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On Perron complements of totally nonnegative matrices

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Abstract

An $n \times n$ matrix is called totally nonnegative if every minor of A is nonnegative. The problem of interest is to describe the Perron complement of a principal submatrix of an irreducible totally nonnegative matrix. We show that the Perron complement of a totally nonnegative matrix is totally nonnegative only if the complementary index set is based on consecutive indices. We also demonstrate a quotient formula for Perron complements analogous to the so-called quotient formula for Schur complements, and verify an ordering between the Perron complement and Schur complement of totally nonnegative matrices, when the Perron complement is totally nonnegative.

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1. Introduction

An $n \times n$ matrix A is called *totally positive*, TP (*totally nonnegative*, TN) if *every* minor of A is positive (nonnegative). Such matrices arise in a variety of applica-

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tions [6], have been studied most of the 20th century, and have received increasing attention of late (see also [1,4,5,11]).

Let $A = [a_{ij}]$ be an $n \times n$ matrix, and let α, β be nonempty ordered subsets of $\langle n \rangle := \{1, 2, \ldots, n\}$, both consisting of strictly increasing integers. By $A[\alpha, \beta]$ we shall denote the submatrix of A lying in rows indexed by α and columns indexed by β. Similarly, $A(\alpha, \beta)$ is the submatrix obtained from A by deleting the rows indexed by α and columns indexed by β. If, in addition, $\alpha = \beta$, then the principal submatrix $A[\alpha, \alpha]$ is abbreviated to $A[\alpha]$, and the complementary principal submatrix is $A(\alpha)$. For any n-vector, x and $\alpha \subset \langle n \rangle$, we let $x[\alpha]$ denote the subvector of x whose coordinates are indexed by α.

Let $\beta \subset \langle n \rangle$. If $A[\beta]$ is nonsingular, then the *Schur-complement* of $A[\beta]$ in A is given by

$$\mathscr{S}(A/A[\beta]) \;=\; A[\alpha] - A[\alpha, \beta](A[\beta])^{-1}A[\beta, \alpha], \tag{1}$$

where $\alpha = \langle n \rangle \backslash \beta$. Schur complements have been well-studied for various classes of matrices, including: positive definite, M-matrices, inverse M-matrices (see, for example, [8,9]), and TN matrices (see [1]). In particular, it is known that the classes of positive definite, M- and inverse M-matrices are all closed under arbitrary Schur complementation. The situation is slightly more subtle for TN matrices. Recall that the dispersion of a given set $S = \{i_1, i_2, \ldots, i_k\}$, where $i_j < i_{j+1} (j = 1, \ldots, k-1)$ is defined to be $d(S) = i_k - i_1 - (k-1)$, with the convention that $d(S) = 0$ whenever S is a singleton. Thus $d(S) = 0$, whenever S is based on consecutive indices, i.e., S is a *contiguous* index set. For TN matrices it is known that $\mathscr{S}(A/A[\beta])$ is TN if $\alpha = \langle n \rangle \backslash \beta$ is a contiguous set (see [1,4]). Otherwise, $\mathscr{S}(A/A[\beta])$ need not be TN in general (see [1]). If A is TN and invertible, then a routine calculation using Jacobi's identity (see [8]) reveals that $SA^{-1}S$ is TN, for $S = \mathrm{diag}(1, -1, \ldots, \pm 1)$.

In connection with a divide and conquer algorithm for computing the stationary distribution vector for a Markov chain, Meyer [12] introduced, for an $n \times n$ nonnegative and irreducible matrix A, the *Perron complement* of $A[\beta]$ in A, which is given by

$$\mathscr{P}(A/A[\beta]) \;=\; A[\alpha] + A[\alpha, \beta](\rho(A)I - A[\beta])^{-1}A[\beta, \alpha], \tag{2}$$

where $\beta \subset \langle n \rangle$, $\alpha = \langle n \rangle \backslash \beta$ and $\rho(\cdot)$ denotes the spectral radius of a matrix. Recall that since A is irreducible and nonnegative, $\rho(A) > \rho(A[\beta])$, so that the expression on the right-hand side of (2) is well-defined. Meyer proved many interesting results regarding $\mathscr{P}(A/A[\beta])$ including, $\mathscr{P}(A/A[\beta])$ is nonnegative and $\rho(\mathscr{P}(A/A[\beta])) = \rho(A)$. (Observe that the matrix $(\rho(A)I - A[\beta])^{-1}$ is an inverse M-matrix.) To avoid any difficulties we will assume throughout that A is irreducible. We also remark that Perron complements have been studied in [10,14]. Some of the results proved in [14] (see also Section 2) motivated this study on Perron complements of TN matrices.

Throughout this paper we will work with a slight extension of the notion of a Perron complement. For any $\beta \subset \langle n \rangle$ and for any $t \geqslant \rho(A)$, let the *extended Perron complement at t* be the matrix

$$\mathscr{P}_t(A/A[\beta]) = A[\alpha] + A[\alpha,\beta](tI - A[\beta])^{-1}A[\beta,\alpha], \tag{3}$$

which is also well-defined since $t \geqslant \rho(A) > \rho(A[\beta])$.

We conclude this introductory section with a few (needed) preliminary facts. The first is an identity for the minors of certain Schur complements. Fix k ($2 \leqslant k \leqslant n$). Then for any subsets $\gamma, \delta \subseteq \langle k-1\rangle$ with $|\gamma| = |\delta|$, we have (see [3])

$$\det \mathscr{S}(A/A[\{k,\dots,n\}])[\gamma,\delta] = \frac{\det A[\gamma \cup \{k,\dots,n\}, \delta \cup \{k,\dots,n\}]}{\det A[\{k,\dots,n\}]}. \tag{4}$$

It is the case that there exist similar formulae for the minors of more general Schur complements, although we do not need them here (see [1, (1.35)]). The next fact is Fischer's inequality. If A is an $n \times n$ TN matrix and $\alpha, \beta \subseteq \langle n\rangle$ with $\alpha \cap \beta = \emptyset$, then

$$\det A[\alpha \cup \beta] \leqslant \det A[\alpha] \det A[\beta]$$

(see [1,4,5]).

For a given $m \times n$ matrix A we denote by $C_k(A)$ the $\binom{m}{k} \times \binom{n}{k}$ matrix whose general entry is $\det A[\alpha,\beta]$; it is called the kth *compound matrix* of A. Here $\alpha \subseteq \langle m\rangle$ and $\beta \subseteq \langle n\rangle$ are index sets of cardinality k, $1 \leqslant k \leqslant \min\{m,n\}$, usually ordered lexicographically. Let A and B be two $m \times n$ real matrices. We say $A \overset{(c)}{\geqslant} B$ if and only if

$$\det A[\alpha,\beta] \geqslant \det B[\alpha,\beta]$$

for all $\alpha \subset \langle m\rangle$, $\beta \subset \langle n\rangle$ with $|\alpha| = |\beta|$. In other words, $A \overset{(c)}{\geqslant} B$ if and only if $C_k(A) \geqslant C_k(B)$ (entrywise) for every $k = 1, 2, \dots, \min\{m,n\}$. (We remark here that in [1] the notation $\overset{(t)}{\geqslant}$ was used for the same ordering; however, to avoid confusion with the t in $\mathscr{P}_t(A/A[\beta])$, we chose the notation $\overset{(c)}{\geqslant}$.) Thus $A \overset{(c)}{\geqslant} 0$ means A is TN. Observe that if $A \overset{(c)}{\geqslant} B \overset{(c)}{\geqslant} 0$ and $C \overset{(c)}{\geqslant} D \overset{(c)}{\geqslant} 0$, then $AC \overset{(c)}{\geqslant} BD \overset{(c)}{\geqslant} 0$, which follows easily from the Cauchy–Binet identity for determinants (assuming the products exist). Unfortunately, $A \overset{(c)}{\geqslant} B$ does not enjoy some of the useful properties that the positive definite or entrywise orderings possess. For example, $A \overset{(c)}{\geqslant} B$ does not imply $A - B \overset{(c)}{\geqslant} 0$, and if, in addition $A \overset{(c)}{\geqslant} B \overset{(c)}{\geqslant} 0$, then it is not true in general that $\mathscr{S}(A/A[\beta]) \overset{(c)}{\geqslant} \mathscr{S}(B/B[\beta])$ or $SB^{-1}S \overset{(c)}{\geqslant} SA^{-1}S$, for $S = \mathrm{diag}(1, -1, \dots, \pm 1)$, in the event B, and hence A, is invertible. The following lemma is proved in [1, Theorem 3.7].

Lemma 1.1 [1]. *If A is an $n \times n$ totally nonnegative matrix, and $\beta = \{1, 2, \dots, k\}$ or $\beta = \{k, k+1, \dots, n\}$, then*

$$A[\alpha] \overset{(c)}{\geqslant} \mathscr{S}(A/A[\beta]) \overset{(c)}{\geqslant} 0,$$

where $\alpha = \langle n\rangle \backslash \beta$ and provided that $A[\beta]$ is invertible.

In this paper we first determine when the Perron complement of a TN matrix is, in turn, TN. In particular, we prove that $\mathscr{P}_t(A/A[\beta])$ is TN only if $\alpha = \langle n\rangle\backslash\beta$ is a contiguous set. Along the way, we verify a quotient formula for the Perron complement that is reminiscent of Haynsworth's quotient formula for Schur complements (this observation may be of independent interest). Finally, we also verify that when $\alpha = \langle n\rangle\backslash\beta$ is a contiguous set not only is $\mathscr{P}_t(A/A[\beta])$ a TN matrix, but, in fact,

$$\mathscr{P}_t(A/A[\beta]) \overset{(c)}{\geqslant} A[\alpha] \overset{(c)}{\geqslant} \mathscr{S}(A/A[\beta]) \overset{(c)}{\geqslant} 0.$$

2. Main results

Since the issue regarding Schur complements of TN matrices is rather subtle, it seems natural to ask: When (if ever) is the Perron complement of a TN matrix TN? A hint comes from prior work. In [14, Corollary 3.3] it is shown that if A is the inverse of a tridiagonal M-matrix (which implies A is TN), then any Perron complement of A is again TN.

Consider also the following special (yet important) class of matrices. It is well known that a tridiagonal matrix is TN if and only if it is entrywise nonnegative and all of its principal minors are nonnegative (see [5]). Suppose D is a positive diagonal matrix, then for any irreducible nonnegative matrix A it follows that

$$\mathscr{P}_t(DAD^{-1}/DAD^{-1}[\beta]) = D[\alpha]\mathscr{P}_t(A/A[\beta])D^{-1}[\alpha],$$

where $\alpha = \langle n\rangle\backslash\beta$ and $t \geqslant \rho(A)$ $(= \rho(DAD^{-1}))$. We are now in a position to state a result on the Perron complements of tridiagonal TN matrices.

Proposition 2.1. *Let A be an $n \times n$ irreducible tridiagonal totally nonnegative matrix. Then, for any singleton $\beta \subset \langle n\rangle$ and $t \geqslant \rho(A)$, the matrix*

$$\mathscr{P}_t(A/A[\beta]) = A[\alpha] + A[\alpha, \beta](tI - A[\beta])^{-1}A[\beta, \alpha],$$

where $\alpha = \langle n\rangle\backslash\beta$, is irreducible tridiagonal and totally nonnegative.

Proof. Let $\beta = \{i\}$, $1 \leqslant i \leqslant n$ and $\alpha = \langle n\rangle\backslash\beta$. First observe that $\mathscr{P}_t(A/A[\beta])$ is an irreducible nonnegative tridiagonal matrix for any irreducible nonnegative tridiagonal matrix A. It is well known (and easy to prove) that there exists a positive diagonal matrix D such that DAD^{-1} is symmetric and hence positive semidefinite, since A is TN. Moreover, by the remark preceding this proposition $\mathscr{P}_t(DAD^{-1}/DAD^{-1}[\beta]) = D[\alpha]\mathscr{P}_t(A/A[\beta])D^{-1}[\alpha]$. Hence the total nonnegativity of $\mathscr{P}_t(A/A[\beta])$ will follow from the total nonnegativity of $\mathscr{P}_t(DAD^{-1}/DAD^{-1}[\beta])$. Since DAD^{-1} is positive semidefinite it follows that $\mathscr{P}_t(DAD^{-1}/DAD^{-1}[\beta])$ is positive semidefinite. (In fact, this observation holds for all positive semidefinite matrices.) Thus we see that $\mathscr{P}_t(DAD^{-1}/DAD^{-1}[\beta])$ is a nonnegative tridiagonal positive semidefinite matrix and hence is totally nonnegative. This completes the proof. □

Unfortunately, for general TN matrices not all Perron complements with respect to singletons are necessarily TN. This leads us to our first result which proves that $\mathscr{P}_t(A/A[\beta])$ is TN, whenever A is irreducible TN and $\beta = \{1\}$ or $\beta = \{n\}$.

Lemma 2.2. *Let A be an $n \times n$ irreducible totally nonnegative matrix, and let $\beta = \{1\}$ or $\beta = \{n\}$ and define $\alpha = \langle n \rangle \backslash \beta$. Then for any $t \in [\rho(A), \infty)$, the matrix*

$$\mathscr{P}_t(A/A[\beta]) \;=\; A[\alpha] + A[\alpha, \beta](tI - A[\beta])^{-1}A[\beta, \alpha]$$

is totally nonnegative. In particular, the Perron complement $\mathscr{P}(A/A[\beta])$ is totally nonnegative, for $\beta = \{1\}$ or $\beta = \{n\}$.

Proof. Assume $\beta = \{n\}$. (The arguments for the case $\beta = \{1\}$ are similar.) Let A be partitioned as follows:

$$A \;=\; \left[\begin{array}{c|c} B & c \\ \hline d^{\mathrm{T}} & e \end{array}\right],$$

where B is $(n-1) \times (n-1)$ and e is a scalar. Then

$$\mathscr{P}_t(A/e) = B + \frac{cd^{\mathrm{T}}}{(t-e)}.$$

Consider the matrix

$$X \;=\; \left[\begin{array}{c|c} B & -c \\ \hline d^{\mathrm{T}} & t-e \end{array}\right].$$

Observe that $\mathscr{S}(X/(t-e)) = \mathscr{P}_t(A/e)$. Thus we can compute any minor of $\mathscr{P}_t(A/e)$ by computing minors of a related Schur complement. Using formula (4) we have

$$\det \mathscr{S}(X/(t-e))[\gamma, \delta] \;=\; \frac{\det X[\gamma \cup \{n\}, \delta \cup \{n\}]}{(t-e)},$$

where $\gamma, \delta \subset \langle n-1 \rangle$. Hence the nonnegativity of any minor of $\mathscr{P}_t(A/e)$ will follow from the nonnegativity of any minor of X of the form $\det X[\gamma \cup \{n\}, \delta \cup \{n\}]$ (since $t - e > 0$). Observe that

$$\begin{aligned}
\det X[\gamma \cup \{n\}, \delta \cup \{n\}] &= \det \left[\begin{array}{c|c} B[\gamma, \delta] & -c[\gamma] \\ \hline d^{\mathrm{T}}[\delta] & t-e \end{array}\right] \\
&= t \det B[\gamma, \delta] + \det \left[\begin{array}{c|c} B[\gamma, \delta] & -c[\gamma] \\ \hline d^{\mathrm{T}}[\delta] & -e \end{array}\right] \\
&= t \det B[\gamma, \delta] - \det \left[\begin{array}{c|c} B[\gamma, \delta] & c[\gamma] \\ \hline d^{\mathrm{T}}[\delta] & e \end{array}\right] \\
&\geqslant t \det B[\gamma, \delta] - e \det B[\gamma, \delta] \\
&= (t-e) \det B[\gamma, \delta] \\
&\geqslant 0.
\end{aligned}$$

The first inequality follows since the matrix on the left is TN and TN matrices satisfy Fischer's inequality (see Section 1). This completes the proof. □

Unfortunately, TN matrices are not closed under arbitrary Perron complementation, even when β is a singleton as the next example demonstrates.

Example 2.3. Let $f(x) = \sum_{i=0}^{n} a_i x^i$ be an nth degree polynomial in x. By the *Routh–Hurwitz matrix* we mean the $n \times n$ matrix given by

$$H = \begin{bmatrix} a_1 & a_3 & a_5 & a_7 & \cdots & 0 & 0 \\ a_0 & a_2 & a_4 & a_6 & \cdots & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & \cdots & 0 & 0 \\ 0 & a_0 & a_2 & a_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & 0 & \cdots & a_{n-2} & a_n \end{bmatrix}.$$

A polynomial $f(x)$ is said to be *stable* if all the zeros of $f(x)$ have nonpositive real parts. It is proved in [2], for example, that if $f(x)$ is stable polynomial, then the Routh–Hurwitz matrix formed from f is totally nonnegative. Consider the following polynomial:

$$\begin{aligned} f(x) = {} & x^{10} + 6.2481x^9 + 17.0677x^8 + 26.7097x^7 + 26.3497x^6 \\ & + 16.9778x^5 + 7.1517x^4 + 1.9122x^3 + 0.3025x^2 \\ & + 0.0244x + 0.0007. \end{aligned}$$

It can be shown that f is a stable polynomial. Hence its associated Routh–Hurwitz array is totally nonnegative, call it H. Let $P \equiv \mathscr{P}(H/H[\{7\}])$ (which is 9×9). Then P is not TN, as $\det P[\{8, 9\}, \{5, 6\}] < 0$, for example.

We note here that the above example is indeed tedious and cumbersome; however, one of the reasons for this is, because, typically (using many random examples) the Perron complement of a TN matrix with respect to a single entry is TN. Thus we expected the collection of counterexamples to be slight. *Moreover, using brute force one can show that for* $n \leqslant 4$, $\mathscr{P}_t(A/A[\beta])$ *is TN for any singleton* β. Hence, to begin searching for a counterexample n has to be at least 5.

The following result is a general observation regarding the Perron complement of nonnegative matrices. Recall Haynsworth's quotient formula (see [7]) which can be stated as follows. For $\emptyset \neq \alpha \subset \beta \subset \langle n \rangle$, we have

$$\mathscr{S}(A/A[\beta]) = \mathscr{S}(\mathscr{S}(A/A[\alpha])/A[\beta]/A[\alpha]).$$

For simplicity of notation we assume that the indexing of any complement (Perron or Schur) is inherited from the indexing of the original matrix. For example, if A is 7×7, and $\beta = \{2, 3, 6\}$, then the rows and columns of $\mathscr{S}(A/A[\beta])$ and $\mathscr{P}(A/A[\beta])$ are indexed by the integers $1, 4, 5, 7$ (ordered). Observe that the indexing of the submatrix $A[\beta]/A[\alpha]$ in $\mathscr{S}(A/A[\alpha])$ is given by $\beta\backslash\alpha$. As a result Haynsworth's quotient formula can be viewed as constructing Schur complements from "smaller"

Schur complements. For example, a Schur complement with respect to a submatrix of order 2 can be computed by taking two Schur complements with respect to submatrices of order 1, albeit using different matrices. For simplicity of the next proof, we will abuse notation as follows. Rather then denote Schur and Perron complements with respect to principal submatrices we will denote them only by the index sets instead. For example if A is $n \times n$ and $\beta \subset \langle n \rangle$, then we will denote $\mathscr{P}_t(A/A[\beta])$ by $\mathscr{P}_t(A/\beta)$ and $\mathscr{S}(A/A[\beta])$ by $\mathscr{S}(A/\beta)$. Assuming the convention stated above with regards to the indexing of Perron and Schur complements, Haynsworth's quotient formula can be rewritten as: For any $\emptyset \neq \alpha \subset \beta \subset \langle n \rangle$, we have

$$\mathscr{S}(A/\beta) = \mathscr{S}(\mathscr{S}(A/\alpha)/\beta\backslash\alpha).$$

We could also state the quotient formula as follows: if $\gamma_1, \gamma_2 \subset \beta$ with $\gamma_1 \cup \gamma_2 = \beta$ and $\gamma_1 \cap \gamma_2 = \emptyset$, then

$$\mathscr{S}(A/\beta) = \mathscr{S}(\mathscr{S}(A/\gamma_1)/\gamma_2).$$

This is the version we state and prove for Perron complements.

Theorem 2.4. *Let A be any $n \times n$ irreducible nonnegative matrix, and fix any non-empty set $\beta \subset \langle n \rangle$. Then for any $\emptyset \neq \gamma_1, \gamma_2 \subset \beta$ with $\gamma_1 \cup \gamma_2 = \beta$ and $\gamma_1 \cap \gamma_2 = \emptyset$, we have*

$$\mathscr{P}_t(A/\beta) = \mathscr{P}_t(\mathscr{P}_t(A/\gamma_1)/\gamma_2),$$

for any $t \in [\rho(A), \infty)$.

Proof. We begin by commenting that according to Meyer [13, Theorem 2.3], the Perron complement of a nonnegative and irreducible matrix is nonnegative and irreducible. Therefore the Perron complement of $\mathscr{P}_t(A/\gamma_1)$ is well defined.

Observe that for any index set $\beta \subset \langle n \rangle$, the following identity holds:

$$\mathscr{P}_t(A/\beta) = tI - \mathscr{S}((tI - A)/\beta).$$

Hence we have

$$\begin{aligned}\mathscr{P}_t(\mathscr{P}_t(A/\gamma_1)/\gamma_2) &= \mathscr{P}_t((tI - \mathscr{S}((tI - A)/\gamma_1))/\gamma_2)\\ &= tI - \mathscr{S}([tI - (tI - \mathscr{S}((tI - A)/\gamma_1))]/\gamma_2)\\ &= tI - \mathscr{S}(\mathscr{S}((tI - A)/\gamma_1)/\gamma_2)\\ &= tI - \mathscr{S}((tI - A)/\beta) = \mathscr{P}_t(A/\beta).\end{aligned}$$

The second to last equality follows from the quotient formula for Schur complements. This completes the proof. □

Using this quotient formula for extended Perron complements and Lemma 2.2 we have the following result.

Theorem 2.5. *Let A be an $n \times n$ irreducible totally nonnegative matrix, and let $\emptyset \neq \beta \subset \langle n \rangle$ such that $\alpha = \langle n \rangle \backslash \beta$ is contiguous. Then for any $t \in [\rho(A), \infty)$, the matrix*

$$\mathscr{P}_t(A/A[\beta]) = A[\alpha] + A[\alpha, \beta](tI - A[\beta])^{-1} A[\beta, \alpha]$$

is totally nonnegative. In particular, the Perron complement $\mathscr{P}(A/A[\beta])$ is totally nonnegative, whenever $\langle n \rangle \backslash \beta$ is contiguous.

Proof. Observe that since α is a contiguous set, the Perron complement $\mathscr{P}_t(A/A[\beta])$ can be obtained by Theorem 2.4 from a sequence of Perron complements with respect to the first or last index at each stage, which are TN by Lemma 2.2. □

Corollary 2.6. *Let A be an $n \times n$ irreducible tridiagonal totally nonnegative matrix. Then, for any $\beta \subset \langle n \rangle$, the matrix*

$$\mathscr{P}_t(A/A[\beta]) = A[\alpha] + A[\alpha, \beta](tI - A[\beta])^{-1} A[\beta, \alpha],$$

where $t \geqslant \rho(A)$, is irreducible tridiagonal and totally nonnegative.

Our next lemma involves an ordering between the compounds of extended Perron complements and Schur complements of TN matrices discussed in Section 1. Recall from Lemma 1.1 that if $\alpha = \{1, 2, \ldots, k\}$ or $\alpha = \{k, k+1, \ldots, n\}$, then

$$A[\alpha] \overset{(c)}{\geqslant} \mathscr{S}(A/A[\beta]) \overset{(c)}{\geqslant} 0,$$

where $\beta = \langle n \rangle \backslash \alpha$ and provided that $A[\beta]$ is invertible. In fact, even more is true, namely $A[\alpha] \overset{(c)}{\geqslant} \mathscr{S}(A/A[\beta]) \overset{(c)}{\geqslant} 0$, where $\alpha = \langle n \rangle \backslash \beta$ is a contiguous set. We note here that this was not explicitly noted in [1], even though it follows from two applications of Lemma 1.1. For example, if $\alpha = \{i, i+1, \ldots, i+k\}$, then

$$\begin{aligned} \mathscr{S}(A/\beta) &= \mathscr{S}(\mathscr{S}(A/\{1, \ldots, i-1\})/\{i+k+1, \ldots, n\}) \\ &\overset{(c)}{\leqslant} \mathscr{S}(A/\{1, \ldots, i-1\})[\{i, \ldots, i+k\}] \\ &\overset{(c)}{\leqslant} A[\alpha]. \end{aligned}$$

We note here that the inequality $A[\alpha] \overset{(c)}{\geqslant} \mathscr{S}(A/A[\beta])$ need not hold in general when $\mathscr{S}(A/A[\beta])$ is not TN. In the same spirit we have the following result.

Theorem 2.7. *Let A be an $n \times n$ irreducible totally nonnegative matrix, and let $\phi \neq \beta \subset \langle n \rangle$ such that $\alpha = \langle n \rangle \backslash \beta$ is a contiguous set. Then for any $t \in [\rho(A), \infty)$,*

$$\mathscr{P}_t(A/A[\beta]) \overset{(c)}{\geqslant} A[\alpha] \overset{(c)}{\geqslant} \mathscr{S}(A/A[\beta]) \overset{(c)}{\geqslant} 0.$$

Proof. We first prove the result for the case when $\alpha = \{1, \dots, k\}$ and $\beta = \{k+1, \dots, n\}$. It is enough to verify the inequality $\mathscr{P}_t(A/A[\beta]) \overset{(c)}{\geqslant} A[\alpha]$, as the remaining two inequalities are contained in Lemma 1.1. We begin, though, with the special case when β is a singleton, namely, $\beta = \{n\}$. Recall from the proof of Lemma 2.2 that if

$$A = \left[\begin{array}{c|c} B & c \\ \hline d^{\mathrm{T}} & e \end{array}\right] \quad \text{and} \quad X = \left[\begin{array}{c|c} B & -c \\ \hline d^{\mathrm{T}} & t-e \end{array}\right],$$

then for any $\gamma, \delta \subset \langle n-1 \rangle$

$$\begin{aligned} \det \mathscr{P}_t[\gamma, \delta] &= \frac{\det X[\gamma \cup \{n\}, \delta \cup \{n\}]}{t-e} \\ &= \frac{t \det B[\gamma, \delta] - \det A[\gamma \cup \{n\}, \delta \cup \{n\}]}{t-e} \\ &\geqslant \frac{t \det B[\gamma, \delta] - e \det B[\gamma, \delta]}{t-e} \quad \text{(by Fischer's inequality)} \\ &= \det B[\gamma, \delta] \quad (\text{here } B = A[\alpha]). \end{aligned}$$

Thus $\mathscr{P}_t(A/A[\beta]) \overset{(c)}{\geqslant} A[\alpha]$, as desired.

We are now ready to proceed with the case when β is not a singleton, namely, $\beta = \{k+1, \dots, n\}, k < n-1$. Let $K = \mathscr{P}_t(A/A[\{n\}])$ and let $\gamma = \{1, \dots, n-2\}$. Then K is irreducible and, by Lemma 2.2, we also know that K is a totally nonnegative matrix. But then applying the initial part of the proof to K we see that

$$\mathscr{P}_t\,(K/[\{n-1\}])\,[\gamma] \overset{(c)}{\geqslant} K[\gamma] \overset{(c)}{\geqslant} A[\gamma],$$

the last inequality follows since $\overset{(c)}{\geqslant}$ is inherited by submatrices. The claim of the theorem now follows by repeating this argument as many times as necessary and making use of the quotient formula in Theorem 2.4.

Thus far we have shown that if $\beta = \{1, \dots, k\}$ or $\beta = \{k+1, \dots, n\}$ and $\alpha = \langle n \rangle \backslash \beta$, then

$$\mathscr{P}_t(A/A[\beta]) \overset{(c)}{\geqslant} A[\alpha] \overset{(c)}{\geqslant} \mathscr{S}(A/A[\beta]) \overset{(c)}{\geqslant} 0. \tag{5}$$

More generally, suppose $\beta \subset \langle n \rangle$ such that $\alpha = \langle n \rangle \backslash \beta$ is a contiguous set. Then $\alpha = \{i, i+1, \dots, i+k\}$, and hence $\beta = \{1, \dots, i-1, i+k+1, \dots, n\}$. Thus, by Theorem 2.4,

$$\mathscr{P}_t(A/\beta) = \mathscr{P}_t(\mathscr{P}_t(A/\{1, \dots, i-1\})/\{i+k+1, \dots, n\}).$$

Applying (5) twice we have

$$\begin{aligned}\mathscr{P}_t(A/\beta) &= \mathscr{P}_t(\mathscr{P}_t(A/\{1,\ldots,i-1\})/\{i+k+1,\ldots,n\})\\ &\overset{(c)}{\geqslant} \mathscr{P}_t(A/\{1,\ldots,i-1\})[\{i+k+1,\ldots,n\}]\\ &\overset{(c)}{\geqslant} A[\alpha],\end{aligned}$$

as desired. The remaining inequalities, namely $A[\alpha] \overset{(c)}{\geqslant} \mathscr{S}(A/A[\beta]) \overset{(c)}{\geqslant} 0$, follow from the remarks preceding Theorem 2.7. This completes the proof. □

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