasing function of $x$ for $x \geq 2$. Therefore, for $x \geq 3$.
\[ n_H \geq \frac{n(2n + 2)}{9} = (3k + 4) \frac{6k + 10}{9} > (3k + 4) \frac{(k + 3)}{2} \geq N_0(G). \]
For $x = 2$, we have $h \geq k + 2$. Since $x = 2$ and the remark following (4), for each index $i$, there is at most one index $j \neq i$ such that $|X_{ij}| = 4$. Therefore, either $h$ is an even number or there is $i$ such that $|X_{ij}| \leq 3 \forall i \neq j$. In both cases, $h \geq k + 3$. Hence,
\[ n_H \geq \frac{(3k + 4)(k + 3)}{2} = \frac{n}{2} \left( \left\lfloor \frac{n}{3} \right\rfloor + 2 \right). \quad \Box \]

**Proposition 7.** For each $n \geq 5$,
\[ \frac{n}{2} \left( \left\lfloor \frac{n}{5} + 1 \right\rfloor \right) \leq L(n, n - 3) \quad \text{and} \quad U(n, n - 3) = \frac{n}{2} \left( \left\lfloor \frac{n}{3} \right\rfloor + 2 \right). \]

**Proof.** If $r = n - 3$ then the complement $\tilde{G}$ of $G$ is 2-regular. Therefore, by Lemma 6
\[ U(n, n - 3) \leq \frac{n}{2} \left( \left\lfloor \frac{n}{3} \right\rfloor + 2 \right). \]
Moreover, the same Lemma gives a family of graphs of degree $n - 3$ reaching this bound.

On the other hand, every subset $X$ of vertices of $G$ induces a graph with minimum degree $\delta_X \geq |X| - 3$. Therefore, no two of these induced subgraphs are packable if $|X| \geq 6$. Hence $\theta \geq n/5$. Therefore,
\[ \frac{n}{2} \left( \left\lfloor \frac{n}{5} + 1 \right\rfloor \right) \leq L(n, n - 3). \]
To see that this bound is tight, consider the complement of a disjoint union of pentagons. Their vertex sets induce a packable partition of $G$, so that $\theta(G) = \tilde{\theta}(G) = n/5$. Thus, when $n \equiv 0(\mod 5)$ we have $L(n, n - 3) = n/2(n/5 + 1)$. \quad \Box
Theorem 8. Let $G$ be a regular graph of order $n$ and degree $n/2 \leq r \leq n - 2$. Then
\[
\frac{n^2}{8} \leq U(n, r) \leq \frac{n}{2} \left( \left\lfloor \frac{n}{2 + \epsilon} \right\rfloor + 1 \right),
\]
where $\epsilon = n - r \pmod{2}$. The bounds are tight.

Proof. For the upper bounds of $U(n, r)$ we consider two cases.

Suppose that $r \equiv n \pmod{2}$.

In this case $n$ and $r$ must be even. Let $M$ be a maximal matching in $\bar{G}$. Then
\[
|E(M)| \geq \left( \left\lfloor \frac{n(r + 1)}{2(r + 2)} \right\rfloor - 1 \right) \geq \frac{n}{3}
\]
(see e.g. [1, Theorem 4.2]).

Let $P = \{V_1, V_2, \ldots, V_m\}$ be a partition of $V(G)$ into $m = n/2$ sets of cardinality two which contains the edges of $M$. Let $m' \in \mathbb{Z}$ be such that $G[V_i]$ has no edges for $1 \leq i \leq m'$. By the choice of $P$, $m' > m/2$. Hence, $G[V_i]$ and $G[V_{m-i}]$ are packable, for $0 \leq i \leq m/2$. Hence $P$ is a packable partition and $\theta(G) = n/2$. Therefore, Theorem 1 gives us $U(n, r) \leq n/2(n/2 + 1)$.

Suppose now that $n$ and $r$ have different parity. Then the complement $\bar{G}$ of a graph $G \in \mathcal{G}_{n,r}$ is regular of even degree. By Petersen’s theorem (see e.g. [2]) $G$ has a 2-factor. The upper bound follows from Lemma 6, in which a graph attaining the bound is presented.

For the lower bound we construct, for each $n$ and $n/2 \leq r \leq n - 3$, a graph $G \in \mathcal{G}_{n,r}$ such that $n^2/8 \leq N_0(G)$.

Let $\{M_1, \ldots, M_m\}$, $m = \lfloor n/2 \rfloor$, be a 1-factorization of the complete bipartite graph $K_{m,m}$, with stable sets $V_1$ and $V_2$.

If $n$ is even, let $G_i$ be the graph consisting of two cliques of size $m$ on the vertices of $V_1$ and $V_2$, and the edges in $E(M_1) \cup \cdots \cup E(M_i)$. The graph $G_i$ is regular of degree $m + i - 1$. Let $X \subset V(G_i)$. Then, the largest stable sets in $G[X]$ have cardinality at most 2. Therefore, if $G_i[X]$ and $G_i[X']$ are packable for some $X, X' \subset V(G_i)$, then $|X| = |X'| \leq 4$. By Theorem 2
\[
\frac{n^2}{8} \leq N_0(G).
\]
A similar construction can be made when $n = 2m + 1$. If $m + i - 1$ is even, let $G_i'$ be the graph obtained from $G_i$ by adding a new vertex $x$ and replacing $m + i - 1/2$ edges of $M_1$ by edges connecting their incident vertices with $x$. The resulting graph is $(m + 1/2)$-regular. If $G_i'[X]$ and $G_i'[X']$ are packable for some $X, X' \subset V(G_i')$, then $|X| = |X'| \leq 4$, as in the previous case of even order or $x \in X \cap X'$. In this case, there is at most one largest stable set in $G_i'[X]$ of size 3 and therefore $|X| = |X'| \leq 6$.

Let $K = G_i'$ and let $H$ be a $G$-decomposable graph, say $H = K_1 \oplus \cdots \oplus K_h$, $K_i \simeq G_i$, $1 \leq i \leq h$ with $z = r_H / r_K$. Let $C = \{(x, i) : X \in V(K_i) \cap V(K_i), i > 1\}$. Since each vertex of $H$ is in $z$ copies of $K$ in the $K$-decomposition, $C = n(z - 1)$. By the above remarks, each copy $K_i$ intersects $K_1$ in at most 4 vertices, except maybe at most one of them.
whose intersection with \( K_1 \) has at most cardinality 6. Therefore, \(|C| \leq 4(h - 1) + 6\).

By Lemma 4
\[
|V(H)| = \frac{nh}{x} \geq \frac{n}{4x}(n(x - 1) - 2).
\]

This is an increasing function of \( x \) for \( n > 0 \). For \( x = 2 \), \(|V(H)| = n^2/8\). \( \square \)

Let us now consider the lower bounds for \( N_0(G) \).

Given two graphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \), the composition product \( G_1[G_2] \) has vertex set \( V_1 \times V_2 \) and each vertex \((x, y) \in V_1 \times V_2 \) is adjacent to the vertices \((x', y') \) such that either \((x, x') \in E_1 \) or \((y, y') \in E_2 \) and \( x = x' \).

Lemma 9 provides some examples of extremal graphs for the lower bounds \( L(n, r) \). The particular case when \( S \) is a stable set was proved by Davitt et al. [3]. We give here a simpler proof of a more general result.

**Lemma 9.** Let \( S \) be a self-packable graph of order \( s \). Then
\[
N_0(K_k[S]) = \frac{ks}{2}(k + 1).
\]

**Proof.** Set \( G = K_k[S] \). Let \( P = \{V_1, \ldots, V_k\} \) be the partition of \( V = V(G) \) such that the subgraph induced by \( V_i \) is isomorphic to \( S \) for each \( i = 1, \ldots, k \). Clearly, \( P \) is a packable partition regardless of the parity of \( k \) (if \( k \) is odd, then each part is self-packable and if \( k \) is even, the parts can be grouped in pairs of packable graphs). Therefore, \( \theta(G) \geq k \) and Theorem 1 implies \( N_0(G) \leq ks/2(k + 1) \).

On the other hand, for each pair of subsets \( X, Y \subset V \) of size \(|X| = |Y| > s\), each of the induced subgraphs \( G[X] \) and \( G[Y] \) is connected. If \( X \cap V_i \neq \emptyset \) then \( G[X \cap V_i] \) is a connected component of the complementary graph of \( G[X] \) and therefore this complementary graph is not connected. Therefore, \( G[X] \) and \( G[Y] \) are not packable. Hence, \( \theta \geq k \) and Theorem 2 implies \( N_0(G) \geq ks/2(k + 1) \). \( \square \)

We next give some examples of self-packable graphs which will be useful to construct dense graphs attaining the lower bounds \( L(n, r) \). Given two disjoint sets \( A, B \) we denote by \( K_{A,B} \) the complete bipartite graph whose stable parts are \( A \) and \( B \). When \(|A| = |B| \), we denote by \( K^l_{A,B} \), the graph \( K_{A,B} \) with the edges of \( l \) edge disjoint perfect matchings removed. We define a family \( S_n \) of self-packable graphs of order \( n \) and even degree. Let \( n = 4m + \beta \), \( 0 \leq \beta < 4 \) and let \( A_1, A_2, B_1, B_2 \) be four sets of cardinality \( m \).

If \( \beta = 0 \) then \( S_n \) is the regular graph of degree \( 2(m - 1) \) which has vertex set \( V(S_n) = A_1 \cup A_2 \cup B_1 \cup B_2 \) such that \( S_n[A_i] \cong S_n[B_i] \cong K_m, i = 1, 2 \) and \( S_n[A_i, B_i] \cong K^1_{A_i, B_i} \) and \( S_n[A_1, A_2] \cong K^1_{A_1, A_2} \). The graph \( S_m \) and its complement are depicted in Fig. 1 which illustrates that \( S_n \) is a self-packable graph.

The examples for \( \beta \neq 0 \) are obtained from \( S_m \) as follows. If \( \beta = 1 \), we add a vertex \( x \) to \( S_m \) and join it to all vertices in \( B_1 \cup B_2 \). We also add the missing perfect matching in each of \( K^1_{A_1, B_1}, K^1_{A_2, B_2}, K^1_{A_1, A_2} \) to obtain a \( 2m \)-regular self-packable graph (actually, this
Fig. 1. The graph $S_n$ for $n = 4m$ and its complement.

Fig. 2. The graph $S_n$ for $\beta = 1$ (a), $\beta = 2$ (b) and $\beta = 3$ (c).

The above graphs provide instances of regular self-packable graphs of all degrees up to the degree of $S_n$. To see this, we need the following case of a more general result proved by Bollobás et al. [2].
Theorem 10 (Bollobás et al. [2]). Let \( G \) be a regular graph of edge connectivity \( \lambda = \lambda(G) \geq 1 \) and even degree \( r \). Then, \( G \) has a \( k \)-factor for all even values of \( k, 2 \leq k \leq r - 2 \). Moreover, if \( \lambda(G) = r \) and \( |V(G)| \) is even, then \( G \) has a \( k \)-factor for all odd values of \( k, 1 \leq k \leq r - 1 \).

According to this result, \( S_n \) has \( r \)-regular spanning subdigraph for all even values of \( r \) from 2 to the degree of \( S_n \). Moreover, it is not difficult to check that the edge connectivity of \( S_n \) equals its degree. Therefore, when \( n \) is even, \( S_n \) contains also \( r \)-factors for each odd value of \( r, 1 \leq r \leq (n/2) - 2 \). We denote by \( S^r_n \) the regular spanning subgraph of \( S_n \) for each suitable value of \( r \).

Theorem 11. Let \( G \) be a regular graph of order \( n \) and degree \( (2k - 3)/2k - 2 \) \( n \leq r \leq (2k - 1)/2k - 2, k \geq 2 \). Then
\[
\frac{n}{2}k < L(n,r) \leq \frac{n}{2}(k + 2).
\]
Moreover, if \( k \leq 4 \), then \( n/2(k + 1) \leq L(n,r) \).

Proof. Suppose that \( H = G_1 \oplus \cdots \oplus G_h \) is a \( G \)-decomposition of graph \( H \) of order \( n_H \). Let \( X, Y \subset V(G) \) be largest subsets such that \( G[X] \) and \( G[Y] \) are packable. Let \( \delta_X \) and \( \delta_Y \) be the minimum degree of \( G[X] \) and \( G[Y] \), respectively, and \( \delta = \min \{\delta_X, \delta_Y\} \). Since \( \delta_X + \delta_Y \leq |X| - 1 \), we have \( \delta \leq (|X| - 1)/2 \). Therefore,
\[
\frac{2k - 3}{2k - 2}n \leq r \leq (n - |X|) + \delta \leq (2n - |X| - 1)/2,
\]
which implies \( |X| \leq n/(k - 1) - 1 \). Therefore, \( \hat{\theta} > k \) and Theorem 2 implies \( (n/2)k < n_H \). Moreover, suppose that \( k \leq 4 \). Since each vertex of \( H \) is in at least two copies of \( G \), the number \( h \) of copies of \( G \) is at least \( t + 1 \). Let \( x = r_H/r \), where \( r_H \) denotes the degree of \( H \). If \( x \geq 3 \), then \( n_H \geq 3r + 1 \geq 3(2k - 3)/2k - 2 \) \( n \geq n/2(k + 1) \). If \( x = 2 \), then, by Lemma 4 \( n_H \geq nh/x \geq n/2(k + 1) \).

We shall now show the upper bound.

Suppose first that \( n \) is an even number and let \( n = x_1 + \cdots + x_k \) be a partition of \( n \) such that \( \lfloor n/k \rfloor \leq x_i \leq \lfloor n/k \rfloor + 1 \). Since \( n \) is even, there is an even number of odd parts in this partition. Therefore, there is a partition \( n = x'_1 + \cdots + x'_k \) such that \( \lfloor n/k \rfloor - 1 \leq x_i \leq \lfloor n/k \rfloor + 2 \) and all parts are even numbers. Let \( P = \{X_1, \ldots, X_k\} \) a partition of a set \( V \) of cardinality \( n \) such that \( |X_i| = x'_i, i = 1, \ldots, k \). Let \( G \) be the graph with vertex set \( V \) such that \( G[V_i, V_j] = K_{x'_i, x'_j} \) for each pair \( i, j \) of different subscripts and \( G[X_i] \simeq S_{x'_i} \), where \( l = r - \sum_{j \neq i} x'_j \). This is possible since \( 0 \leq l = r - (n - x'_i) = x'_i - (n/2k) - 2 \leq x'_i/2 - 2 \) and \( x'_i \) is an even number. Since \( S_{x'_i} \) is self-packable, if \( k \) is odd, then \( P \) is a packable partition. If \( k \) is even, the partition obtained by splitting one of the parts of \( P \) into two non empty subsets is packable. Therefore, \( N_0(G) \leq n/2(k + 2) \).

Suppose now that \( n \) is odd, then, \( r \) is an even number. Let \( n = x_1 + \cdots + x_k \) be a partition of \( n \) such that \( \lfloor n/k \rfloor \leq x_i \leq \lfloor n/k \rfloor + 1 \) and \( P = \{X_1, \ldots, X_k\} \) a partition of a set \( V \) of cardinality \( n \) such that \( |X_i| = x_i, i = 1, \ldots, t \). Let \( G \) be the graph with vertex set
such that \( G[V_i, V_j] = K_{V_i, V_j} \) for each pair \( i, j \) of different subscripts and \( G[X_i] \simeq S_{x_i}', \) where \( l = r - \sum_{j \neq i} x_j. \) This is possible since \( 0 \leq l = r - (n - x_i') - x_i' - (n/2k) - 2 \leq x_i'/2 - 2 \) and, if \( x_i' \) is odd, then \( l \) is an even number. Since \( S_{x_i}' \) is self-packable, if \( k \) is odd, then \( P \) is a packable partition. If \( k \) is even, the partition obtained by splitting one of the parts of \( P \) into two nonempty subsets is packable. Therefore, \( N_0(G) \leq n/2(k + 2). \)

In particular, when \( r = n/2 \), the values of \( L(n, r) \) and \( U(n, r) \) can be given.

**Corollary 12.**

\[
L \left( n, \frac{n}{2} \right) = \frac{3n}{2} \quad \text{and} \quad U \left( n, \frac{n}{2} \right) = \frac{n}{2} \left( \left\lceil \frac{n}{4} \right\rceil + 1 \right).
\]

**Proof.** When \( r = n/2 \), the complementary graph \( \tilde{G} \) is Hamiltonian. Then \( V(G) \) can be partitioned into either \( n/4 \) paths of length 3 when \( n \equiv 0 \) (mod 4) or into \( \lceil n/4 \rceil - 1 \) paths of length 3 and two paths of length 2 when \( n \equiv 2 \) (mod 4). In either case, each part induces a self-packable subgraph of \( G \) and we have a packable partition \( P \) regardless of the parity of \( |P| \). By Theorem 1, \( N_0(G) \leq n/2 \left( \left\lceil n/4 \right\rceil + 1 \right). \) On the other hand, as shown in the proof of Theorem 2 the cartesian product \( K_m \times K_2 \) attains this bound. For the lower bound, Theorem 11 shows that \( L(n, n/2) \geq 3n/2 \) and \( K_2[Km] \) attains this bound.

As a final remark, note that the regularity of \( G \) involves the relation \( n_H r_H = nr \), where \( h \) is the number of copies of a \( G \)-decomposition of \( H \) and \( n_H, r_H \) are the order and the degree of \( H \), and \( n \) and \( r \), the order and the degree of \( G \). In all the bounds obtained for \( N_0(G) \), it turns out that \( r_H/r = 2 \), the minimum possible value. Therefore, the number \( h \) of copies is minimum as well. In other words, a \( G \)-decomposable graph \( H \) of minimum order, has also the minimum number of edges and the minimum number of copies of \( G \) among all non trivial \( G \)-decomposable graphs.

**References**