On multiple fixed-sign solutions of a discrete system with Hermite boundary conditions

Johnny Henderson, Patricia J.Y. Wong

Abstract
We consider the discrete system

\[ \Delta^m u_i(k) = P_i(k, u_1(k), u_2(k), \ldots, u_n(k)), \quad k \in \{0, 1, \ldots, N\}, \ 1 \leq i \leq n, \]

together with Hermite boundary conditions,

\[ \Delta^j u_i(k_v) = 0, \quad j = 0, \ldots, m_v - 1, \ v = 1, \ldots, r, \ 1 \leq i \leq n, \]

where \( r \geq 2, \ m_v \geq 1 \) for \( v = 1, \ldots, r, \) \( \sum_{v=1}^{r} m_v = m, \) and \( k_v \)'s are integers such that \( 0 = k_1 < k_1 + m_1 < k_2 < k_2 + m_2 < \cdots < k_r + m_r - 1 = N + m. \) By using the Leggett–Williams fixed point theorem and the Five functionals fixed point theorem, we establish two different sets of criteria for the existence of three solutions of the system which are of ‘fixed-signs.' Examples are also included to illustrate the results obtained.

Keywords: Fixed-sign solutions; System of difference equations; Boundary value problems

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1. Introduction

Throughout we shall use the notation $Z[c, d] = \{c, c + 1, \ldots, d\}$ where $c, d (\geq c)$ are integers. In this paper we shall consider a system of difference equations subject to Hermite boundary conditions at multipoints $k_\nu, \nu = 1, 2, \ldots, r$. To be exact, our system is

$$\begin{align*}
\Delta^n u_i(k) &= P_i(k, u_1(k), u_2(k), \ldots, u_n(k)), \quad k \in Z[0, N], \\
\Delta^j u_i(k_\nu) &= 0, \quad j = 0, \ldots, m_\nu - 1, \quad \nu = 1, 2, \ldots, r,
\end{align*}$$

(H)

where $r \geq 2, m_\nu \geq 1$ for $\nu = 1, 2, \ldots, r$, $\sum_{\nu=1}^r m_\nu = m$, and $k_\nu$'s are integers such that $0 = k_1 < k_1 + m_1 < k_2 < k_2 + m_2 < \cdots < k_r < k_r + m_r - 1 = N + m$.

Assume that for each $1 \leq i \leq n$, $P_i : Z[0, N] \times R^n \to R$ is continuous.

A solution $u = (u_1, u_2, \ldots, u_n)$ of (H) will be sought in $(Z[0, N + m])^n = Z[0, N + m]$ (n times). If $u$ is a solution of (H), then it is clear that for each $1 \leq i \leq n$,

$$u_i(k) = 0, \quad k \in Z[k_\nu, k_\nu + m_\nu - 1], \quad \nu = 1, \ldots, r.$$

We say that $u = (u_1, u_2, \ldots, u_n)$ is a solution of fixed-sign if for each $1 \leq i \leq n$, we have

$$(-1)^{m_1 + m_2 + \cdots + m_r} \theta_i u_i(k) \geq 0 \quad \text{for} \quad k \in Z[k_\nu, k_\nu + 1], \quad \nu = 1, \ldots, r - 1,$$

where $\theta_i \in \{1, -1\}$ is fixed. We remark that although in many practical two-point problems, it is only meaningful to have positive solutions, our definition of fixed-sign solution gives extra flexibility.

We shall establish criteria so that the system (H) has at least three fixed-sign solutions. In addition, we also provide estimates on the norms of these solutions.

The present work is motivated by the fact that multipoint boundary value problem of the type (H) models various dynamic systems with $m$ degrees of freedom in which $m$ states are observed at $m$ times [23]. In fact, when $m = r = 2$ the boundary value problem (H) describes a vast spectrum of nonlinear phenomena which include gas diffusion through porous media, nonlinear diffusion generated by nonlinear sources, thermal self-ignition of a chemically active mixture of gases in a vessel, catalysis theory, chemically reacting systems, adiabatic tubular reactor processes, as well as concentration in chemical or biological problems, e.g., see [7,11–13,24]. Since boundary value problems model many physical phenomena, it is not surprising that they have received numerous attention in the recent literature, and a vast amount of work has focused on the existence of positive solutions (single, double and triple), e.g., see [2,5,6,9,10,14–20,22,25,26] and the monographs [3,4].

In the present work, fixed point theorems of both Leggett and Williams [21] and Avery [8] are used to derive sets of criteria for the existence of three fixed-sign solutions. Not only are new results obtained, but we also discuss the relation between the results in terms of generality and illustrate through some examples. Moreover, we have generalized a single-dependent-variable boundary value problem, the usual consideration in the literature, to a system of boundary value problems, which is a much more appropriate and robust model for most physical phenomena.
The plan of the paper is as follows. In Section 2 we state the relevant definitions and fixed point theorems. The main results and discussion are presented in Section 3. Finally, we illustrate the results through some examples in Section 5.

2. Preliminaries

In this section we shall state some necessary definitions and the relevant fixed point theorems. Let $B$ be a real Banach space with norm $\|\cdot\|$.

Definition 2.1. Let $C \subset B$ be a nonempty closed convex set. We say that $C$ is a cone provided the following conditions are satisfied:

(a) if $u \in C$ and $\alpha \geq 0$, then $\alpha u \in C$;
(b) if $u \in C$ and $-u \in C$, then $u = 0$.

Definition 2.2. Let $C \subset B$ be a cone. A map $\psi$ is a nonnegative continuous concave functional on $C$ if the following conditions are satisfied:

(a) $\psi : C \rightarrow [0, \infty)$ is continuous;
(b) $\psi(ty + (1 - t)z) \geq t\psi(y) + (1 - t)\psi(z)$ for all $y, z \in C$ and $0 \leq t \leq 1$.

Definition 2.3. Let $C \subset B$ be a cone. A map $\beta$ is a nonnegative continuous convex functional on $C$ if the following conditions are satisfied:

(a) $\beta : C \rightarrow [0, \infty)$ is continuous;
(b) $\beta(ty + (1 - t)z) \leq t\beta(y) + (1 - t)\beta(z)$ for all $y, z \in C$ and $0 \leq t \leq 1$.

Let $\gamma, \beta, \Theta$ be nonnegative continuous convex functionals on $C$ and let $\alpha, \psi$ be nonnegative continuous concave functionals on $C$. For nonnegative numbers $w_i$, $1 \leq i \leq 3$, we shall introduce the following notations:

\[ C(w_1) = \{ u \in C \mid \|u\| < w_1 \}, \]
\[ C(\psi, w_1, w_2) = \{ u \in C \mid \psi(u) \geq w_1 \text{ and } \|u\| \leq w_2 \}, \]
\[ P(\gamma, w_1) = \{ u \in C \mid \gamma(u) < w_1 \}, \]
\[ P(\gamma, \alpha, w_1, w_2) = \{ u \in C \mid \alpha(u) \geq w_1 \text{ and } \gamma(u) \leq w_2 \}, \]
\[ Q(\gamma, \beta, w_1) = \{ u \in C \mid \beta(u) \leq w_1 \text{ and } \gamma(u) \leq w_2 \}, \]
\[ P(\gamma, \Theta, \alpha, w_1, w_2, w_3) = \{ u \in C \mid \alpha(u) \geq w_1, \Theta(u) \leq w_2, \text{ and } \gamma(u) \leq w_3 \}, \]
\[ Q(\gamma, \beta, \psi, w_1, w_2, w_3) = \{ u \in C \mid \psi(u) \geq w_1, \beta(u) \leq w_2, \text{ and } \gamma(u) \leq w_3 \}. \]

The following fixed point theorems are needed later. The first is usually called Leggett–Williams fixed point theorem, and the second is known as the Five-functionals fixed point theorem.
Theorem 2.1 [21]. Let $C \subset B$ be a cone, and $w_4 > 0$ be given. Assume that $\psi$ is a nonnegative continuous concave functional on $C$ such that $\psi(u) \leq \|u\|$ for all $u \in C(w_4)$, and let $S : C(w_4) \to C(w_4)$ be a continuous and completely continuous operator. Suppose there exist numbers $w_1, w_2, w_3$, where $0 < w_1 < w_2 < w_3 \leq w_4$, such that

(a) $\{u \in C(\psi, w_2, w_3) \mid \psi(u) > w_2\} \neq \emptyset$, and $\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_3)$;
(b) $\|Su\| < w_1$ for all $u \in C(w_1)$;
(c) $\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_4)$ with $\|Su\| > w_3$.

Then, $S$ has at least three fixed points $u^1$, $u^2$, and $u^3$ in $C(w_4)$. Furthermore, we have

$$u^1 \in C(w_1), \quad u^2 \in \{u \in C(\psi, w_2, w_4) \mid \psi(u) > w_2\}, \quad \text{and} \quad u^3 \in \bar{C}(w_4) \setminus (C(\psi, w_2, w_4) \cup C(w_1)).$$

(2.1)

Theorem 2.2 [8]. Let $C \subset B$ be a cone. Assume there exist positive numbers $w_5$, $M$, nonnegative continuous convex functionals $\gamma$, $\beta$, $\Theta$ on $C$, and nonnegative continuous concave functionals $\alpha$, $\psi$ on $C$, with

$$\alpha(u) \leq \beta(u) \quad \text{and} \quad \|u\| \leq M \gamma(u)$$

for all $u \in \bar{P}(\gamma, w_5)$. Let $S : \bar{P}(\gamma, w_5) \to \bar{P}(\gamma, w_5)$ be a continuous and completely continuous operator. Suppose that there exist nonnegative numbers $w_i$, $1 \leq i \leq 4$, with $0 < w_2 < w_3$, such that

(a) $\{u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(u) > w_3\} \neq \emptyset$, and $\alpha(Su) > w_3$, for all $u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$;
(b) $\{u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(u) < w_2\} \neq \emptyset$, and $\beta(Su) < w_2$, for all $u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5)$;
(c) $\alpha(Su) > w_3$ for all $u \in P(\gamma, \alpha, w_3, w_5)$ with $\Theta(Su) > w_4$;
(d) $\beta(Su) < w_2$ for all $u \in Q(\gamma, \beta, w_2, w_5)$ with $\psi(Su) < w_1$.

Then, $S$ has at least three fixed points $u^1$, $u^2$, and $u^3$ in $\bar{P}(\gamma, w_5)$. Furthermore, we have

$$\beta(u^1) < w_2, \quad \alpha(u^2) > w_3, \quad \text{and} \quad \beta(u^3) > w_2 \quad \text{with} \quad \alpha(u^2) < w_3.$$ (2.2)

3. Main results

For each $v = 1, \ldots, r - 1$, define the constant $\delta_v$ and the interval $I_v \subset Z[k_v, k_{v+1}]$ by

$$\delta_v = \sum_{j=v+1}^{r} m_j \quad \text{and} \quad I_v = Z[k_v + m_v, k_{v+1} - 1].$$ (3.1)

Throughout we shall denote $u = (u_1, u_2, \ldots, u_n)$. Let the Banach space

$$B = \{u \mid u : (Z[0, N + m])^n \to \mathbb{R}^n\}$$ (3.2)
be equipped with norm
\[ \|u\| = \max_{1 \leq i \leq n} \max_{k \in \mathbb{Z}[0,N+m]} |u_i(k)| = \max_{1 \leq i \leq n} |u_i|, \] (3.3)
where we let \( |u_i|_0 = \max_{k \in \mathbb{Z}[0,N+m]} |u_i(k)|, 1 \leq i \leq n. \)

To apply the fixed point theorems in Section 2, we need to define an operator \( S : B \to B \) so that a solution \( u \) of the system (H) is a fixed point of \( S \), i.e., \( u = Su \). For this, let \( G(k,\ell) \) be the Green’s function of the scalar boundary value problem
\[ \begin{align*}
\Delta^m y(k) &= 0, \quad k \in \mathbb{Z}[0,N], \\
\Delta^j y(k_v) &= 0, \quad j = 0, \ldots, m_v - 1, \quad v = 1, \ldots, r.
\end{align*} \] (3.4)

If \( u \) is a solution of (H), then \( u \) can be represented as
\[ u_i(k) = \sum_{\ell=0}^{N} G(k,\ell) P_i(\ell,u(\ell)), \quad k \in \mathbb{Z}[0,N+m], \quad 1 \leq i \leq n. \] (3.5)

Hence, we shall define the operator \( S : B \to B \) by
\[ Su(k) = (Su_1(k), Su_2(k), \ldots, Su_n(k)), \quad k \in \mathbb{Z}[0,N+m], \] (3.6)
where
\[ S_{ui}(k) = \sum_{\ell=0}^{N} G(k,\ell) P_i(\ell,u(\ell)), \quad k \in \mathbb{Z}[0,N+m], \quad 1 \leq i \leq n. \]

Then, a fixed point of the operator \( S \) is a solution of the system (H).

Our first lemma gives the properties of the Green’s function \( G(k,\ell) \) which will be used later.

**Lemma 3.1** [27]. Let \( G(k,\ell) \) be the Green’s function for (3.4). Then

(a) the signs of \( G(k,\ell) \) are:
\[ (-1)^v G(k,\ell) \geq 0, \quad (k,\ell) \in \mathbb{Z}[k_v,k_v+1] \times \mathbb{Z}[0,N], \quad v = 1, \ldots, r - 1; \]
\[ G(k,\ell) = 0, \quad (k,\ell) \in \mathbb{Z}[k_r,N+m] \times \mathbb{Z}[0,N]; \]

(b) for \( (k,\ell) \in I_v \times \mathbb{Z}[0,N], \quad v = 1, \ldots, r - 1, \)
\[ (-1)^v G(k,\ell) \geq L_v \|G(\cdot,\ell)\|, \]
where
\[ \|G(\cdot,\ell)\| = \max_{k \in \mathbb{Z}[0,N+m]} |G(k,\ell)| = \max_{1 \leq v, r \leq r-1} \max_{k \in [k_v,k_{v+1}]} (-1)^v G(k,\ell), \]
\[ L_v = \min \left\{ \frac{\min[p(k_v + m_v), p(k_{v+1} - 1)]}{\max_{k \in \mathbb{Z}[0,N+m]} p(k)}, \frac{\min[q(k_v + m_v), q(k_{v+1} - 1)]}{\max_{k \in \mathbb{Z}[0,N+m]} q(k)} \right\}, \]
and the functions \( p \) and \( q \) are defined as
\[
p(k) = \left| \prod_{j=1}^{r-1} (k - k_j)^{(m_j)} \right| (N + m - k)^{(m_r - 1)},
q(k) = k^{(m_1 - 1)} \left| \prod_{j=2}^{r} (k - k_j)^{(m_j)} \right|;
\]

(c) \((-1)^h \mathcal{G}(k, \ell) \leq \|G(\cdot, \ell)\|, (k, \ell) \in Z[k_v, k_v + 1] \times Z[0, N], v = 1, \ldots, r - 1.\)

**Lemma 3.2.** The operator \(S\) defined in (3.5) is continuous and completely continuous.

**Proof.** Since each \(P_i : Z[0, N] \times \mathbb{R}^n \to \mathbb{R}\) is continuous, using the Ascoli–Arzela theorem, we can show that \(S\) is continuous and completely continuous. \(\Box\)

For clarity, we shall now list some conditions that are needed later. In these conditions, \(\theta_i \in \{1, -1\}, 1 \leq i \leq n\), are fixed, and the sets \(\tilde{K}\) and \(K\) are given by

\[
\tilde{K} = \{ u \in B \mid \text{for each } 1 \leq i \leq n, (-1)^h \theta_i u_i(k) \geq 0 \text{ for } k \in Z[k_v, k_v + 1], v = 1, 2, \ldots, r - 1 \},
\]

and

\[
K = \{ u \in \tilde{K} \mid \text{for some } j \in \{1, 2, \ldots, n\}, (-1)^h \theta_j u_j(k) > 0 \text{ for some } k \in Z[0, N + m] \}
\]

\(= \tilde{K} \setminus \{0\}\).

(C1) For each \(1 \leq i \leq n\), assume that

\[
\theta_i P_i(\ell, u) \geq 0, \quad u \in \tilde{K}, \quad \ell \in Z[0, N] \quad \text{and} \quad \theta_i P_i(\ell, u) > 0, \quad u \in K, \quad \ell \in Z[0, N].
\]

(C2) There exist continuous functions \(f, b\) and \(a_i, 1 \leq i \leq n\) with \(f : \mathbb{R}^n \to [0, \infty)\) and \(b, a_i : Z[0, N] \to [0, \infty)\) such that for each \(1 \leq i \leq n\),

\[
a_i(\ell) f(u) \leq \theta_i P_i(\ell, u) \leq b(\ell) f(u), \quad u \in \tilde{K}, \quad \ell \in Z[0, N].
\]

(C3) For each \(1 \leq i \leq n\), there exists a number \(0 < \rho_i \leq 1\) such that

\[
a_i(\ell) \geq \rho_i b(\ell), \quad \ell \in Z[0, N].
\]

Next, we define a cone \(C\) in \(B\) as

\[
C = \{ u \in B \mid \text{for each } 1 \leq i \leq n, u_i(k) = 0 \text{ for } k \in Z[k_r, N + m],
(-1)^h \theta_i u_i(k) \geq 0 \text{ for } k \in Z[k_v, k_v + 1], v = 1, 2, \ldots, r - 1
\]

\(\text{and } \min_{k \in I_v} (-1)^h \theta_i u_i(k) \geq L_v \rho_i |u_i|_0, v = 1, 2, \ldots, r - 1 \}.
\]

where \(L_v\) and \(\rho_i\) are defined in Lemma 3.1(b) and (C3), respectively. Note that \(C \subseteq \tilde{K}\).

Moreover, a fixed point of \(S\) obtained in \(C\) will be a fixed-sign solution of the system (H).
If (C1) and (C2) hold, then it follows from (3.6) and Lemma 3.1(a) that, for \( u \in \tilde{K} \) and \( k \in \mathbb{Z}[k_\nu, k_{\nu+1}], 1 \leq \nu \leq r - 1, \)

\[
\sum_{\ell=0}^{N} (-1)^{k_\nu} G(k, \ell) a_\ell(f(u(\ell))) \leq (-1)^{\nu} \theta_i S_{ui}(k) \leq \sum_{\ell=0}^{N} (-1)^{k_\nu} G(k, \ell) b(\ell) f(u(\ell)), \quad 1 \leq i \leq n.
\]

(3.8)

**Lemma 3.3.** Let (C1)–(C3) hold. Then, the operator \( S \) maps \( C \) into itself.

**Proof.** Let \( u \in C \). From Lemma 3.1(a) and (3.6), it is clear that

\[
S_{ui}(k) = 0, \quad k \in \mathbb{Z}[k_r, N + m], 1 \leq i \leq n.
\]

Next, from (3.8) and Lemma 3.1(a), we have for \( 1 \leq i \leq n \) and \( k \in \mathbb{Z}[k_\nu, k_{\nu+1}], 1 \leq \nu \leq r - 1, \)

\[
(-1)^{k_\nu} \theta_i S_{ui}(k) \geq \sum_{\ell=0}^{N} (-1)^{k_\nu} G(k, \ell) a_\ell(f(u(\ell))) \geq 0.
\]

(3.10)

Using (3.10), (3.8), and Lemma 3.1(c), we obtain for \( 1 \leq i \leq n \) and \( k \in \mathbb{Z}[k_\nu, k_{\nu+1}], 1 \leq \nu \leq r - 1, \)

\[
|S_{ui}(k)| \geq (-1)^{k_\nu} \theta_i S_{ui}(k) \leq \sum_{\ell=0}^{N} (-1)^{k_\nu} G(k, \ell) b(\ell) f(u(\ell)) \leq \sum_{\ell=0}^{N} \|G(\cdot, \ell)\| b(\ell) f(u(\ell)).
\]

Therefore, together with (3.9) we have

\[
|S_{ui}|_0 = \max_{k \in \mathbb{Z}[0, N + m]} |S_{ui}(k)| = \max_{1 \leq \nu \leq r - 1} \max_{k \in \mathbb{Z}[k_\nu, k_{\nu+1}]} |S_{ui}(k)|
\]

\[
\leq \sum_{\ell=0}^{N} \|G(\cdot, \ell)\| b(\ell) f(u(\ell)), \quad 1 \leq i \leq n,
\]

which immediately yields

\[
\|Su\| = \max_{1 \leq i \leq n} |S_{ui}|_0 \leq \sum_{\ell=0}^{N} \|G(\cdot, \ell)\| b(\ell) f(u(\ell)).
\]

(3.11)

Now, applying (3.8), Lemma 3.1(b), (C3), (3.10), and (3.11), we find for \( 1 \leq i \leq n \) and \( k \in I_\nu, 1 \leq \nu \leq r - 1, \)

\[
(-1)^{k_\nu} \theta_i S_{ui}(k) \geq \sum_{\ell=0}^{N} (-1)^{k_\nu} G(k, \ell) a_\ell(f(u(\ell)))
\]
\[ \sum_{\ell=0}^{N} L_\nu \| G(\cdot, \ell) \|_0 b(\ell) f (u(\ell)) \geq L_\nu \rho_i \| Su \|_0 \geq L_\nu \rho_i |Su|_0. \]

This leads to
\[ \min_{k \in I_\nu} (-1)^{k_i} \theta_i u_i(k) \geq L_\nu \rho_i |S u_i|_0, \quad 1 \leq i \leq n, \quad 1 \leq v \leq r - 1. \quad (3.12) \]

With (3.9), (3.10), and (3.12) established, we have shown that \( Su \in C. \)

**Remark 3.1.** From the proof of Lemma 3.3, we see that it is possible to use another cone \( C' \) given by
\[
\begin{align*}
C' &= \left\{ u \in B \mid \text{for each } 1 \leq i \leq n, u_i(k) = 0 \text{ for } k \in Z[k_r, N + m], \right. \\
&\quad \left. (-1)^{k_i} \theta_i u_i(k) \geq 0 \text{ for } k \in Z[k_v, k_{v+1}], \quad v = 1, 2, \ldots, r - 1, \right. \\
&\quad \left. \text{and } \min_{k \in I_\nu} (-1)^{k_i} \theta_i u_i(k) \geq L_\nu \rho_i \| u \|, \quad v = 1, 2, \ldots, r - 1. \right\}
\end{align*}
\]

The arguments used would be similar.

Let \( \tau_1, \tau_2, \tau_3, \tau_4 \in Z[0, N + m], \quad 1 \leq v \leq r - 1 \) be fixed integers. For subsequent results, we define the following constants for each \( 1 \leq i \leq n \):

\[
q = \max_{1 \leq v \leq r - 1} \max_{k \in Z[k_v, k_{v+1}]} \sum_{\ell=0}^{N} (-1)^{v_\nu} G(k, \ell) b(\ell),
\]

\[
r_i = \min_{1 \leq v \leq r - 1} \min_{k \in I_\nu} \sum_{\ell=k_v+m_v}^{\infty} (-1)^{v_\nu} G(k, \ell) a_i(\ell),
\]

\[
d_{1,i} = \min_{1 \leq v \leq r - 1} \min_{k \in Z[\tau_2, \tau_3]} \sum_{\ell=\tau_2}^{\infty} (-1)^{v_\nu} G(k, \ell) a_i(\ell),
\]

\[
d_2 = \max_{1 \leq v \leq r - 1} \max_{1 \leq j \leq r - 1} \max_{k \in Z[\tau_1, \tau_4]} \max_{\ell=\tau_1}^{\tau_1+1} (-1)^{v_\nu} G(k, \ell) b(\ell),
\]

\[
d_3 = \max_{1 \leq v \leq r - 1} \max_{1 \leq j \leq r - 1} \max_{k \in Z[\tau_1, \tau_4]} \max_{\ell=\tau_1}^{\tau_1+1} \sum_{\ell=0}^{N} (-1)^{v_\nu} G(k, \ell) b(\ell)
\]

\[
+ \sum_{\ell=\min[\tau_4, N]+1}^{\tau_1+1} (-1)^{v_\nu} G(k, \ell) b(\ell),
\]

\[
d_4 = \max_{1 \leq v \leq r - 1} \max_{k \in Z[\tau_1, \tau_4]} \sum_{\ell=\tau_1}^{\infty} (-1)^{v_\nu} G(k, \ell) b(\ell),
\]
\[ d_5 = \max_{1 \leq \nu \leq r - 1} \max_{k \in \mathbb{Z}[k_1, k_4]} \left[ \sum_{\ell=0}^{t_1 - 1} (-1)^{\nu \ell} G(k, \ell)b(\ell) \right. \]
\[ + \left. \sum_{\ell=\min\{t_1, N\} + 1}^N (-1)^{\nu \ell} G(k, \ell)b(\ell) \right]. \tag{3.13} \]

**Lemma 3.4.** Let (C1)–(C3) hold, and assume

(C4) for each \( 1 \leq \nu \leq r - 1 \) and each \( k \in \mathbb{Z}[k_1, k_{\nu + 1}] \), the function \( G(k, \ell)b(\ell) \) is nonzero for some \( \ell \in \mathbb{Z}[0, N] \).

Suppose that there exists a number \( d > 0 \) such that for \( |u_j| \leq d, 1 \leq j \leq n \),

\[ f(u_1, u_2, \ldots, u_n) < \frac{d}{q}. \tag{3.14} \]

Then,

\[ S(C(d)) \subseteq \bar{C}(d) \subseteq C(d). \tag{3.15} \]

**Proof.** Let \( u \in \bar{C}(d) \). Then, it is clear that \( |u_j| \leq d, 1 \leq j \leq n \). Applying (3.10), (3.8), (C4), (3.14), and (3.13), we find for \( 1 \leq i \leq n \) and \( k \in \mathbb{Z}[k_1, k_{\nu + 1}], 1 \leq \nu \leq r - 1 \),

\[ |S_{\nu i}(k)| = (-1)^{\nu i} S_{\nu i}(k) \leq \sum_{\ell=0}^N (-1)^{\nu \ell} G(k, \ell)b(\ell)f(u(\ell)) \]
\[ < \sum_{\ell=0}^N (-1)^{\nu \ell} G(k, \ell)b(\ell)\frac{d}{q} \leq q \frac{d}{q} = d. \]

This implies \( |S_{\nu i}| < d, 1 \leq i \leq n \) and so \( \|Su\| < d \). Coupling with the fact that \( Su \in C \) (Lemma 3.3), we have \( Su \in C(d) \). The conclusion (3.15) is now immediate. \( \square \)

The next lemma is similar to Lemma 3.4 and its proof is omitted.

**Lemma 3.5.** Let (C1)–(C3) hold. Suppose that there exists a number \( d > 0 \) such that for \( |u_j| \leq d, 1 \leq j \leq n \),

\[ f(u_1, u_2, \ldots, u_n) \leq \frac{d}{q}. \]

Then,

\[ S(\bar{C}(d)) \subseteq \bar{C}(d). \]

We are now ready to establish existence criteria for three fixed-sign solutions. Our first result employs Theorem 2.1.
Theorem 3.1. Assume \(kr_1 + mr_1 \leq N\). Let (C1)--(C4) hold, and assume (C5) for each \(1 \leq i \leq n\), each \(1 \leq v \leq r - 1\), and each \(k \in I_v\), the function \(G(k, \ell) a_i(\ell)\) is nonzero for some \(\ell \in Z[k + m_v, \min\{k + 1 - 1, N\}]\). Suppose that there exist numbers \(w_1, w_2, w_3\) with

\[
0 < w_1 < w_2 < \frac{w_2}{\min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} L_v p_i} \leq w_3
\]
such that the following hold:

(P) \(f(u_1, u_2, \ldots, u_n) < w_1/q\) for \(|u_j| \leq w_1, 1 \leq j \leq n\);

(Q) one of the following holds:

(Q1) \(\limsup_{|u_1|, |u_2|, \ldots, |u_n| \to \infty} f(u_1, u_2, \ldots, u_n)/|u_j| < 1/q\) for some \(j \in \{1, 2, \ldots, n\}\);

(Q2) there exists a number \(\eta (\geq w_3)\) such that \(f(u_1, u_2, \ldots, u_n) \leq \eta/q\) for \(|u_j| \leq \eta, 1 \leq j \leq n\);

(R) for each \(1 \leq i \leq n\), \(f(u_1, u_2, \ldots, u_n) > w_2/r_i\) for \(w_2 \leq |u_j| \leq w_3, 1 \leq j \leq n\).

Then, the system (H) has at least three fixed-sign solutions \(u^1, u^2, u^3 \in C\) such that

\[
\|u^1\| < w_1; \\
\|u^2(k)\| > w_2, \quad k \in I_v, 1 \leq v \leq r - 1, 1 \leq i \leq n; \\
\|u^3\| > w_1, \quad \text{and} \quad \min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} \min_{k \in I_v}\|u^3(k)\| < w_2.
\]

Proof. We shall employ Theorem 2.1. First, we shall prove that condition (Q) implies the existence of a number \(w_4\) where \(w_4 \geq w_3\) such that

\[
S(\overline{C}(w_4)) \subseteq \overline{C}(w_4). \tag{3.17}
\]

Suppose that (Q2) holds. Then, by Lemma 3.5 we immediately have (3.17) where we pick \(w_4 = \eta\). Suppose now that (Q1) is satisfied. Then, there exist \(R > 0\) and \(\varepsilon < 1/q\) such that for some \(j \in \{1, 2, \ldots, n\}\),

\[
\frac{f(u_1, u_2, \ldots, u_n)}{|u_j|} < \varepsilon, \quad |u_1|, |u_2|, \ldots, |u_n| > R. \tag{3.18}
\]

Define

\[
M = \max_{|u_j| \in [0, R], 1 \leq i \leq n} f(u_1, u_2, \ldots, u_n).
\]

In view of (3.18), it is clear that for some \(j \in \{1, 2, \ldots, n\}\), the following holds for all \((u_1, u_2, \ldots, u_n) \in \mathbb{R}^n\),

\[
f(u_1, u_2, \ldots, u_n) \leq M + \varepsilon|u_j|.
\]

Now, pick the number \(w_4\) so that

\[
w_4 > \max \left\{ w_3, M \left( \frac{1}{q} - \varepsilon \right)^{-1} \right\}. \tag{3.20}
\]
Let \( u \in \tilde{C}(w_4) \). Then, using (3.8), (3.19), and (3.20) we find for \( 1 \leq i \leq n \) and \( k \in Z[k_v, k_{v+1}] \), \( 1 \leq v \leq r - 1 \),

\[
|S_{u_i}(k)| = (-1)^{k_v} \theta_i S_{u_i}(k)
\]

\[
\leq \sum_{\ell=0}^{N} (-1)^{k_v} G(k, \ell) b(\ell) f(u(\ell)) \leq \sum_{\ell=0}^{N} (-1)^{k_v} G(k, \ell) [M + \varepsilon |u_j(\ell)|]
\]

\[
\leq \sum_{\ell=0}^{N} (-1)^{k_v} G(k, \ell) b(\ell) (M + \varepsilon w_k) \leq q(M + \varepsilon w_k)
\]

\[
< q \left[ w_4 \left( \frac{1}{q} - \varepsilon \right) + \varepsilon w_k \right] = w_4.
\]

This leads to \( |S_{u_i}| < w_4 \), \( 1 \leq i \leq n \). Hence, \( \|S u\| < w_4 \) and so \( S u \in C(w_4) \subset \tilde{C}(w_4) \). Thus, (3.17) follows immediately.

Let \( \psi : C \to [0, \infty) \) be defined by

\[
\psi(u) = \min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} \min_{k \in I_v} \min(-1)^{k_v} \theta_i u_i(k).
\] (3.21)

Clearly, \( \psi \) is a nonnegative continuous concave functional on \( C \) and \( \psi(u) \leq \|u\| \) for all \( u \in C \).

We shall verify that condition (a) of Theorem 2.1 is satisfied. First, we note that

\[
u^* = (u_1^*, u_2^*, \ldots, u_n^*) \in \{ u \in C(\psi, w_2, w_3) \mid \psi(u) > w_2 \} \neq \emptyset,
\]

where

\[
u_i^*(k) = \begin{cases} (-1)^{k_v} \theta_i \frac{w_2 + w_3}{2}, & k \in I_v, 1 \leq v \leq r - 1, \\ 0, & k \in Z[0, N + m] \setminus \bigcup_{v=1}^{r-1} I_v, \end{cases}
\]

for \( 1 \leq i \leq n \). Next, let \( u \in C(\psi, w_2, w_3) \). Then, \( w_2 \leq \psi(u) \leq \|u\| \leq w_3 \) provides

\[
w_2 \leq \min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} \min_{k \in I_v} (-1)^{k_v} \theta_i u_i(\ell) = |u_j(\ell)| \leq w_3, \quad \ell \in I_v, \quad 1 \leq v \leq r - 1, \quad 1 \leq j \leq n. \] (3.22)

Applying (3.8), (3.22), (C5), (R), and (3.13), we find

\[
\psi(Su) = \min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} \min_{k \in I_v} \min(-1)^{k_v} \theta_i (Su_i)(k)
\]

\[
\geq \min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} \min_{k \in I_v} \min_{\ell = k_v + m_v}^{\min[k_v+1-N]} (-1)^{k_v} G(k, \ell) a_i(\ell) f(u(\ell))
\]

\[
\geq \min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} \min_{k \in I_v} \min_{\ell = k_v + m_v}^{\min[k_v+1-N]} (-1)^{k_v} G(k, \ell) a_i(\ell) f(u(\ell))
\]

\[
\geq \min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} \min_{k \in I_v} \sum_{\ell = k_v + m_v}^{\min[k_v+1-N]} (-1)^{k_v} G(k, \ell) a_i(\ell) \frac{w_2}{r_i}
\]

\[
= \min_{1 \leq i \leq n} r_i \frac{w_2}{r_i} = w_2.
\]
Therefore, we have shown that $\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_3)$.

Next, condition (b) of Theorem 2.1 is fulfilled, since by Lemma 3.4 and condition (P), we have $S(\bar{C}(w_1)) \subseteq C(w_1)$.

Finally, we shall show that condition (c) of Theorem 2.1 holds. Let $u \in C(\psi, w_2, w_4)$ with $\|Su\| > w_3$. Using (3.8), Lemma 3.1(b), (C3), and (3.11), we get

$$
\psi(Su) = \min_{1 \leq i \leq n} \min_{1 \leq \nu \leq r-1} \min_{k \in I_\nu} \psi_i(Su_i)(k)
$$

$$
\geq \min_{1 \leq i \leq n} \min_{1 \leq \nu \leq r-1} \min_{k \in I_\nu} \sum_{\ell=0}^{N} (-1)^{\delta_i} G(k, \ell) a_i(\ell) f\left(u(\ell)\right)
$$

$$

\geq \min_{1 \leq i \leq n} \min_{1 \leq \nu \leq r-1} \sum_{\ell=0}^{N} L_{\nu} \|G(\cdot, \ell)\|_p b(\ell) f\left(u(\ell)\right)
$$

$$
\geq \min_{1 \leq i \leq n} \min_{1 \leq \nu \leq r-1} L_{\nu} \|Su\| w_3
$$

$$
\geq \min_{1 \leq i \leq n} \min_{1 \leq \nu \leq r-1} L_{\nu} \|Su\| w_3
$$

Hence, we have proved that $\psi(Su) > w_2$ for all $u \in C(\psi, w_2, w_4)$ with $w_3 > w_2$.

It now follows from Theorem 2.1 that the system (H) has at least three fixed-sign solutions $u^1, u^2, u^3 \in \bar{C}(w_4)$ satisfying (2.1). It is easy to see that here (2.1) reduces to (3.16).

We shall now employ Theorem 2.2 to give another criterion for existence of at least three fixed-sign solutions of (H). In applying Theorem 2.2 it is possible to choose the functionals and constants in many different ways. We shall present two results to show the arguments involved. In particular the first result is a generalization of Theorem 3.1.

**Theorem 3.2.** Let (C1)–(C3) hold. Assume there exist integers $\tau_{1,v}, \tau_{2,v}, \tau_{3,v}, \tau_{4,v}, 1 \leq v \leq r-1$ with

$$
0 \leq \tau_{1,v} \leq k_v + m_v + \tau_{2,v} \leq \tau_{3,v} \leq k_{v+1} - 1 \leq \tau_{4,v} \leq N + m \quad \text{and} \quad \tau_{2,v} \leq N,
$$

such that

(C6) for each $1 \leq i \leq n$, each $1 \leq \nu \leq r-1$, and each $k \in Z[\tau_{2,v}, \tau_{3,v}]$, the function $G(k, \ell) a_i(\ell)$ is nonzero for some $\ell \in Z[\tau_{2,v}, \min[\tau_{3,v}, N]]$;

(C7) for each $v, j \in \{1, 2, \ldots, r-1\}$ such that $Z[\tau_{1,v}, \tau_{4,v}] \cap Z[k_j, k_{j+1}] \neq \emptyset$, and each $k \in Z[\tau_{1,v}, \tau_{4,v}] \cap Z[k_j, k_{j+1}]$, the function $G(k, \ell) b(\ell)$ is nonzero for some $\ell \in Z[\tau_{1,v}, \min[\tau_{4,v}, N]]$.

Suppose there exist numbers $w_3, 2 \leq i \leq 5$ with

$$
0 < w_2 < w_3 < \frac{w_3}{\min_{1 \leq i \leq n} \min_{1 \leq \nu \leq r-1} L_{\nu} \rho_i} \leq w_4 \leq w_5
$$

and $w_2 > w_3d_3/q$ such that the following hold:
(P) $f(u_1, u_2, \ldots, u_n) < (1/d_2)(w_2 - w_3d_3/q)$ for $|u_j| \leq w_2$, $1 \leq j \leq n$;

(Q) $f(u_1, u_2, \ldots, u_n) \leq w_5/q$ for $|u_j| \leq w_5$, $1 \leq j \leq n$;

(R) for each $1 \leq i \leq n$, $f(u_1, u_2, \ldots, u_n) > w_3/d_{1,j}$ for $w_3 \leq |u_j| \leq w_4$, $1 \leq j \leq n$.

Then, the system (H) has at least three fixed-sign solutions $u^1, u^2, u^3 \in \mathcal{C}(w_5)$ such that

\[
\begin{align*}
|u_i^1(k)| &< w_2, \quad k \in \mathbb{Z}[\tau_1, \tau_4, \nu], \ 1 \leq i \leq r - 1, \ 1 \leq i \leq n; \\
|u_i^2(k)| &> w_3, \quad k \in \mathbb{Z}[\tau_2, \tau_3, \nu], \ 1 \leq i \leq r - 1, \ 1 \leq i \leq n; \\
\max_{1 \leq i \leq n} \max_{1 \leq v \leq r - 1} \max_{k \in \mathbb{Z}[\tau_1, \tau_4, \nu]} |u_i^3(k)| &> w_2, \quad \text{and} \\
\min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} \min_{k \in \mathbb{Z}[\tau_2, \tau_3, \nu]} |u_i^3(k)| &< w_3. 
\end{align*}
\]

(3.23)

**Proof.** In the context of Theorem 2.2, we define functionals on $C$ by,

\[
\gamma(u) = \|u\|,
\psi(u) = \min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} \min_{k \in I_v} |u_i(k)|,
\beta(u) = \Theta(u) = \max_{1 \leq i \leq n} \max_{1 \leq v \leq r - 1} \max_{k \in \mathbb{Z}[\tau_1, \tau_4, \nu]} |u_i(k)|,
\alpha(u) = \min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} \min_{k \in \mathbb{Z}[\tau_2, \tau_3, \nu]} |u_i(k)|. 
\]

(3.24)

First, we shall show that the operator $S$ maps $\bar{P}(\gamma, w_5)$ into $\bar{P}(\gamma, w_5)$. Let $u \in \bar{P}(\gamma, w_5) = \mathcal{C}(w_5)$. Then, we have $|u_j| \leq w_5$, $1 \leq j \leq n$. Using (Q) and Lemma 3.5, it follows that $Su \in \mathcal{C}(w_5)$. Hence, we have shown that $S : \bar{P}(\gamma, w_5) \rightarrow \bar{P}(\gamma, w_5)$.

Next, we shall prove that condition (a) of Theorem 2.2 is fulfilled. Clearly, $u^* = (u^*_1, u^*_2, \ldots, u^*_n) \in \left\{ u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \mid \alpha(u) > w_3 \right\} \neq \emptyset$, where

\[
u_i(k) = \begin{cases} (-1)^{k} \theta_{i} \frac{u_i^{3} + u_i^{4}}{2}, & k \in I_v, \ 1 \leq v \leq r - 1, \\
0, & k \in \mathbb{Z}[0, N+m] \setminus \bigcup_{v=1}^{r} I_v, \end{cases}
\]

for $1 \leq i \leq n$. Now, let $u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5)$. Then, we have $\alpha(u) \geq w_3$ and $\Theta(u) \leq w_4$ which imply

\[
w_3 \leq |u_j(\ell)| \leq w_4, \quad \ell \in \mathbb{Z}[\tau_2, \tau_3, \nu], \ 1 \leq v \leq r - 1, \ 1 \leq j \leq n.
\]

(3.25)

Applying (3.8), (3.25), (C6), (R), and (3.13), we obtain

\[
\alpha(Su) = \min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} \min_{k \in \mathbb{Z}[\tau_2, \tau_3, \nu]} |Su_i(k)| \\
= \min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} \min_{k \in \mathbb{Z}[\tau_2, \tau_3, \nu]} (-1)^{k} \theta_{i} Su_i(k) \\
\geq \min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} \min_{k \in \mathbb{Z}[\tau_2, \tau_3, \nu]} \sum_{\ell=0}^{N} (-1)^{k} G(k, \ell) a_i(\ell) f(u(\ell))
\]
Hence, \( \alpha(Su) > w_3 \) for all \( u \in P(\gamma, \Theta, \alpha, w_3, w_4, w_5) \).

We shall now verify that condition (b) of Theorem 2.2 is satisfied. Let \( w_1 \) be such that \( 0 < w_1 < w_2 \). It is noted that

\[
u^* = (u_1^*, u_2^*, \ldots, u_n^*) \in \left\{ u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(u) < w_2 \right\} \neq \emptyset,
\]

where

\[
u_i^*(k) = \begin{cases}
-1^{k_i} \theta_{ij} \left( \frac{u_i - u_j + w_j}{2} \right), & k \in I_{ij}, \; 1 \leq v \leq r - 1, \\
0, & k \in Z[0, N + m] \backslash \bigcup_{v=1}^{r-1} I_v,
\end{cases}
\]

for \( 1 \leq i \leq n \). Let \( u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \). Then, we have \( \beta(u) \leq w_2 \) and \( \gamma(u) \leq w_5 \) which give

\[
\begin{align*}
\left| u_j(\ell) \right| & \leq w_2, \quad \ell \in Z[t_{1,v}, t_{4,v}], \quad 1 \leq v \leq r - 1, \quad 1 \leq j \leq n; \\
\left| u_j(\ell) \right| & \leq w_5, \quad \ell \in Z[0, N], \quad 1 \leq j \leq n.
\end{align*}
\]

Noting (3.8), (3.26), (C7), (P), (Q), and (3.13), we find

\[
\begin{align*}
\beta(Su) &= \max_{1 \leq i \leq n} \max_{1 \leq v \leq r-1} \max_{k \in Z[t_{1,v}, t_{4,v}]} \left\{ |S_{u_i}(k)| \right\} \\
&= \max_{1 \leq i \leq n} \max_{1 \leq v \leq r-1} \max_{1 \leq j \leq r-1} \max_{k \in Z[t_{1,v}, t_{4,v}] \cap Z[k, k_{j+1}]} \left\{ |S_{u_i}(k)| \right\} \\
&= \max_{1 \leq i \leq n} \max_{1 \leq v \leq r-1} \max_{1 \leq j \leq r-1} \max_{k \in Z[t_{1,v}, t_{4,v}] \cap Z[k, k_{j+1}]} \left( -1 \right)^{\delta_j} \theta_{ij} S_{u_i}(k) \\
&\leq \max_{1 \leq i \leq n} \max_{1 \leq v \leq r-1} \max_{1 \leq j \leq r-1} \max_{k \in Z[t_{1,v}, t_{4,v}] \cap Z[k, k_{j+1}]} \sum_{\ell=0}^{N} \left( -1 \right)^{\delta_j} G(k, \ell) \\
&\times b(\ell) f(u(\ell)) \\
&= \max_{1 \leq i \leq n} \max_{1 \leq v \leq r-1} \max_{1 \leq j \leq r-1} \max_{k \in Z[t_{1,v}, t_{4,v}] \cap Z[k, k_{j+1}]} \left[ \min_{t_{1,v}, N} \sum_{\ell=0}^{t_{1,v} - 1} \left( -1 \right)^{\delta_j} G(k, \ell) \times b(\ell) f(u(\ell)) + \sum_{\ell=0}^{N} (-1)^{\delta_j} G(k, \ell) b(\ell) f(u(\ell)) \right] \\
&\leq \max_{1 \leq i \leq n} \max_{1 \leq v \leq r-1} \max_{1 \leq j \leq r-1} \max_{k \in Z[t_{1,v}, t_{4,v}] \cap Z[k, k_{j+1}]} \left[ \min_{t_{1,v}, N+1} \sum_{\ell=0}^{t_{1,v} - 1} \left( -1 \right)^{\delta_j} G(k, \ell) \times b(\ell) f(u(\ell)) + \sum_{\ell=0}^{N} (-1)^{\delta_j} G(k, \ell) b(\ell) f(u(\ell)) \right]
\end{align*}
\]
Moreover, (C3) and Lemma 3.1(b) yield for $u_\beta(Su) < w_2$. Therefore, $u_\alpha(Su) = \Theta(Su) = 2$ for all $\min\{\tau_{k_1}, \ldots, \tau_{k_N}\}\geq d_\max\leq 1$.

Next, we shall show that condition (c) of Theorem 2.2 is met. Using Lemma 3.1(c), we observe that for $u \in C$,

$$\Theta(Su) = \max_{1 \leq i \leq n} \max_{1 \leq v \leq r-1} \max_{1 \leq j \leq r-1} \max_{k \in \mathbb{Z}[t_{k_1}, t_{k_2}]} \|Su_i(k)\|$$

$$= \max_{1 \leq i \leq n} \max_{1 \leq v \leq r-1} \max_{1 \leq j \leq r-1} \max_{k \in \mathbb{Z}[t_{k_1}, t_{k_2}]} \| -1^{\beta} \theta_i Su_i(k) \|$$

$$\leq \max_{1 \leq i \leq n} \max_{1 \leq v \leq r-1} \max_{1 \leq j \leq r-1} \max_{k \in \mathbb{Z}[t_{k_1}, t_{k_2}]} \sum_{\ell=0}^{N} (-1)^{\beta} G(k, \ell) \|b(\ell) f(u(\ell))\|$$

$$= \sum_{\ell=0}^{N} |G(\cdot, \ell)| \|b(\ell) f(u(\ell))\|. \quad (3.27)$$

Moreover, (C3) and Lemma 3.1(b) yield for $u \in C$,

$$\alpha(Su) = \min_{1 \leq i \leq n} \min_{1 \leq v \leq r-1} \min_{k \in \mathbb{Z}[t_{k_1}, t_{k_2}]} \|Su_i(k)\|$$

$$= \min_{1 \leq i \leq n} \min_{1 \leq v \leq r-1} \min_{k \in \mathbb{Z}[t_{k_1}, t_{k_2}]} ( -1)^{\beta} \theta_i Su_i(k)$$

$$\geq \min_{1 \leq i \leq n} \min_{1 \leq v \leq r-1} \min_{k \in \mathbb{Z}[t_{k_1}, t_{k_2}]} \sum_{\ell=0}^{N} (-1)^{\beta} G(k, \ell) a_i(\ell) f(u(\ell))$$

$$\geq \min_{1 \leq i \leq n} \min_{1 \leq v \leq r-1} \min_{k \in \mathbb{Z}[t_{k_1}, t_{k_2}]} \sum_{\ell=0}^{N} (-1)^{\beta} G(k, \ell) a_i(\ell) f(u(\ell))$$
\[
\geq \min_{1 \leq i \leq n} \min_{1 \leq v \leq r-1} \sum_{\ell=0}^{N} L_v \| G(\cdot, \ell) \| \rho_i b(\ell) f(u(\ell)).
\]  
(3.28)

A combination of (3.27) and (3.28) gives
\[
\alpha(Su) \geq \min_{1 \leq i \leq n} \min_{1 \leq v \leq r-1} L_v \rho_i \Theta(Su), \quad u \in C.
\]  
(3.29)

Let \( u \in P(\gamma, \alpha, w_3, w_5) \) with \( \Theta(Su) > w_4 \). Then, it follows from (3.29) that
\[
\alpha(Su) \geq \min_{1 \leq i \leq n} \min_{1 \leq v \leq r-1} L_v \rho_i \Theta(Su) > \min_{1 \leq i \leq n} \min_{1 \leq v \leq r-1} L_v \rho_i w_4
\]
\[
\geq \min_{1 \leq i \leq n} \min_{1 \leq v \leq r-1} L_v \rho_i \frac{w_3}{\min_{1 \leq i \leq n} \min_{1 \leq v \leq r-1} L_v \rho_i} = w_3.
\]

Thus, \( \alpha(Su) > w_3 \) for all \( u \in P(\gamma, \alpha, w_3, w_5) \) with \( \Theta(Su) > w_4 \).

Finally, we shall prove that condition (d) of Theorem 2.2 is fulfilled. Let \( u \in Q(\gamma, \beta, w_2, w_5) \) with \( \psi(Su) < w_1 \). Then, we have \( \beta(u) \leq w_2 \) and \( \gamma(u) \leq w_5 \) which give (3.26). Using (3.8), (3.26), (C7), (P), (Q), and (3.13), we get as in an earlier part \( \beta(Su) < w_2 \) for all \( u \in Q(\gamma, \beta, w_2, w_5) \) with \( \psi(Su) < w_1 \).

It now follows from Theorem 2.2 that the system (H) has (at least) three fixed-sign solutions \( u^1, u^2, u^3 \in P(\gamma, w_5) = \tilde{C}(w_5) \) satisfying (2.2). It is clear that (2.2) reduces to (3.23) immediately. \( \square \)

For each \( 1 \leq v \leq r-1 \), if we choose \( \tau_{1,v} = 0, \quad \tau_{4,v} = N + m, \quad \tau_{2,v} = k_v + m_v, \) and \( \tau_{3,v} = k_{v+1} - 1, \)
then
\[
d_{1,i} = r_i, \quad 1 \leq i \leq n, \quad d_2 = q, \quad \text{and} \quad d_3 = 0.
\]  
(3.30)

In this case Theorem 3.2 yields the following corollary.

**Corollary 3.1.** Assume \( k_{r-1} + m_{r-1} \leq N \). Let (C1)–(C3) hold, and assume

(C6)’ for each \( 1 \leq i \leq n \), each \( 1 \leq v \leq r-1 \), and each \( k \in I_v \), the function \( G(k, \ell) \) is nonzero for some \( \ell \in Z[k_v + m_v, \min[k_{v+1} - 1, N]] \);

(C7)’ for each \( 1 \leq v \leq r-1 \) and each \( k \in Z[k_v, k_{v+1}] \), the function \( G(k, \ell) b(\ell) \) is nonzero for some \( \ell \in Z[0, N] \).

Suppose there exist numbers \( w_1, 2 \leq i \leq 5 \) with
\[
0 < w_2 < w_3 < \frac{w_3}{\min_{1 \leq i \leq n} \min_{1 \leq v \leq r-1} L_v \rho_i} \leq w_4 \leq w_5
\]
such that the following hold:

(P) \( f(u_1, u_2, \ldots, u_n) < w_2/q \) for \( |u_j| \leq w_2, 1 \leq j \leq n; \)

(Q) \( f(u_1, u_2, \ldots, u_n) \leq w_5/q \) for \( |u_j| \leq w_5, 1 \leq j \leq n; \)

(R) for each \( 1 \leq i \leq n \), \( f(u_1, u_2, \ldots, u_n) > w_3/r_i \) for \( w_3 \leq |u_j| \leq w_4, 1 \leq j \leq n. \)
Then, the system (H) has at least three fixed-sign solutions \( u^1, u^2, u^3 \in \tilde{C}(w_5) \) such that

\[
\|u^1\| < w_2; \\
|u^2_i(k)| > w_3, \quad k \in I_v, \ 1 \leq v \leq r - 1, \ 1 \leq i \leq n; \\
\|u^3\| > w_2, \quad \text{and} \quad \min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} |u^3_i(k)| < w_3.
\]

(3.31)

**Remark 3.2.** Corollary 3.1 is actually Theorem 3.1. Hence, Theorem 3.2 is more general than Theorem 3.1.

The next result illustrates another application of Theorem 2.2.

**Theorem 3.3.** Let (C1)–(C3) hold. Assume there exist integers \( \tau_{1,v}, \tau_{2,v}, \tau_{3,v}, \tau_{4,v}, 1 \leq v \leq r - 1 \), with

\[
k_v + m_v \leq \tau_{1,v} \leq \tau_{2,v} < \tau_{3,v} < \tau_{4,v} \leq k_{v+1} - 1, \quad \text{and} \quad \tau_{2,v} \leq N,
\]

such that (C6) holds and

\[
(3.32)
\]

Suppose that there exist numbers \( w_i, 1 \leq i \leq 5 \) with

\[
0 < w_1 \leq w_2 \leq \min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} L_{v,i} \rho_i < w_3 < w_4 \leq w_5
\]

and \( w_2 > w_5d_5/q \) such that the following hold:

\[
\begin{align*}
(P) & \quad f(u_1, u_2, \ldots, u_n) < \frac{1}{2} (w_2 - w_5d_5/q), \quad \text{for} \ w_1 \leq |u_j| \leq w_2, \ 1 \leq j \leq n; \\
(Q) & \quad f(u_1, u_2, \ldots, u_n) \leq w_5/q \text{ for } |u_j| \leq w_5, \ 1 \leq j \leq n; \\
(R) & \quad \text{for each } 1 \leq i \leq n, \ f(u_1, u_2, \ldots, u_n) > w_3/d_{1,i} \text{ for } w_3 \leq |u_j| \leq w_4, \ 1 \leq j \leq n.
\end{align*}
\]

Then, the system (H) has at least three fixed-sign solutions \( u^1, u^2, u^3 \in \tilde{C}(w_5) \) such that

\[
|u^1_i(k)| < w_2, \quad k \in Z[\tau_{1,v}, \tau_{4,v}], \ 1 \leq v \leq r - 1, \ 1 \leq i \leq n; \\
|u^2_i(k)| > w_3, \quad k \in Z[\tau_{2,v}, \tau_{3,v}], \ 1 \leq v \leq r - 1, \ 1 \leq i \leq n; \\
\max_{1 \leq i \leq n} \max_{1 \leq v \leq r - 1} k \in Z[\tau_{1,v}, \tau_{4,v}] |u^3_i(k)| > w_2, \quad \text{and} \quad \min_{1 \leq i \leq n} \min_{1 \leq v \leq r - 1} k \in Z[\tau_{2,v}, \tau_{3,v}] |u^3_i(k)| < w_3.
\]

(3.32)

**Proof.** In the context of Theorem 2.2, we define functionals on \( C \) by,
\( \gamma(u) = \|u\|, \)

\( \psi(u) = \min_{1 \leq i \leq n} \min_{1 \leq v \leq r-1} \min_{k \in Z[\ell_1, \ell_2]} |u_i(k)|, \)

\( \beta(u) = \max_{1 \leq i \leq n} \max_{1 \leq v \leq r-1} \max_{k \in Z[\ell_1, \ell_2]} |u_i(k)|, \)

\( \alpha(u) = \min_{1 \leq i \leq n} \min_{1 \leq v \leq r-1} \min_{k \in Z[\ell_1, \ell_2]} |u_i(k)|, \)

\( \Theta(u) = \max_{1 \leq i \leq n} \max_{1 \leq v \leq r-1} \max_{k \in Z[\ell_1, \ell_2]} |u_i(k)|. \)

The proof of \( S \) maps \( \tilde{P}(\gamma, w_5) \) into \( \tilde{P}(\gamma, w_5) \) and also condition (a) of Theorem 2.2 proceeds much along the lines of the proof for Theorem 3.2.

Now, we shall check that condition (b) of Theorem 2.2 is satisfied. As in the proof of Theorem 3.2, we note that \( \{u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \mid \beta(u) < w_2 \} \neq \emptyset \). Let \( u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \). Then, we have (3.26). Using (3.8), (3.26), (C8), (P), (Q), and (3.13), we find

\[
\beta(Su) = \max_{1 \leq i \leq n} \max_{1 \leq v \leq r-1} \max_{k \in Z[\ell_1, \ell_2]} |Su_i(k)|
\]

\[
= \max_{1 \leq i \leq n} \max_{1 \leq v \leq r-1} \max_{k \in Z[\ell_1, \ell_2]} (-1)^{\delta_i} \theta_i Su_i(k)
\]

\[
< \max_{1 \leq v \leq r-1} \max_{k \in Z[\ell_1, \ell_2]} \sum_{\ell=0}^{N} (-1)^{\delta_i} G(k, \ell) b(\ell)f(u(\ell))
\]

\[
= \max_{1 \leq v \leq r-1} \max_{k \in Z[\ell_1, \ell_2]} \left[ \sum_{\ell=\ell_1}^{\ell_1-1} (-1)^{\delta_i} G(k, \ell) b(\ell)f(u(\ell)) + \sum_{\ell=\ell_1}^{N} (-1)^{\delta_i} G(k, \ell) b(\ell)f(u(\ell)) \right]
\]

\[
< \max_{1 \leq v \leq r-1} \max_{k \in Z[\ell_1, \ell_2]} \sum_{\ell=\ell_1}^{\ell_1-1} (-1)^{\delta_i} G(k, \ell) b(\ell)
\]

\[
+ \max_{1 \leq v \leq r-1} \max_{k \in Z[\ell_1, \ell_2]} \sum_{\ell=\ell_1}^{N} (-1)^{\delta_i} G(k, \ell) b(\ell)
\]

\[
= d_4 \frac{1}{d_4} \left( w_2 - \frac{w_5 d_5}{q} \right) + d_5 \frac{w_5}{q} = w_2.
\]
Therefore, \( \beta(Su) < w_2 \) for all \( u \in Q(\gamma, \beta, \psi, w_1, w_2, w_5) \).

Next, we shall show that condition (c) of Theorem 2.2 is met. We observe that, by (3.8) and Lemma 3.1(c), for \( u \in C \),

\[
\Theta(Su) = \max_{1 \leq i \leq n} \max_{1 \leq \nu \leq r-1} \max_{k \in Z_{[\tau_2, \cdots, \tau_5, \nu]}} |S_{ui}(k)| \\
= \max_{1 \leq i \leq n} \max_{1 \leq \nu \leq r-1} \max_{k \in Z_{[\tau_2, \cdots, \tau_5, \nu]}} (-1)^{\delta_i} \theta_i S_{ui}(k) \\
\leq \max_{1 \leq i \leq n} \max_{1 \leq \nu \leq r-1} \max_{k \in Z_{[\tau_2, \cdots, \tau_5, \nu]}} \sum_{\ell=0}^{N} (-1)^{\delta_i} G(k, \ell) b(\ell) f(u(\ell)) \\
\leq \max_{1 \leq i \leq n} \max_{1 \leq \nu \leq r-1} \max_{k \in Z_{[\tau_2, \cdots, \tau_5, \nu]}} \sum_{\ell=0}^{N} \|G(\cdot, \ell)\| b(\ell) f(u(\ell)) \\
= N \sum_{\ell=0}^{N} \|G(\cdot, \ell)\| b(\ell) f(u(\ell)).
\]

(3.34)

Moreover, using (3.8), (C3), and Lemma 3.1(b), we obtain (3.28) for \( u \in C \). A combination of (3.28) and (3.34) yields (3.29). Following a similar argument as in the proof of Theorem 3.2, we get \( \alpha(Su) > w_3 \) for all \( u \in P(\gamma, \alpha, w_3, w_5) \) with \( \Theta(Su) > w_4 \).

Finally, we shall prove that condition (d) of Theorem 2.2 is fulfilled. As in (3.34), by (3.8) and Lemma 3.1(c), we see that for \( u \in C \),

\[
\beta(Su) = \max_{1 \leq i \leq n} \max_{1 \leq \nu \leq r-1} \max_{k \in Z_{[\tau_2, \cdots, \tau_4, \nu]}} |S_{ui}(k)| \\
= \max_{1 \leq i \leq n} \max_{1 \leq \nu \leq r-1} \max_{k \in Z_{[\tau_2, \cdots, \tau_4, \nu]}} (-1)^{\delta_i} \theta_i S_{ui}(k) \\
\leq \max_{1 \leq i \leq n} \max_{1 \leq \nu \leq r-1} \max_{k \in Z_{[\tau_2, \cdots, \tau_4, \nu]}} \sum_{\ell=0}^{N} (-1)^{\delta_i} G(k, \ell) a_i(\ell) f(u(\ell)) \\
\geq \min_{1 \leq i \leq n} \min_{1 \leq \nu \leq r-1} \min_{k \in Z_{[\tau_2, \cdots, \tau_4, \nu]}} \sum_{\ell=0}^{N} \|G(\cdot, \ell)\| a_i(\ell) f(u(\ell)) \\
\geq \min_{1 \leq i \leq n} \min_{1 \leq \nu \leq r-1} \min_{k \in \ell_v} \sum_{\ell=0}^{N} (-1)^{\delta_i} G(k, \ell) a_i(\ell) f(u(\ell)) \\
\geq \min_{1 \leq i \leq n} \min_{1 \leq \nu \leq r-1} \sum_{\ell=0}^{N} L(\|G(\cdot, \ell)\| \rho_i b(\ell) f(u(\ell))).
\]

(3.36)
A combination of (3.35) and (3.36) gives
\[
\psi(Su) \geq \min_{1 \leq i \leq n} \min_{1 \leq v \leq s-1} L_v \rho_i \beta(Su), \quad u \in C.
\] (3.37)

Let \( u \in Q(\gamma, \beta, w_2, w_5) \) with \( \psi(Su) < w_1 \). Then, (3.37) leads to
\[
\beta(Su) \leq \frac{1}{\min_{1 \leq i \leq n} \min_{1 \leq v \leq s-1} L_v \rho_i} \psi(Su)
\leq \frac{1}{\min_{1 \leq i \leq n} \min_{1 \leq v \leq s-1} L_v \rho_i} w_1
\leq \min_{1 \leq i \leq n} \min_{1 \leq v \leq s-1} L_v \rho_i w_2 \cdot \min_{1 \leq i \leq n} \min_{1 \leq v \leq s-1} L_v \rho_i = w_2.
\]
Thus, \( \beta(Su) < w_2 \) for all \( u \in Q(\gamma, \beta, w_2, w_5) \) with \( \psi(Su) < w_1 \).

It now follows from Theorem 2.2 that the system (H) has (at least) three fixed-sign solutions \( u^1, u^2, u^3 \in \bar{P}(\gamma, w_5) = \bar{C}(w_5) \) satisfying (2.2). Furthermore, (2.2) reduces to (3.32) immediately. \( \square \)

4. Examples

In this section we shall provide examples to illustrate the results obtained in Section 3.

**Example 4.1.** Consider the boundary value problem
\[
\begin{aligned}
\Delta^3 u_i(k) = P_i(k, u_1(k), u_2(k)), & \quad k \in Z[0, 5], \\
u_i(0) = u_i(2) = u_i(8) = 0, & \quad i = 1, 2,
\end{aligned}
\] (4.1)

where
\[
P_i(k, u_1, u_2) = P_2(k, u_1, u_2) = f(u_1, u_2)
= \begin{cases} 
\frac{\eta u_2}{2}, & (u_1, u_2) \in [0, w_1] \times [0, w_1] \equiv E_1, \\
\left\{ \frac{\eta}{q} \frac{u_2}{\min(r_1, r_2)} \right\}, & (u_1, u_2) \in [w_2, \infty) \times [w_2, \infty) \equiv E_2, \\
h(u_1, u_2), & (u_1, u_2) \in \mathbb{R}^2 \setminus (E_1 \cup E_2),
\end{cases}
\]
and \( h(u_1, u_2) \) is continuous in each argument and satisfies
\[
\begin{aligned}
h(0, u_2) = h(w_1, u_2) = h(u_1, 0) = h(u_1, w_1) = \frac{w_1}{2q}, & \quad u_1, u_2 \in [0, w_1]; \\
h(w_2, u_2) = h(u_1, w_2) = \frac{1}{2} \left( \frac{\eta}{q} + \frac{w_2}{\min(r_1, r_2)} \right), & \quad u_1, u_2 \in [w_2, \infty); \\
0 \leq h(u_1, u_2) \leq \frac{1}{2} \left( \frac{\eta}{q} + \frac{w_2}{\min(r_1, r_2)} \right), & \quad (u_1, u_2) \in \mathbb{R}^2 \setminus (E_1 \cup E_2);
\end{aligned}
\] (4.2)

and \( w_1 \)’s and \( \eta \) are as in the context of Theorem 3.1 and fulfill
\[
0 < w_1 < w_2 < \frac{w_2}{\min_{i=1,2} \min_{i=1,2} L_i \rho_i} \leq w_3 \leq \eta.
\] (4.3)
Clearly, in this example we have \( n = 2, m = 3, N = 5, r = 3, k_1 = 0, k_2 = 2, k_3 = 8, m_1 = m_2 = m_3 = 1, I_1 = [1], \) and \( I_2 = [3, 7]. \) The Green’s function associated with (4.1) is given by [1, Section 9.9]

\[
G(k, \ell) = \frac{1}{2} \left\{ \frac{k(k-2)(7-\ell)^2}{48} - (k - \ell - 1)^2, \quad 0 \leq \ell \leq k - 3,
\right.

\[
\ \quad \left. k - 2 \leq \ell \leq 5. \right.
\]

Take \( \theta_1 = \theta_2 = 1 \) and the functions \( q_1 = q_2 = b = 1 \) (this implies \( \rho_1 = \rho_2 = 1 \)). By direct computation we have \( q = 8, \ r_1 = r_2 = 5/16, \ L_1 = 1/48, \) and \( L_2 = 3/48. \) Thus, (4.4) reduces to

\[
0 < w_1 < w_2 < 48w_2 \leq w_3 \leq \eta. \tag{4.5}
\]

We shall check the conditions of Theorem 3.1. First, it is clear that (C1)–(C5) are fulfilled. Next, condition (P) is obviously satisfied. Noting that \( w_2/\min\{r_1, r_2\} < \eta/q \) (i.e., \( \eta > 25.6w_2 \)), we find for \( (u_1, u_2) \in [0, \eta] \times [0, \eta], \)

\[
f(u_1, u_2) \leq \frac{1}{2} \left( \frac{\eta}{q} + \frac{w_2}{\min\{r_1, r_2\}} \right) < \frac{1}{2} \left( \frac{\eta}{q} + \frac{\eta}{q} \right) = \frac{\eta}{q}.
\]

Thus, condition (Q2) is met. Finally, (R) is satisfied since for \( (u_1, u_2) \in [w_2, w_3] \times [w_2, w_3] \), we have

\[
f(u_1, u_2) = \frac{1}{2} \left( \frac{\eta}{q} + \frac{w_2}{\min\{r_1, r_2\}} \right) > \frac{1}{2} \left( \frac{w_2}{\min\{r_1, r_2\}} + \frac{w_2}{\min\{r_1, r_2\}} \right) = \frac{w_2}{\min\{r_1, r_2\}}.
\]

By Theorem 3.1, the boundary value problem (4.1)–(4.3), (4.5) has at least three positive solutions \( u^1, u^2, u^3 \in C \) such that

\[
\|u^1\| < w_1; \quad |u^2_i(k)| > w_2, \quad k \in \{1\} \cup Z[3, 7], \quad i = 1, 2; \quad \|u^3\| > w_1, \quad \text{and} \quad \min_{i=1,2} \min_{k \in \{1\} \cup Z[3, 7]} |u^i(k)| < w_2. \tag{4.6}
\]

**Example 4.2.** Consider the boundary value problem

\[
\Delta^3 u_i(k) = P_i(k, u_1(k), u_2(k)), \quad k \in Z[0, 8],
\]

\[
u_i(0) = u_i(5) = u_i(11) = 0, \quad i = 1, 2, \tag{4.7}
\]

where

\[
P_i(k, u_1, u_2) = P_2(k, u_1, u_2) = f(u_1, u_2)
\]

\[
= \begin{cases} \frac{1}{2} w_2, & (u_1, u_2) \in [0, w_2] \times [0, w_2] \equiv E_3, \\ \frac{1}{2} \left( \frac{w_3}{\min\{\ell_1, \ell_2\}} + \frac{w_3}{\min\{r_1, r_2\}} \right), & (u_1, u_2) \in [w_3, \infty) \times [w_3, \infty] \equiv E_4, \\ f(u_1, u_2), & (u_1, u_2) \in \mathbb{R}^2 \setminus (E_3 \cup E_4), \end{cases}
\]

and \( f(u_1, u_2) \) is continuous in each argument and satisfies
By direct computation we have fulfilled. Next, condition (P) is obviously satisfied. Noting that $min(d_{1,1}, d_{1,2}) = \frac{w_3}{2d_2}$ and $w_i = 1$, we have

$$f(u_1, u_2) = l(u_1, u_2) = l(u_1, 0) = l(0, u_2) = l(w_2, u_2) = \frac{1}{2d_2}w_2, \quad u_1, u_2 \in [0, w_2];$$

$$l(w_3, u_2) = l(u_1, w_3) = \frac{1}{2} \left( \frac{w_3}{\min[d_{1,1}, d_{1,2}]} + \frac{w_3}{\min[r_1, r_2]} \right), \quad u_1, u_2 \in [w_3, \infty);$$

$$0 \leq l(u_1, u_2) = \frac{1}{2} \left( \frac{w_3}{\min[d_{1,1}, d_{1,2}]} + \frac{w_3}{\min[r_1, r_2]} \right), \quad (u_1, u_2) \in \mathbb{R}^2 \setminus [E_3 \cup E_4];$$

and $w_i$'s are as in the context of Theorem 3.2 and satisfy

$$0 < w_2 < w_3 < \frac{w_3}{\min_{i=1,2} \min_{i=1,2} L_i, \rho_i} \leq w_4 \leq w_5, \quad \frac{w_5}{q} > \frac{w_3}{\min[r_1, r_2]}. \quad (4.10)$$

Here we have $n = 2, m = 3, N = 8, r = 3, k_1 = 0, k_2 = 5, k_3 = 11, m_1 = m_2 = m_3 = 1, I_1 = [1, 4], \text{ and } I_2 = [6, 10].$ The Green's function associated with (4.7) is given by [1, Section 9.9]

$$G(k, \ell) = -\frac{1}{2} \left\{ \begin{array}{ll}
\frac{(11-k)}{30} (4-\ell)^2 + \frac{k(k-5)}{66}(10-\ell)^2 & , 0 \leq \ell \leq \min[k, 5] - 3,
\frac{k(1-k)}{66}(10-\ell)^2 & , \max[k, 5] - 2 \leq \ell \leq 8,
\frac{(11-k)}{30}(4-\ell)^2 + \frac{k(k-5)}{66}(10-\ell)^2 & , 3 \leq \ell \leq k - 3,
\frac{k(1-k)}{66}(10-\ell)^2 & , k - 2 \leq \ell \leq 2.
\end{array} \right.$$  

Take $\theta_1 = \theta_2 = 1, a_1 = a_2 = b = 1$ (this implies $\rho_1 = \rho_2 = 1$), and

$$r_{1,1} = r_{1,2} = 0, \quad r_{4,1} = r_{4,2} = 11, \quad r_{2,1} = 2, \quad r_{3,1} = 3, \quad r_{2,2} = 7, \quad r_{3,2} = 8.$$

By direct computation we have $q = d_2 = 125/6, r_1 = r_2 = 10/11, d_{1,1} = d_{1,2} = 201/55, d_3 = 0, L_1 = 2/33, \text{ and } L_2 = 1/11.$ Hence, (4.10) reduces to

$$0 < w_2 < w_3 < \frac{33}{2} \leq w_4 \leq w_5 \quad \text{and} \quad w_5 > \frac{275}{12} w_3. \quad (4.11)$$

We shall check the conditions of Theorem 3.2. Clearly, (C1)–(C3), (C6), and (C7) are fulfilled. Next, condition (P) is obviously satisfied. Noting that $\min[r_1, r_2] < \min[d_{1,1}, d_{1,2}] < d_2$ and $w_3/\min[r_1, r_2] < w_5/q$, we find for $(u_1, u_2) \in [0, w_3] \times [0, w_5],

$$f(u_1, u_2) \leq \frac{1}{2} \left( \frac{w_3}{\min[r_1, r_2]} + \frac{w_3}{\min[d_{1,1}, d_{1,2}]} \right) \leq \frac{1}{2} \left( \frac{w_3}{\min[r_1, r_2]} + \frac{w_3}{\min[d_{1,1}, d_{1,2}]} \right) = \frac{w_3}{\min[r_1, r_2]} < \frac{w_5}{q}. $$

Hence, condition (Q) is met. Finally, (R) is satisfied since for $(u_1, u_2) \in [w_3, w_4] \times [w_3, w_4]$, we have

$$f(u_1, u_2) = \frac{1}{2} \left( \frac{w_3}{\min[r_1, r_2]} + \frac{w_3}{\min[d_{1,1}, d_{1,2}]} \right) > \frac{1}{2} \left( \frac{w_3}{\min[d_{1,1}, d_{1,2}]} + \frac{w_3}{\min[d_{1,1}, d_{1,2}]} \right) = \frac{w_3}{\min[d_{1,1}, d_{1,2}]}.$$
It follows from Theorem 3.2 that the boundary value problem (4.7)–(4.9), (4.11) has at least three positive solutions \( u^1, u^2, u^3 \in \bar{C}(w_5) \) such that
\[
\begin{align*}
|u^1_i(k)| &< w_2, \quad k \in \mathbb{Z}[0, 11], \ i = 1, 2; \\
|u^2_i(k)| &> w_3, \quad k \in \{2, 3, 7, 8\}, \ i = 1, 2; \\
\max_{i=1,2} \max_{k \in \mathbb{Z}[0,11]} |u^3_i(k)| &> w_2, \quad \text{and} \quad \min_{i=1,2} \min_{k \in \{2,3,7,8\}} |u^3_i(k)| < w_3. \quad (4.12)
\end{align*}
\]

Remark 4.1. In Example 4.2, we see that for \((u_1, u_2) \in [w_3, w_4] \times [w_3, w_4]\),
\[
f(u_1, u_2) = \frac{1}{2} \left( \frac{w_3}{\min\{r_1, r_2\}} + \frac{w_3}{\min\{d_{1,1}, d_{1,2}\}} \right) < \frac{1}{2} \left( \frac{w_3}{\min\{r_1, r_2\}} + \frac{w_3}{\min\{r_1, r_2\}} \right)
= \frac{w_3}{\min\{r_1, r_2\}}.
\]
Thus, condition (R) of Corollary 3.1 is not satisfied. Recalling that Corollary 3.1 is actually Theorem 3.1, Example 4.2 illustrates the case when Theorem 3.2 is applicable but not Theorem 3.1. Hence, this example shows that Theorem 3.2 is indeed more general than Theorem 3.1.

References