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Small loop spaces and covering theory of non-homotopically Hausdorff spaces

Ali Pakdaman, Hamid Torabi, Behrooz Mashayekhy *

Department of Pure Mathematics, Center of Excellence in Analysis on Algebraic Structures, Ferdowsi University of Mashhad, P.O. Box 1159-91775, Mashhad, Iran

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ABSTRACT

This paper is devoted to spaces that are not homotopically Hausdorff and study their covering spaces. We introduce the notion of small covering and prove that every small covering of X is the universal covering in the categorical sense. Also, we introduce the notion of semi-locally small loop space which is the necessary and sufficient condition for existence of universal covering for non-homotopically Hausdorff spaces, equivalently existence of small covering spaces. Moreover, we prove that for semi-locally small loop spaces, X is a small loop space if and only if every covering of X is trivial if and only if $\pi_1^{top}(X)$ is an indiscrete topological group.

1. Introduction and motivation

We recall that a continuous map $p:\widetilde{X} \longrightarrow X$ is a *covering* of X, and $\widetilde{X}$ is called a *covering space* of X, if for every $x \in X$ there exists an open subset U of X with $x \in U$ such that U is *evenly covered* by p, that is, $p^{-1}(U)$ is a disjoint union of open subsets of $\widetilde{X}$ each of which is mapped homeomorphically onto U by p.

In the classical covering theory, one assumes that X is, in addition, connected, locally path connected, semi-locally simply connected and wishes to classify all path connected covering spaces of X and to find among them the *universal covering* in the categorical sense, that is, a covering $p:\widetilde{X} \longrightarrow X$ with the property that for every covering $q:\widetilde{Y} \longrightarrow X$ with a path connected space $\widetilde{Y}$ there exists a covering $f:\widetilde{X} \longrightarrow \widetilde{Y}$ such that $q \circ f = p$. We have the following well-known result which can be found for example in [7].

Every simply connected covering space of X is a universal covering space. Moreover, X admits a simply connected covering space if and only if X is semi-locally simply connected, in which case the coverings $p:(\widetilde{X},\tilde{x}) \longrightarrow (X,x)$ with path connected $\widetilde{X}$ are in direct correspondence with the conjugacy classes of subgroups of $\pi_1(X,x)$ via the monomorphism $p_*:\pi_1(\widetilde{X},\tilde{x}) \longrightarrow \pi_1(X,x)$.

Outside of locally nice spaces, the traditional theory of covering is not as pleasant. H. Fischer and A. Zastrow in [5] studied universal coverings of homotopically Hausdorff spaces. They defined a generalized regular covering of a topological space X as a continuous map $p:\widetilde{X} \longrightarrow X$ satisfies the following conditions for some normal subgroup H of $\pi_1(X)$:

* Corresponding author.
E-mail addresses: Alipaky@yahoo.com (A. Pakdaman), hamid_torabi86@yahoo.com (H. Torabi), bmashf@um.ac.ir (B. Mashayekhy).

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U_1. $\widetilde{X}$ is a connected and locally path connected space.
U_2. The map $p:\widetilde{X} \longrightarrow X$ is surjective and $p_* = \pi_1(p):\pi_1(\widetilde{X}) \longrightarrow \pi_1(X)$ is a monomorphism onto H.
U_3. For every connected, locally path connected space Y, for every continuous map $f:(Y,y) \longrightarrow (X,x)$ with $f_*(\pi_1(Y,y)) \subseteq H$, and for every $\tilde{x} \in \widetilde{X}$ with $p(\tilde{x}) = x$, there exists a unique continuous lift $g:(Y,y) \longrightarrow (\widetilde{X},\tilde{x})$ with $p \circ g = f$.

Note that the last property implies that every generalized regular covering $p:\widetilde{X} \longrightarrow X$ has the homotopy lifting property with respect to $Y = [0,1]^n$ for all $n > 0$ but does not have the homotopy lifting property with respect to every space.

This generalized notion of covering $p:\widetilde{X} \longrightarrow X$ enjoys most of the usual properties, with the possible exception of evenly covered neighborhoodness. If X is connected, locally path connected and semi-locally simply connected, then $p:\widetilde{X} \longrightarrow X$ agrees with the classical universal covering. A necessary condition for the standard construction to yield a generalized universal covering is that X to be homotopically Hausdorff. A space X is *homotopically Hausdorff* if given any point x in X and any nontrivial homotopy class $[\alpha] \in \pi_1(X,x)$, then there is a neighborhood U of x which contains no representative for $[\alpha]$. Accordingly, we would like to study coverings of non-homotopically Hausdorff spaces and investigate the topology type of their universal covering spaces.

Definition 1.1. ([8]) A loop $\alpha:(I,\partial I) \longrightarrow (X,x)$ is *small* if and only if there exists a representative of the homotopy class $[\alpha] \in \pi_1(X,x)$ in every open neighborhood U of x. A *small loop* is a nontrivial small loop if it is not homotopically trivial. A non-simply connected space X is called *small loop space* if for every $x \in X$, every loop $\alpha:(I,\partial I) \longrightarrow (X,x)$ is small.

Definition 1.2. ([8]) The *small loop group* $\pi_1^s(X,x)$ of (X,x) is the subgroup of the fundamental group $\pi_1(X,x)$ consisting of all homotopy classes of small loops. The SG subgroup of $\pi_1(X,x)$, denoted by $\pi_1^{sg}(X,x)$, is the subgroup generated by the following set

$$\big\{\big[\alpha * \beta * \alpha^{-1}\big] \bigm| [\beta] \in \pi_1^s\big(X,\alpha(1)\big),\ \alpha \in P(X,x)\big\},$$

where $P(X,x)$ is the space of all paths in X with initial point x.

As it is remarked in [8], the set $\pi_1^s(X,x)$ forms a subgroup of $\pi_1(X,x)$ which is not necessarily a normal subgroup. Also, we can consider π_1^s as a functor from *hTop*$_*$ to *Groups*. Note that the presence of small loops is equivalent to the absence of semi-locally simply connectedness and homotopically Hausdorffness. Although non-semi-locally simply connected spaces may have non-homotopic nontrivial loops at every neighborhood of a point but in the case of a small loop, the homotopy type of one loop can be chosen for all neighborhoods of the base point.

In Section 2, we find some relations between small loop groups, SG subgroups and fundamental groups. Z. Virk in [8] constructed a small loop space by using Harmonic Archipelago and studied the relation between covering spaces and small loops. For example, for the universal covering $p:\widetilde{X} \longrightarrow X$ of a non-homotopically Hausdorff space X, $p_*\pi_1(\widetilde{X},\tilde{x})$ contains $\pi_1^{sg}(X,x)$ as a subgroup and so X has no simply connected covering space. We show that this fact holds for every covering of X.

In Section 3, we show that if $\widetilde{X}$ is a small loop space, then $\widetilde{X}$ is a universal covering space of X. Also, in this case X is not homotopically Hausdorff and $\pi_1^s(X,x) = \pi_1^{sg}(X,x) = p_*\pi_1(\widetilde{X},\tilde{x})$ which is a normal subgroup of $\pi_1(X,x)$.

Finally, in Section 4, we present the main result of this article which states that a space X has a small loop space as covering space if and only if X is a semi-locally small loop space, that is, for every $x \in X$ there exists an open neighborhood U such that $i_*\pi_1(U,y) = \pi_1^s(X,y)$, for all $y \in U$, where $i:U \longrightarrow X$ is the inclusion map. Moreover, we show that if X is a semi-locally small loop space, then X is a small loop space if and only if every covering of X is trivial if and only if the topological fundamental group of X is indiscrete.

Throughout this article, all the homotopies between two paths are relative to end points, X is a connected and locally path connected space with the base point $x \in X$, and $p:\widetilde{X} \longrightarrow X$ is a covering of X with $\tilde{x} \in p^{-1}(\{x\})$ as the base point of $\widetilde{X}$.

2. Small loop groups

The importance of small loops in the covering space theory was pointed out by Brodskiy, Dydak, Labuz, and Mitra [2,3] and by Virk [8]. In this section, we study some basic properties of small loops, small loop groups and SG subgroups of the fundamental group of a non-homotopically Hausdorff space X and their relations to the covering spaces of X. It is obvious that if the topological space X is not homotopically Hausdorff, then there exists $x \in X$ such that $\pi_1^s(X,x) \neq 1$.

Theorem 2.1. *For every covering $p:\widetilde{X} \longrightarrow X$ and $x \in X$ the following relations hold*:

$$\pi_1^s(X,x) \leqslant \pi_1^{sg}(X,x) \leqslant p_*\pi_1(\widetilde{X},\tilde{x}).$$

Proof. By definitions $\pi_1^s(X,x) \leqslant \pi_1^{sg}(X,x)$. Let $[\alpha] \in \pi_1^s(X,x)$ and let U be an evenly covered open neighborhood of x such that $p^{-1}(U) = \bigcup_{j \in J} V_j$, where V_j's are disjoint and $p|_{V_j}:V_j \longrightarrow U$ is a homeomorphism, for every $j \in J$. Since α is a small loop, there exists a loop $\alpha_U : I \longrightarrow U$ such that $[\alpha] = [\alpha_U]$. Assume that $\tilde{x} \in V_j$, then $\widetilde{\alpha_j} := (p|_{V_j})^{-1} \circ \alpha_U$ is a loop in $\widetilde{X}$ based at $\tilde{x}$ such that $[\alpha] = p_*([\widetilde{\alpha_j}])$ and hence $\pi_1^s(X,x) \subseteq p_*\pi_1(\widetilde{X},\tilde{x})$. Let $[\alpha * \beta * \alpha^{-1}] \in \pi_1^{sg}(X,x)$, where $[\beta] \in \pi_1^s(X,\alpha(1))$. If $\widetilde{\alpha}$ is the lift of α with initial point $\tilde{x}$ and end point $\tilde{y}$, then $y := p(\tilde{y}) = \alpha(1)$. Since $\pi_1^s(X,y) \leqslant p_*\pi_1(\widetilde{X},\tilde{y})$, there exists $[\widetilde{\beta}] \in \pi_1(\widetilde{X},\tilde{y})$ such that $p_*([\widetilde{\beta}]) = [\beta]$. Hence $p_*([\widetilde{\alpha} * \widetilde{\beta} * \widetilde{\alpha}^{-1}]) = [\alpha * \beta * \alpha^{-1}]$ which implies that $\pi_1^{sg}(X,x) \leqslant p_*\pi_1(\widetilde{X},\tilde{x})$. □

Corollary 2.2. *If there exists $x \in X$ such that $\pi_1^s(X,x) \neq 1$, then X does not admit a simply connected covering space.*

Corollary 2.3. *Let X be a connected, locally path connected and simply connected space. If the action of a group G on X is properly discontinuous, then X/G has no small loop and therefore it is homotopically Hausdorff.*

Corollary 2.4. *For every covering $p : \widetilde{X} \longrightarrow X$, $\pi_1^{sg}(X,x)$ acts trivially on $p^{-1}(\{x\})$, that is, $\hat{x}.[\alpha] = \hat{x}$, for all $\hat{x} \in p^{-1}(\{x\})$ and $[\alpha] \in \pi_1^{sg}(X,x)$.*

Example 2.5. Fischer and Zastrow [5, Examples 4.13, 4.14, 4.16] introduced some spaces that admit generalized universal covering. Since homotopically Hausdorffness is necessary for the existence of generalized universal coverings, spaces which admit generalized universal covering have no small loop. Some examples of spaces with this property are subsets of closed surfaces, 1-dimensional compact Hausdorff spaces, 1-dimensional separable and metrizable spaces, and trees of manifolds.

Remark 2.6. If X is a small loop space, then $\pi_1^s(X,x) = \pi_1(X,x)$. Since $\pi_1^s(X,x) \leqslant \pi_1^{sg}(X,x) \leqslant \pi_1(X,x)$, we have $\pi_1^s(X,x) = \pi_1^{sg}(X,x) = \pi_1(X,x)$. Note that if $\pi_1^s(X,x) = \pi_1^{sg}(X,x)$, then the equality $\pi_1^{sg}(X,x) = \pi_1(X,x)$ does not hold in general. As an example, consider a non-simply connected, semi-locally simply connected space.

3. Small loop spaces and coverings

In keeping with modern nomenclature, the term universal covering space will always mean a categorical universal object, that is, a covering $p : \widetilde{X} \longrightarrow X$ with the property that for every covering $q : \widetilde{Y} \longrightarrow X$ with a path connected space $\widetilde{Y}$ there exists a covering $f : \widetilde{X} \longrightarrow \widetilde{Y}$ such that $q \circ f = p$. The following theorem shows that small loop spaces have no nontrivial covering.

Theorem 3.1. *Every covering space of a small loop space X is homeomorphic to X.*

Proof. Let $p : \widetilde{X} \longrightarrow X$ be a covering of a small loop space X. Then by Theorem 2.1 $\pi_1^s(X,x) \leqslant p_*\pi_1(\widetilde{X},\tilde{x}) \leqslant \pi_1(X,x) = \pi_1^s(X,x)$, for each $x \in X$ which implies that $p : \widetilde{X} \longrightarrow X$ is a one sheeted covering of X. Hence $\widetilde{X}$ is homeomorphic to X. □

Definition 3.2. By a *small covering* of a topological space X we mean a covering $p : \widetilde{X} \longrightarrow X$ such that $\widetilde{X}$ is a small loop space.

Lemma 3.3. *Let X have a small covering $p : \widetilde{X} \longrightarrow X$ and α be a loop in X. Then*:

(i) *If α is a small loop, then every lift $\widetilde{\alpha}$ of α in $\widetilde{X}$ is closed (equivalently, every lift $\widetilde{\alpha}$ of α is a loop in $\widetilde{X}$).*
(ii) *If there exists a closed lift $\widetilde{\alpha}$ of α in $\widetilde{X}$, then α is a small loop in X.*

Consequently, if a loop α in X has a closed lift $\widetilde{\alpha}$ in $\widetilde{X}$, then every lift of α in $\widetilde{X}$ is also closed.

Proof. (i) Let α be a small loop based at x in X and $\widetilde{\alpha}$ be a lift of α by initial point $\tilde{x} \in p^{-1}(\{x\})$. Let U be an evenly covered open subset of X containing x and α_U be a loop based at x in U that is homotopic to α. If V is the homeomorphic copy of U in $p^{-1}(U)$ which contains $\tilde{x}$, then $\widetilde{\alpha_U} = (p|_V)^{-1} \circ \alpha_U$ is a loop based at $\tilde{x}$ in $\widetilde{X}$ which is a small loop. Since $\alpha \simeq \alpha_U$, $\widetilde{\alpha} \simeq \widetilde{\alpha_U}$ and $\widetilde{\alpha}(1) = \widetilde{\alpha_U}(1) = \tilde{x}$ which implies that $\widetilde{\alpha}$ is a loop.

(ii) Assume that a lift $\widetilde{\alpha}$ of α is closed and U, V are chosen as above. Let $\widetilde{\alpha}_V$ be a loop based at $\tilde{x}$ in V which is homotopic to $\widetilde{\alpha}$, then $\alpha_U = p \circ \widetilde{\alpha}_V$ is a loop based at x which is homotopic to α. Since every open subset of an evenly covered open subset is evenly covered, the result holds. □

Corollary 3.4. *If X has a small covering $p : \widetilde{X} \longrightarrow X$, then X is not homotopically Hausdorff.*

Proposition 3.5. *If X has a small covering $p : \widetilde{X} \longrightarrow X$, then $\pi_1^s(X,x) = \pi_1^{sg}(X,x) = p_*\pi_1(\widetilde{X},\tilde{x})$.*

Proof. Assume that $[\alpha] = [p \circ \beta]$ for $[\beta] \in \pi_1(\widetilde{X},\tilde{x})$, U is an evenly covered open subset of X containing x, and V is the homeomorphic copy of U in $\widetilde{X}$ containing $\tilde{x}$. Since $\widetilde{X}$ is a small loop space, there exists a loop β_V in V such that

$\beta \simeq \beta_V$ and hence $\alpha \simeq \alpha_U := p \circ \beta_V$. Also, U and therefore V can be chosen to be arbitrarily small which implies that $p_*\pi_1(\widetilde{X}, \tilde{x}) \leqslant \pi_1^s(X, x)$. Hence by Theorem 2.1 the result holds. □

Theorem 3.6. *If X has a small covering $p : \widetilde{X} \longrightarrow X$, then $p_*\pi_1(\widetilde{X}, \tilde{x})$ is a normal subgroup of $\pi_1(X, x)$.*

Proof. Let $[\alpha] \in p_*\pi_1(\widetilde{X}, \tilde{x})$ and $[\beta] \in \pi_1(X, x)$. Proposition 3.5 implies that α is a small loop and hence every lift of α is closed, by Lemma 3.3. Let $\widetilde{\beta}$ be the lift of β with initial point $\tilde{x}$, $\widetilde{\beta^{-1}}$ be the lift of β^{-1} with initial point $\widetilde{\beta}(1)$, and $\widetilde{\alpha}$ be the lift of α with initial point $\widetilde{\beta}(1)$. Then $\widetilde{\beta^{-1}} = \widetilde{\beta}^{-1}$ and $\widetilde{\beta} * \widetilde{\alpha} * \widetilde{\beta^{-1}}$ is a loop and hence $p \circ (\widetilde{\beta} * \widetilde{\alpha} * \widetilde{\beta^{-1}}) = \beta * \alpha * \beta^{-1}$ which implies that $[\beta][\alpha][\beta]^{-1} = [\beta * \alpha * \beta^{-1}] = p_*([\widetilde{\beta} * \widetilde{\alpha} * \widetilde{\beta^{-1}}])$. Therefore $p_*\pi_1(\widetilde{X}, \tilde{x})$ is a normal subgroup of $\pi_1(X, x)$. □

Corollary 3.7. *If X has a small covering, then $\pi_1^s(X, x)$ is a normal subgroup of $\pi_1(X, x)$, for every $x \in X$.*

We denote by $\mathcal{COV}(X)$ the category of all coverings of X as objects and covering maps between them as morphisms.

Theorem 3.8. *A small covering of a space X is the universal object in the category $\mathcal{COV}(X)$.*

Proof. Assume that $p : \widetilde{X} \longrightarrow X$ is a small covering of X and $q : \widetilde{Y} \longrightarrow X$ is another covering. By Proposition 3.5 and Theorem 2.1 $p_*\pi_1(\widetilde{X}) = \pi_1^{sg}(X) \leqslant q_*\pi_1(\widetilde{Y})$ which implies that there exists $f : \widetilde{X} \longrightarrow \widetilde{Y}$ such that $q \circ f = p$. □

4. Existence

We now consider the following natural question: *Given a non-homotopically Hausdorff space X, does X have a small covering space*? First, we derive a rather simple necessary condition. Let $p : \widetilde{X} \longrightarrow X$ be a small covering of X, x be an arbitrary point of X, $\tilde{x} \in p^{-1}(\{x\})$, U be an evenly covered open neighborhood of x, and V be the homeomorphic copy of U in $p^{-1}(U)$ which contains the point $\tilde{x}$. We then have the following commutative diagram involving fundamental groups:

$$\begin{array}{ccc} \pi_1(V, \tilde{x}) & \xrightarrow{\ j_*\ } & \pi_1(\widetilde{X}, \tilde{x}) \\ \Big\downarrow {\scriptstyle (p|_V)_*} & & \Big\downarrow {\scriptstyle p_*} \\ \pi_1(U, x) & \xrightarrow{\ i_*\ } & \pi_1(X, x). \end{array}$$

Obviously $\pi_1^s(X, x) \subseteq i_*\pi_1(U, x)$. Let α be a loop based at x in U such that $i_*([\alpha]) \neq 1$ and let $\widetilde{\alpha} = (p|_V)^{-1} \circ \alpha$. Since $(p|_V)_*$ is an isomorphism and p_* is injective, commutativity of diagram implies that $j_*([\widetilde{\alpha}]) \neq 1$. Therefore $\widetilde{\alpha}$ is a nontrivial small loop in $\widetilde{X}$ and since $p_*\pi_1(\widetilde{X}, \tilde{x}) = \pi_1^s(X, x)$, $[\alpha] \in \pi_1^s(X, x)$. Thus we conclude that the space X has the following property: Every point $x \in X$ has a neighborhood U such that $i_*\pi_1(U, x) = \pi_1^s(X, x)$. We call a space with this property a *semi-locally small loop space*. Since for every $y \in X$, we have $i_*\pi_1(U, y) = \pi_1^s(X, y)$, this definition can also be rephrased as follows:

Definition 4.1. A space X is a semi-locally small loop space if and only if for each $x \in X$ there exists an open neighborhood U of x such that $i_*\pi_1(U, y) = \pi_1^s(X, y)$, for all $y \in U$, where $i : U \longrightarrow X$ is the inclusion map.

For example, every small loop space is semi-locally small loop space. Also, the products $X \times Y$, where X is a small loop space and Y is either locally simply connected or locally path connected and semi-locally simply connected are semi-locally small loop spaces.

Lemma 4.2. *If X is a semi-locally small loop space, then $\pi_1^{sg}(X, x) = \pi_1^s(X, x)$, for every $x \in X$. Also $\pi_1^s(X, x) \cong \pi_1^s(X, y)$, for every $x, y \in X$.*

Proof. By definitions $\pi_1^s(X, x) \leqslant \pi_1^{sg}(X, x)$, for every $x \in X$. Conversely, assume that $[\alpha * \beta * \alpha^{-1}] \in \pi_1^{sg}(X, x)$, where $[\beta] \in \pi_1^s(X, \alpha(1))$. For every $t \in I$, let W_t be an open neighborhood of $\alpha(t)$ such that $i_*\pi_1(W_t, y) = \pi_1^s(X, y)$, for all $y \in W_t$. The sets $\alpha^{-1}(W_t)$, $t \in I$, form an open cover of I. Let $\lambda > 0$ be the Lebesgue number for this cover. Choose $N \in \mathbb{N}$ such that $1/N < \lambda$. For each $1 \leqslant n \leqslant N$ let

$$I_n = \left[\frac{n-1}{N}, \frac{n}{N}\right] \subseteq I.$$

For every $1 \leqslant n \leqslant N$, let U_n denote an open neighborhood W_t such that $\alpha(I_n) \subseteq U_n$. The U_n's are not necessarily distinct, nor does the proof require this condition. For each $0 \leqslant n \leqslant N$, denote $y_n = \alpha(\frac{n}{N})$ and for each $1 \leqslant n \leqslant N$, denote $\alpha_n = \alpha|_{I_n} \circ \varphi_n$, where $\varphi_n : I \longrightarrow I_n$ is the linear homeomorphism. Since $\beta \in \pi_1^s(X, y_N) = i_*\pi_1(U_N, y_N)$, there exists a loop $\beta_N : I \longrightarrow U_N$ such that $\beta_N \simeq \beta$. Then $\alpha_N * \beta_N * \alpha_N{}^{-1}$ is a loop in U_N based at $y_{N-1} \in U_{N-1} \cap U_N$. Hence there exists a loop $\beta_{N-1} : I \longrightarrow$

$U_{N-1} \cap U_N$ based at y_{N-1} such that $\beta_{N-1} \simeq \alpha_N * \beta_N * {\alpha_N}^{-1}$ since $i_*\pi_1(U_N, y_{N-1}) = \pi_1^s(X, y_{N-1})$. Similarly, for every $1 \leqslant n < N$, there exists $\beta_n : I \longrightarrow U_n \cap U_{n+1}$ such that $\beta_n \simeq \alpha_{n+1} * \beta_{n+1} * {\alpha_{n+1}}^{-1}$. Therefore we have

$$\begin{aligned}\alpha * \beta * \alpha^{-1} &\simeq \alpha_1 * \alpha_2 * \cdots * \alpha_N * \beta * {\alpha_N}^{-1} * \cdots * {\alpha_1}^{-1} \\ &\simeq \alpha_1 * \alpha_2 * \cdots * \alpha_N * \beta_N * {\alpha_N}^{-1} * \cdots * {\alpha_1}^{-1} \\ &\simeq \alpha_1 * \cdots * \alpha_{N-1} * \beta_{N-1} * {\alpha_{N-1}}^{-1} * \cdots * {\alpha_1}^{-1} \simeq \cdots \simeq \alpha_1 * \beta_1 * {\alpha_1}^{-1}.\end{aligned}$$

Since $i_*\pi_1(U_1, x) = \pi_1^s(X, x)$, there exists a small loop α' such that $\alpha_1 * \beta_1 * \alpha_1^{-1} \simeq \alpha'$ which implies that $[\alpha * \beta * \alpha^{-1}] \in \pi_1^s(X, x)$ and hence the first assertion holds. Since X is path connected, $\pi_1^{sg}(X, x) \cong \pi_1^{sg}(X, y)$, for every $x, y \in X$, and hence the second assertion holds. □

Lemma 4.3. *Let X be a semi-locally small loop space. If for at least one point $x \in X$ there exists a covering $p : \widetilde{X} \longrightarrow X$ such that $p_*\pi_1(\widetilde{X}, \tilde{x}) = \pi_1^s(X, x)$, then $\widetilde{X}$ is a small loop space.*

Proof. First we show that for every $y \in X$ and $\tilde{y} \in p^{-1}(\{y\})$, $p_*\pi_1(\widetilde{X}, \tilde{y}) = \pi_1^s(X, y)$. Let $\widetilde{\alpha}$ be a path from $\tilde{x}$ to $\tilde{y}$ and $\alpha = p \circ \widetilde{\alpha}$, then $\theta : p_*\pi_1(\widetilde{X}, \tilde{y}) \longrightarrow p_*\pi_1(\widetilde{X}, \tilde{x})$ by $\theta([\beta]) = [\alpha * \beta * \alpha^{-1}]$ is an isomorphism. Also, by Lemma 4.2 $\eta : \pi_1^s(X, x) \longrightarrow \pi_1^s(X, y)$ defined by $\eta([\gamma]) = [\alpha^{-1} * \gamma * \alpha]$ is an isomorphism. Since $p_*\pi_1(\widetilde{X}, \tilde{x}) = \pi_1^s(X, x)$, $\eta \circ \theta$ is well defined and $\eta \circ \theta([\beta]) = [\beta]$, which implies that $p_*\pi_1(\widetilde{X}, \tilde{y}) = \pi_1^s(X, y)$, for every $y \in X$.

Now let $\widetilde{\gamma} : I \longrightarrow \widetilde{X}$ be a loop based at $\tilde{y}$ and $\widetilde{U}$ be an open neighborhood of $\tilde{y}$. We show that $[\widetilde{\gamma}]$ has a representative in $\widetilde{U}$. For this, let U be an evenly covered open neighborhood of $y = p(\tilde{y})$ such that $U \subseteq p(\widetilde{U})$ and $i_*\pi_1(U, z) = \pi_1^s(X, z)$, for every $z \in U$. Since $\gamma = p \circ \widetilde{\gamma} \in p_*\pi_1(\widetilde{X}, \tilde{y}) = \pi_1^s(X, y)$, there exists a loop $\gamma_U : I \longrightarrow U$ based at y such that $\gamma \simeq \gamma_U$. If $V \subseteq \widetilde{U}$ is the homeomorphic copy of U in $\widetilde{X}$ which contains $\tilde{y}$, then $\widetilde{\gamma_U} := (p|_V)^{-1} \circ \gamma_U$ is a loop based at $\tilde{y}$ in V and $\widetilde{\gamma} \simeq \widetilde{\gamma_U}$, as desired. □

Now we are in a position to state and prove the main result of the paper.

Theorem 4.4. *A connected, locally path connected space X has a small covering space if and only if X is a semi-locally small loop space.*

Proof. If X has a small covering space, then by the argument at the beginning of this section, the result holds. Conversely, assume that X is a semi-locally small loop space. Choose a base point $x \in X$ and let $P(X, x)$ be the family of all paths $\alpha : I \longrightarrow X$ with $\alpha(0) = x$. We define an equivalence relation $\sim$ on $P(X, x)$ as follows. For any $\alpha_1, \alpha_2 \in P(X, x)$, $\alpha_1 \sim \alpha_2$ if and only if α_1 and α_2 have the same end point and $\alpha_1 * \alpha_2^{-1}$ is a small loop. We denote the equivalence class of α by $\langle\alpha\rangle$. Define $\widetilde{X}$ to be the set of all equivalence classes of paths $\alpha \in P(X, x)$. Define a function $p : \widetilde{X} \longrightarrow X$ by setting $p(\langle\alpha\rangle)$ equal to the terminal point of the path class α. We shall now show how to topologize $\widetilde{X}$ so that it is a small loop space and $p : \widetilde{X} \longrightarrow X$ is a covering of X.

Observe that our hypotheses imply that the topology on X has a basis consisting of open sets U with the following properties: U is path connected and every loop in U is small. For brevity, let us agree to call such an open set U basic. Note that, if x and y are any two points in a basic open set U, then for any two paths f and g in U with initial point x and terminal point y, $f * g^{-1}$ is a small loop.

Given any $\langle\alpha\rangle \in \widetilde{X}$ and any basic open set U which contains the end point $p(\langle\alpha\rangle)$, denote by $(\langle\alpha\rangle, U)$ the set of all equivalence classes $\langle\beta\rangle$ such that for some path class $[\alpha']$ with $Im(\alpha') \subseteq U$, $\beta = \alpha * \alpha'$. Then $(\langle\alpha\rangle, U)$ is a subset of $\widetilde{X}$. We topologize $\widetilde{X}$ by choosing the family of all such sets $(\langle\alpha\rangle, U)$ as a basis of open sets. In order that the family of all sets of the form $(\langle\alpha\rangle, U)$ can be a basis for some topology on $\widetilde{X}$, it is enough to show that if $\gamma \in (\langle\alpha\rangle, U) \cap (\langle\beta\rangle, V)$, then there exists a basic open set W such that $(\langle\gamma\rangle, W) \subseteq (\langle\alpha\rangle, U) \cap (\langle\beta\rangle, V)$. For this, we choose W to be any basic open set such that $p(\gamma) \in W \subseteq U \cap V$.

Before proceeding with the proof that $p : \widetilde{X} \longrightarrow X$ is a small covering of X, it is convenient to make the following two observations:

(a) Let $\langle\alpha\rangle \in \widetilde{X}$, and let U be a basic open neighborhood of $p(\langle\alpha\rangle)$. Then $p|_{(\langle\alpha\rangle, U)}$ is a one-to-one map from $(\langle\alpha\rangle, U)$ onto U.
(b) Let U be any basic open set, and let y be any point of U. Then

$$p^{-1}(U) = \bigcup_{\lambda} \big(\langle\alpha_\lambda\rangle, U\big),$$

where $\langle\alpha_\lambda\rangle$ denotes the totality of all path classes in X with initial point x and terminal point y. Moreover, the sets $(\langle\alpha_\lambda\rangle, U)$ are pairwise disjoint.

For proving (a) suppose that $\tilde{y}, \tilde{z} \in (\langle\alpha\rangle, U)$ and $p(\tilde{y}) = p(\tilde{z})$. Now $\tilde{z} = \langle\alpha * \mu\rangle$, where $\mu(0) = \alpha(1) = x_1$ and $\mu(I) \subseteq U$. Similarly, $\tilde{y} = \langle\alpha * \eta\rangle$, where $\eta(0) = x_1$ and $\eta(I) \subseteq U$. Since $p(\tilde{y}) = p(\tilde{z})$, we have $\eta(1) = \mu(1)$, so that $\eta * \mu^{-1}$ is a closed path in U at x_1. By the choice of U, $\eta * \mu^{-1}$ is a small loop in X. Hence $[\alpha * \eta * \mu^{-1} * \alpha^{-1}] \in \pi_1^{sg}(X, x)$ which implies that $\alpha * \eta * \mu^{-1} * \alpha^{-1}$ is a small loop based at x, since $\pi_1^{sg}(X, x) = \pi_1^{s}(X, x)$ by Lemma 4.2. Therefore $\langle \alpha * \eta \rangle = \langle \alpha * \mu \rangle$, that is, $\tilde{y} = \tilde{z}$. Surjectivity of $p|_{(\langle \alpha \rangle, U)}$ follows from path connectedness of U.

Clearly, $p^{-1}(U)$ contains $\bigcup_\lambda (\langle \alpha_\lambda \rangle, U)$. For the reverse inclusion, let $\tilde{y} \in \widetilde{X}$ such that $p(\tilde{y}) \in U$, that is, $\tilde{y} = \langle \alpha \rangle$ and $\alpha(1) \in U$. Since U is path connected, there is a path λ in U from $\alpha(1)$ to y. Then $\langle \alpha_\lambda \rangle = \langle \alpha * \lambda \rangle$ lies in the fiber over y. Since $\langle \alpha \rangle = \langle \alpha * \lambda * \lambda^{-1} \rangle$ and $\langle \alpha * \lambda * \lambda^{-1} \rangle \in (\langle \alpha * \lambda \rangle, U)$, $\langle \alpha \rangle \in (\langle \alpha_\lambda \rangle, U)$. Therefore (b) holds.

Note that it follows from (b) that p is continuous. Hence $p|_{(\langle \alpha \rangle, U)}$ is a one-to-one continuous map from $(\langle \alpha \rangle, U)$ onto U, by (a). We assert that $p|_{(\langle \alpha \rangle, U)}$ is an open map from $(\langle \alpha \rangle, U)$ onto U. For, any open subset of $(\langle \alpha \rangle, U)$ is a union of sets of the form $(\langle \beta \rangle, V)$, where $V \subseteq U$, and hence the fact that $p|_{(\langle \alpha \rangle, U)}$ is open also follows from (a). Thus, p maps $(\langle \alpha \rangle, U)$ homeomorphically onto U. Since U is path connected, so is $(\langle \alpha \rangle, U)$. Because the sets $(\langle \alpha_\lambda \rangle, U)$ occurring in statement (b) are pairwise disjoint, it follows that any basic open set $U \subseteq X$ has all the properties required of an evenly covered neighborhood.

Next, we shall prove that the space $\widetilde{X}$ is path connected. Let $\tilde{x} \in \widetilde{X}$ denote the equivalence class of the constant path at x. Given any point $\langle \alpha \rangle \in \widetilde{X}$, it suffices to exhibit a path joining the points $\tilde{x}$ and $\langle \alpha \rangle$. For any real number $s \in I$, define $\alpha_s : I \longrightarrow X$ by $\alpha_s(t) = \alpha(st)$, $t \in I$. Then $\alpha_1 = \alpha$ and α_0 is the constant path at x. We assert that the map $s \longmapsto \langle \alpha_s \rangle$ is a continuous map $I \longrightarrow \widetilde{X}$, i.e. a path in $\widetilde{X}$. To prove this assertion, we must check that, for any $s_0 \in I$ and any basic neighborhood U of $\alpha(s_0)$, there exists a real number $\delta > 0$ such that if $|s - so| < \delta$, then $\langle \alpha_s \rangle \in (\langle \alpha_{s_0} \rangle, U)$. For this purpose, we choose δ so that if $|s - so| < \delta$, then $\alpha(s) \in U$; such a number δ exists because α is continuous. Thus $s \longmapsto \langle \alpha_s \rangle$ is a path in $\widetilde{X}$ with initial point $\tilde{x}$ and terminal point $\langle \alpha \rangle$, as required.

Finally, we must prove that $\widetilde{X}$ is a small loop space. For this, we first show that $p_*\pi_1(\widetilde{X}, \tilde{x}) = \pi_1^s(X, x)$. We know that $p_*\pi_1(\widetilde{X}, \tilde{x})$ is the isotropy subgroup corresponding to the point $\tilde{x}$ for the action of $\pi_1(X, x)$ on $p^{-1}(\{x\})$. By Theorem 2.1 it suffices to prove that $p_*\pi_1(\widetilde{X}, \tilde{x}) \leqslant \pi_1^s(X, x)$. Let $[\alpha] \in p_*\pi_1(\widetilde{X}, \tilde{x})$ which implies that $\tilde{x}.[\alpha] = \tilde{x}$. The path $\widetilde{\alpha} : I \longrightarrow \widetilde{X}$ defined by $\widetilde{\alpha}(s) = \langle \alpha_s \rangle$ is a lifting of α. This path in $\widetilde{X}$ has $\tilde{x}$ as initial point and $\langle \alpha \rangle \in \widetilde{X}$ as terminal point. Hence, by the definition of the action of $\pi_1(X, x)$ on $p^{-1}(\{x\})$, $\tilde{x}.[\alpha] = \tilde{x}$ if and only if $\tilde{x} = \langle \alpha \rangle$ or equivalently α is a small loop. Therefore $p_*\pi_1(\widetilde{X}, \tilde{x}) = \pi_1^s(X, x)$ as required. Now by Lemma 4.3 $\widetilde{X}$ is a small loop space. □

Theorem 4.5. *Suppose X is a connected, locally path connected and semi-locally small loop space. Then for every subgroup $H \leqslant \pi_1(X, x)$ containing $\pi_1^{sg}(X, x)$, there exists a covering $p : \widetilde{X}_H \longrightarrow X$ such that $p_*\pi_1(\widetilde{X}_H, \tilde{x}) = H$, for a suitably chosen base point $\tilde{x} \in \widetilde{X}_H$.*

Proof. For points $\langle \alpha \rangle$, $\langle \alpha' \rangle$ in the small covering space $\widetilde{X}$ of X constructed above, define $\langle \alpha \rangle \sim \langle \alpha' \rangle$ to mean $\alpha(1) = \alpha'(1)$ and $[\alpha * \alpha^{-1}] \in H$. It is easy to see that this is an equivalence relation. Let $\widetilde{X}_H$ be the quotient space of $\widetilde{X}$ obtained by identifying $\langle \alpha \rangle$ with $\langle \alpha' \rangle$ if $\langle \alpha \rangle \sim \langle \alpha' \rangle$. Note that if $\alpha(1) = \alpha'(1)$, then $\langle \alpha \rangle = \langle \alpha' \rangle$ if and only if $\langle \alpha * \beta \rangle = \langle \alpha' * \beta \rangle$, for arbitrary loop β based at $\alpha(1)$. This means that if any two points in basic neighborhoods $(\langle \alpha \rangle, U)$ and $(\langle \alpha' \rangle, U)$ are identified in $\widetilde{X}_H$, then the whole neighborhoods are identified. Hence the natural projection $\widetilde{X}_H \longrightarrow X$ induced by $\langle \alpha \rangle \longmapsto \alpha(1)$ is a covering. If we choose for the base point $\tilde{x} \in \widetilde{X}_H$ the equivalence class of the constant path e_x, then the image of $p_* : \pi_1(\widetilde{X}_H, \tilde{x}) \longrightarrow \pi_1(X, x)$ is exactly H. This is because for a loop α in X based at x, its lift to $\widetilde{X}$ starting at e_x and ends to $\langle \alpha \rangle$, so the image of this lifted path in $\widetilde{X}_H$ is a loop if and only if $\langle \alpha \rangle = \langle e_x \rangle$, or equivalently $[\alpha] \in H$. □

The authors [6] proved that the topological fundamental group of a small loop space is an indiscrete topological group. Also, in Section 3, it is proved that every covering of a small loop space is trivial. By [1, Theorem 5.5] the connected coverings of X are classified by conjugacy classes of open subgroups of $\pi_1^{top}(X, x)$, so if $\pi_1^{top}(X, x)$ is an indiscrete topological group, then every covering of X is trivial. Hence there is an interesting closed relation between the small loop properties of a given space X, the topological fundamental group $\pi_1^{top}(X, x)$, and the number of covering space of X as in the following theorem.

Theorem 4.6. *If X is a connected, locally path connected and semi-locally small loop space, then the following statements are equivalent.*

(i) *X is a small loop space.*
(ii) *$\pi_1^{top}(X, x)$ is an indiscrete topological group.*
(iii) *Every covering of X is trivial.*

Proof. It suffices to prove (iii) $\longrightarrow$ (i). Since X is a semi-locally small loop space, it has a small covering by Theorem 4.4 and therefore X is a small loop space, as desired. □

If $p : \widetilde{X} \longrightarrow X$ is a covering of X such that $p_*\pi_1(\widetilde{X}, \tilde{x}) = \pi_1^{sg}(X, x)$, then $\widetilde{X}$ is the universal covering space of X. For, if $q : \widetilde{Y} \longrightarrow X$ is another covering, then by Theorem 2.1 $\pi_1^{sg}(X, x) \leqslant q_*\pi_1(\widetilde{Y}, \tilde{y})$ which implies that $p_*\pi_1(\widetilde{X}, \tilde{x}) \leqslant q_*\pi_1(\widetilde{Y}, \tilde{y})$ and therefore there exists a covering $r : \widetilde{X} \longrightarrow \widetilde{Y}$ such that $q \circ r = p$. As we have shown in Theorem 4.4, the space X admits such universal covering if X is a semi-locally small loop space. But with this assumption we have $\pi_1^{sg}(X, x) = \pi_1^s(X, x)$, for every $x \in X$. Since $\pi_1^{sg}(X,x)$ does not depend on the base point, we are interested to have a necessary and sufficient condition for the existence of universal covering space outside of the category of semi-locally small loop spaces.

Theorem 4.7. *A space X has a covering $p : \widetilde{X} \longrightarrow X$ with $p_*\pi_1(\widetilde{X},\tilde{x}) = \pi_1^{sg}(X,x)$ if and only if $\pi_1^{sg}(X,x)$ is an open subgroup of $\pi_1^{top}(X,x)$. Moreover, in this case $\pi_1(\widetilde{X},\tilde{x}) = \pi_1^{sg}(\widetilde{X},\tilde{x})$.*

Proof. By [1, Theorem 5.5], the connected coverings of X are classified by conjugacy classes of open subgroups of $\pi_1^{top}(X,x)$ and hence the first assertion holds. For the second assertion, let $[\alpha] \in \pi_1(\widetilde{X},\tilde{x})$. Since $p_*\pi_1(\widetilde{X},\tilde{x}) = \pi_1^{sg}(X,x)$, $[p \circ \alpha] \in \pi_1^{sg}(X,x)$ and hence there are elements $g_1, g_2, \ldots, g_n$ of $\pi_1^{sg}(X,x)$ such that $[p \circ \alpha] = g_1 g_2 \cdots g_n$, where $g_i = [\alpha_i * \beta_i * \alpha_i^{-1}]$, $\alpha_i(0) = x$ and $[\beta_i] \in \pi_1^{s}(X, \alpha_i(1))$, for $i = 1, 2, \ldots, n$. Let U_i be an evenly covered neighborhood of $\alpha_i(1)$. Since β_i's are small loops, we can assume that $Im\,\beta_i \subseteq U_i$, for every $i = 1, 2, \ldots, n$. Let $\widetilde{\alpha_i}$ be the lift of α_i by initial point $\tilde{x}$ and $\widetilde{\beta_i} = (p|_{V_i})^{-1} \circ \beta_i$ be the loop with base point $\widetilde{\alpha_i}(1)$, where V_i is the homeomorphic copy of U_i in $\widetilde{X}$ which contains $\widetilde{\alpha_i}(1)$. Since β_i is a small loop, so is $\widetilde{\beta_i}$ which implies that

$$\left(\widetilde{\alpha_1} * \widetilde{\beta_1} * \widetilde{\alpha_1}^{-1}\right) * \cdots * \left(\widetilde{\alpha_n} * \widetilde{\beta_n} * \widetilde{\alpha_n}^{-1}\right) \in \pi_1^{sg}(\widetilde{X},\tilde{x}).$$

If $\widetilde{\alpha_i^{-1}}$ is the lift of α_i^{-1} with initial point $\widetilde{\beta_i}(1)$, then $\widetilde{\alpha_i^{-1}} = \widetilde{\alpha_i}^{-1}$ and hence $p \circ \alpha \simeq (\alpha_1 * \beta_1 * \alpha_1^{-1}) * \cdots * (\alpha_n * \beta_n * \alpha_n^{-1})$. Since p_* is injective we have

$$[\alpha] = \left[\left(\widetilde{\alpha_1} * \widetilde{\beta_1} * \widetilde{\alpha_1}^{-1}\right) * \cdots * \left(\widetilde{\alpha_n} * \widetilde{\beta_n} * \widetilde{\alpha_n}^{-1}\right)\right] \in \pi_1^{sg}(\widetilde{X},\tilde{x}). \qquad \square$$

Remark 4.8. Note that if $\pi_1^{sg}(X,x) = 1$, then by [4, Lemmas 1, 2] $\pi_1^{sg}(X,x)$ is an open subgroup of $\pi_1(X,x)$ if and only if X is semi-locally simply connected which is equivalent to the existence of a simply connected covering space of X.

Corollary 4.9. *If X is a semi-locally small loop space, then $\pi_1^{sg}(X,x)$ is an open subgroup of $\pi_1^{top}(X,x)$.*

Note that the converse of the above corollary does not hold. In fact, there exists a space X which is not semi-locally small loop and $\pi_1^{sg}(X,x)$ is an open subgroup of $\pi_1^{top}(X,x)$. As an example, if $X = \mathcal{HA}$ is the Harmonic Archipelago, then $\pi_1^{sg}(X,x) \neq \pi_1^{s}(X,x)$, where x is not the common point of the boundary circles and hence $\mathcal{HA}$ is not a semi-locally small loop space.

Question. Is there any topological condition on a space X which is equivalent to the openness of $\pi_1^{sg}(X,x)$ in $\pi_1^{top}(X,x)$?

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