Theoretical Computer Science 411 (2010) 3625–3634

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

The continuous Skolem-Pisot problem

Paul C. Bell *, Jean-Charles Delvenne, Raphaël M. Jungers, Vincent D. Blondel

Department of Mathematical Engineering, Centre CESAME, Université catholique de Louvain, Bâtiment Euler, Avenue Georges Lemaître, 4, B-1348 Louvain-la-Neuve, Belgium

ARTICLE INFO

Article history:
Received 3 September 2009
Received in revised form 30 April 2010
Accepted 6 June 2010
Communicated by F. Cucker

Keywords:
Skolem-Pisot problem
Exponential polynomials
Continuous time dynamical system
Decidability
Ordinary differential equations

ABSTRACT

We study decidability and complexity questions related to a continuous analogue of the Skolem-Pisot problem concerning the zeros and nonnegativity of a linear recurrent sequence. In particular, we show that the continuous version of the nonnegativity problem is NP-hard in general and we show that the presence of a zero is decidable for several subcases, including instances of depth two or less, although the decidability in general is left open. The problems may also be stated as reachability problems related to real zeros of *exponential polynomials* or solutions to initial value problems of linear differential equations, which are interesting problems in their own right.

1. Introduction

Skolem's problem (also known in the literature as Pisot's problem) asks whether it is algorithmically decidable if a given linear recurrent sequence (LRS) has a zero or not. A LRS may be written in the form:

$$u_k = a_{n-1}u_{k-1} + a_{n-2}u_{k-2} + \cdots + a_0u_{k-n},$$

for $k \geq n$, where $u_0, u_1, \ldots, u_{n-1} \in \mathbb{Z}$ are the initial inputs and $a_0, a_1, \ldots, a_{n-1} \in \mathbb{Z}$ are the recurrence coefficients, see also [9]. This forms the infinite sequence $(u_k)_{k=0}^{\infty} \subseteq \mathbb{Z}$. We may assume a_0 is nonzero, otherwise a shorter and equivalent recurrence exists. Such a recurrence sequence is said to be of *depth n*.

For a linear recurrent sequence $u = (u_k)_{k=0}^{\infty} \subseteq \mathbb{Z}$ the *zero set* of u is given by $Z(u) = \{i \in \mathbb{N} | u_i = 0\}$. One of the first results concerning the zeros of LRS's was by Skolem in [23], when he proved that the zero set is semilinear (i.e., the union of finitely many periodic sets and a finite set). This result was also later shown by Mahler [18] and Lech [17] and is now often referred to as the *Skolem–Mahler–Lech* theorem. It is known that determining if $Z(u)$ is an infinite set is decidable, as was proven by Berstel and Mignotte [3].

It was shown by Vereshchagin in 1985 in [26] that Skolem's problem (i.e., the problem "is the zero set of a LRS empty?") is decidable when the depth of the linear recurrent sequence is less than or equal to four. It was also shown recently in [13] that Skolem's problem is decidable for depth five, but the general decidability status is open. It is also known that determining if a given linear recurrent sequence has a zero is NP-hard, see [4].

Note that we may always encode a linear recurrent sequence of depth n into an integral matrix $A \in \mathbb{Z}^{(n+1)\times(n+1)}$ such that $u_k = A^k_{1,n+1}$ for $k \geq 1$. This follows since given the initial vector $u = (u_0, u_1, \ldots, u_{n-1})^T$ and the recurrence coefficients, $a_0, a_1, \ldots, a_{n-1}$, we first define matrix $A' \in \mathbb{Z}^{n\times n}$:

* Corresponding author. Tel.: +44 151 795 4275; fax: +44 151 795 4235.
E-mail addresses: p.c.bell@liverpool.ac.uk, pcbell@liverpool.ac.uk (P.C. Bell), jean-charles.delvenne@uclouvain.be (J.-C. Delvenne), raphael.jungers@uclouvain.be (R.M. Jungers), vincent.blondel@uclouvain.be (V.D. Blondel).

0304-3975/$ – see front matter
doi:10.1016/j.tcs.2010.06.005

$$A' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_0 & a_1 & a_2 & \cdots & a_{n-1} \end{pmatrix}.$$

Note that $(A')^k u = (u_k, u_{k+1}, \dots, u_{k+n-1})$. Now we shall extend this matrix by 1 dimension to give:

$$A = \begin{pmatrix} A' & A'u \\ \overline{0} & 0 \end{pmatrix} \in \mathbb{Z}^{(n+1)\times(n+1)},$$

where $\overline{0}$ is the zero vector of appropriate size. It is not difficult to now see that $u_k = A^k_{1,n}$ for $k \geq 1$ as required. Skolem's problem in this context is therefore to determine if the upper right entry of a positive power of an integral matrix is zero. More generally, one can show that Skolem's problem is equivalent to the following problem: given a matrix $A \in \mathbb{Z}^{n\times n}$ and two vectors $c, x_0 \in \mathbb{Z}^n$, is there a nonnegative integer t such that $c^T A^t x_0 = 0$? We add that a generalization of this problem where we may take any product of two integral matrices of dimension 10 is known to be undecidable, see [14].

In this paper we shall consider a dynamical system whose updating trajectory is given by $\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t)$ where $A \in \mathbb{R}^{n\times n}$ and the initial point $x(0) \in \mathbb{R}^n$ are given and all entries of A and $x(0)$ are algebraic numbers. We shall be interested in determining whether this trajectory ever reaches a given hyperplane, thus the problem is equivalent to determining if there exists $t \in \mathbb{R}_{\geq 0}$ such that $c^T \exp(At)x(0) = 0$, where $c \in \mathbb{R}^n$ defines the hyperplane (entries of c are also assumed to be algebraic numbers). We consider this as the Skolem-Pisot problem in continuous time. We show that for instances of size two or less this problem is decidable.

We shall also show that determining if $c^T \exp(At)x(0)$ reaches zero is *computationally equivalent* to determining whether a given real-valued exponential polynomial $f(z) = \sum_{j=1}^m P_j(z)\exp(\theta_j z)$, where each P_j is a polynomial, ever reaches zero for a *positive real value*. This is also equivalent to determining if the solution $y(t)$ of an ordinary differential equation $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = 0$ with given initial conditions $y^{(k-1)}(0), y^{(k-2)}(0), \dots, y(0)$ ever reaches zero.

From 1920, Pólya and others characterized the asymptotic distribution of complex zeros of exponential polynomials [19,21,22,24,25,28]. Upper bounds were also found on the number of zeros in a finite region of the complex plane, using the argument principle. Less is known about real zeros. Upper and lower bounds on the number of zeros in a real interval are given in [27]. A formula for the asymptotic density of real zeros for a restricted class of exponential polynomials was found in [15]. Some observations on the first sign change of a sum of cosines are collected in [20]. However, no criterion has been proposed to check the existence of a real zero for a real exponential polynomial.

A related problem, determining whether a given linear recurrent sequence has only nonnegative terms, the *nonnegativity problem*, is decidable for dimension 2, see [12]. The authors note that if the nonnegativity problem is decidable in general, it implies Skolem's problem is decidable. This follows since if $(u_k)_{k=0}^\infty$ is recurrent, then so is $(u_k^2 - 1)_{k=0}^\infty$.

We may note that using the linear recurrent sequence $(u_k)_{k=0}^\infty$ from the proof of NP-hardness of Skolem's problem in [4], and converting it to the form $(u_k^2 - 1)_{k=0}^\infty$, allows one to easily derive the following result:

Theorem 1. *It is NP-hard to decide if a given linear recurrent sequence is nonnegative, i.e., the nonnegativity problem is NP-hard.*

This holds since if $(u_k)_{k=0}^\infty$ is represented by a matrix in $\mathbb{Z}^{n\times n}$, then $(u_k^2 - 1)_{k=0}^\infty$ may be represented by a matrix in $\mathbb{Z}^{(n^2+1)\times(n^2+1)}$ and thus we have a polynomial time reduction. In this paper we show that the nonnegativity problem in the continuous setting is also NP-hard.

Given a matrix $M \in \mathbb{R}^{n\times n}$ and vectors $u, v \in \mathbb{R}^n$, the *orbit problem* asks if there exists a power $k \in \mathbb{N}$ such that $M^k u = v$. Thus it is a type of *reachability problem*, see [5]. This was shown to be decidable even in polynomial time, see [16]. The corresponding version of this problem for continuous time asks whether for a given $M \in \mathbb{R}^{n\times n}$ and vectors $a, b \in \mathbb{R}^n$ there exists some $t \in \mathbb{R}_{\geq 0}$ such that $\exp(Mt)a = b$. This problem was proved to be decidable in [11].

2. Preliminaries

Let $A \in \mathbb{F}^{n\times n}$ denote an $n \times n$ matrix over the field $\mathbb{F}$ and $\sigma(A)$ the set of eigenvalues of A. For a complex number $z \in \mathbb{C}$ we denote by $\Re(z)$ the *real* part of z and by $\Im(z)$ the *imaginary* part of z. We use the notation $\mathbb{R}_{\geq 0}$ to denote the nonnegative real numbers.

We shall denote an *exponential polynomial* $f : \mathbb{C} \to \mathbb{C}$ by a sum of the form: $f(z) = \sum_{j=1}^m P_j(z)\exp(\theta_j z)$, where $P_j \in \mathbb{C}[X]$ and $\theta_j \in \mathbb{C}$.

Given a matrix $A \in \mathbb{C}^{n\times n}$ we shall denote by the *dominant eigenvalues of A* the set of eigenvalues of A with maximum real part, i.e.,

$$\{\theta \in \sigma(A) | \Re(\theta) \geq \Re(\theta'), \theta' \in \sigma(A)\}.$$

We will later require the following theorem from Diophantine approximation [1]:

Theorem 2 (Baker). *Let $\alpha_1, \ldots, \alpha_k, \beta_0, \ldots, \beta_k$ be algebraic numbers. Then the combination*

$$\Lambda = \beta_0 + \sum_i \beta_i \ln \alpha_i$$

is either zero or satisfies $|\Lambda| > h^{-N}$, where h is the largest height of $\beta_1, \ldots, \beta_k$, and N is a computable constant depending only on $\ln \alpha_1, \ldots, \ln \alpha_k$ and the maximum degree of $\beta_0, \ldots, \beta_k$.

Recall that for an algebraic number β with minimal polynomial

$$p(x) = \sum_{0 \le i \le d} a_i x^i,$$

its degree is d and its height is $\max |a_i|$. We shall also use the following theorem regarding the transcendence degree of the field extension of algebraic numbers when considering their exponentials:

Theorem 3 (Lindemann–Weierstrass). *Let $\alpha_j, \lambda_j \in \mathbb{C}$ for $0 \le j \le n-1$ be algebraic numbers such that no $\alpha_j = 0$ and each λ_j is distinct. Then:*

$$\sum_{j=0}^{n-1} \alpha_j e^{\lambda_j} \neq 0$$

and the following theorem about algebraic powers of algebraic numbers:

Theorem 4. *(Gelfond–Schneider) Let $\alpha, \beta \in \mathbb{C}$ be algebraic numbers such that $\alpha \neq 0, 1$ and β is irrational. Then any value of $\alpha^\beta = \exp(\beta \log(\alpha))$ is transcendental.*

The following theorem concerns simultaneous Diophantine approximation of algebraic numbers which are linearly independent over the rationals.

Theorem 5 (Kronecker, *see [7]*). *Let $1, \lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ be real algebraic numbers which are linearly independent over $\mathbb{Q}$. Then for any $\alpha \in \mathbb{R}^n$ and $\epsilon > 0$, there exists $p \in \mathbb{Z}^n$ and $k \in \mathbb{N}$ such that $|(k\lambda_i - \alpha_i - p_i)| < \epsilon$ for all $1 \le i \le n$.*

3. Skolem's problem in continuous time

We shall consider continuous time systems governed by the rule $\frac{dx(t)}{dt} = Ax(t)$, where A is a real matrix and $x(t)$ is a real vector.[1] We are interested in the decidability of whether from an initial vector $x(0)$, we intersect a given hyperplane. We may consider this as a "point-to-set" reachability problem in a dynamical system, see [5] for other examples.

Let $\frac{dx(t)}{dt} = Ax(t)$, where $A \in \mathbb{R}^{n\times n}$ and $x(t) \in \mathbb{R}^n$. Given the initial vector $x(0) \in \mathbb{R}^n$, then $x(t)$ is given by:

$$x(t) = \exp(At) \cdot x(0) = \sum_{j=0}^{\infty} \frac{t^j}{j!} A^j \cdot x(0).$$

Given a vector $c \in \mathbb{R}^n$ defining a hyperplane, we would like to determine if there exists some $t \in \mathbb{R}_{\ge 0}$ such that $c^T x(t) = 0$. In other words, whether the flow of the point $x(0)$ ever intersects the hyperplane. If such a t exists, we say that there exists a solution to the instance[2]. An instance of the Continuous Skolem Problem therefore consists of the matrix $A \in \mathbb{R}^{n\times n}$, the initial point $x(0) \in \mathbb{R}^n$ and the hyperplane vector $c \in \mathbb{R}^n$, *where all entries of A, $x(0)$ and c are algebraic numbers and thus have a finite description*.

3.1. Equivalent formulations

To analyze the behaviour of the system, we will convert a given instance of the Continuous Skolem Problem into various forms which have different properties but which are essentially equivalent to the original problem.

Given such an instance, the following lemma shows that the problem is equivalent to determining if the upper right entry of the exponential of a matrix equals some constant real. A similar construction is known in the discrete case as shown in Section 1.

Theorem 6. *Given an instance of the* Continuous Skolem Problem *defined by $f(t) = c^T \exp(At)x(0)$, where $A \in \mathbb{R}^{n\times n}$ and $c, x(0) \in \mathbb{R}^n$, there exists a polynomial-time computable matrix $B \in \mathbb{R}^{(n+2)\times(n+2)}$ such that $f(t) = \exp(Bt)_{1,n+2} + \lambda$, where $\lambda = c^T x(0) \in \mathbb{R}$ is constant.*

1 We shall consider all problem instances to be defined over *algebraic numbers* throughout this paper.

2 Note that, in the style of Skolem's problem, we shall be more interested in determining whether any solution exists, rather than trying to find an algebraic description of the solution.

Proof. We are given the function $f(t) = c^T \exp(At)x(0)$. Let $B \in \mathbb{R}^{(n+2)\times(n+2)}$ be given by:

$$B \triangleq \begin{pmatrix} 0 & c^T A & c^T Ax(0) \\ \bar{0} & A & Ax(0) \\ 0 & \bar{0}^T & 0 \end{pmatrix},$$

where $\bar{0} = (0, 0, \ldots, 0)^T \in \mathbb{R}^n$, thus:

$$\exp(B) = \begin{pmatrix} 1 & c^T \exp(A) - c^T & c^T \exp(A)x(0) - \lambda \\ \bar{0} & \exp(A) & \exp(A)x(0) - x(0) \\ 0 & \bar{0}^T & 1 \end{pmatrix},$$

where $\lambda = c^T x(0)$ is constant. This can be seen from the power series representation $\exp(tB) = \sum_{j=0}^{\infty} \frac{t^j}{j!} B^j$. Therefore $f(t) = \exp(Bt)_{1,(n+2)} + \lambda$ and thus an instance of the Continuous Skolem Problem can also be given by a single real matrix B and a real number λ, and the problem of whether $f(t)$ reaches zero for $t \in \mathbb{R}_{\geq 0}$ is equivalent to whether $\exp(Bt)_{1,(n+2)}$ ever equals $-\lambda$. □

Theorem 7. *The following problems are computationally equivalent, with polynomial time reductions (where all* parameters *are taken over algebraic numbers):*

(i) *Does there exist a solution to a given instance of the* Continuous Skolem Problem*?*
(ii) *Determine if a real-valued exponential polynomial:*

$$f(t) = \sum_{j=1}^{m} P_j(t)e^{\theta_j t},$$

has a nonnegative real zero (where $\theta_j \in \mathbb{C}$ and $P_j \in \mathbb{C}[X]$).
(iii) *Determine if a function of the form:*

$$f(t) = \sum_{j=1}^{m} e^{r_j t}(P_{1,j}(t)\cos(\lambda_j t) + P_{2,j}(t)\sin(\lambda_j t))$$

has a nonnegative real zero (where $r_j, \lambda_j \in \mathbb{R}$ and $P_{i,j} \in \mathbb{R}[X]$).
(iv) *Determining whether the solution $y(t)$ to an ordinary differential equation $y^{(k)} + a_{k-1}y^{(k-1)} + \cdots + a_0 y = 0$ with the given initial conditions $y^{(k-1)}(0), y^{(k-2)}(0), \ldots, y(0)$ reaches zero for a nonnegative real t.*

Proof. (i) ⇒ (ii): Let $J \in \mathbb{C}^{n\times n}$ be the Jordan matrix for A, thus we may write $A = PJP^{-1}$ for some $P \in GL(n, \mathbb{C})$.[3] Since $\exp(PJP^{-1}) = P\exp(J)P^{-1}$, we can ask the equivalent problem, does there exist a time $t \geq 0$ at which:

$$\begin{aligned} c^T x(t) &= c^T \exp(tA)x(0) \\ &= u^T \exp(tJ)v = 0, \end{aligned}$$

where $u, v \in \mathbb{C}^n$ are defined by $u^T = c^T P$ and $v = P^{-1}x(0)$?

Let $J = J_1 \oplus J_2 \oplus \cdots \oplus J_m$ be a decomposition of J into a direct sum of Jordan blocks with $J_i \in \mathbb{C}^{n_i\times n_i}$ and $\sum_{i=1}^{m} n_i = n$. Each Jordan block may be written $J_i = \theta_i I_{n_i} + M_i$, where $\theta_i \in \mathbb{C}$ is the associated eigenvalue, $I_{n_i} \in \mathbb{Z}^{n_i\times n_i}$ is the identity matrix and $M_i \in \mathbb{Z}^{n_i\times n_i}$ has 1 on the super-diagonal and 0 elsewhere.

For $1 \leq i \leq m$, we see that $\theta_i I_{n_i}$ and M_i commute and therefore $\exp(tJ_i) = \exp(t\theta_i I_{n_i})\exp(tM_i)$. The value of $\exp(t\theta_i I_{n_i})$ is $e^{t\theta_i} I_{n_i}$. Let $\exp(tM_i) = [m_{jk}] \in \mathbb{Q}^{n\times n}$, then

$$m_{jk} = \begin{cases} \frac{t^{(k-j)}}{(k-j)!}; & \text{if } j \leq k \\ 0; & \text{otherwise.} \end{cases} \tag{1}$$

Therefore we may convert our problem equivalently into deciding whether there exists a $t \in \mathbb{R}_{\geq 0}$ such that $f(t) = 0$, where $f : \mathbb{R} \to \mathbb{C}$ is defined by:

$$f(t) = u^T \exp(Jt)v = \sum_{j=1}^{m} P_j(t)e^{\theta_j t}, \tag{2}$$

and $P_j \in \mathbb{C}[X]$ are polynomials whose degree depends upon the size of the corresponding Jordan block and $\theta_j \in \mathbb{C}$. The polynomials P_j can be derived from Eq. (1). Note that each of these steps is effective and can be computed in polynomial time for algebraic entries of the initial matrix A.

[3] These matrices can be effectively found since we only need *algebraic descriptions* of the Jordan normal form J and the similarity matrix P and this is known to be polynomial-time computable for a matrix with algebraic entries, see [8].

(ii) ⇒ (iii): This results from Euler's formula for the complex exponential and the fact that $f(t)$ is a real valued function.
(iii) ⇒ (iv): Functions of the type

$$f(t) = \sum_{j=1}^{m} \mathrm{e}^{r_j t}(P_{1,j}(t)\cos(\lambda_j t) + P_{2,j}(t)\sin(\lambda_j t))$$

where $r_j, \lambda_j \in \mathbb{R}$ are fixed and $P_{k,j}$ are arbitrary real polynomials of degree $\le d_j$ form a real vector space of dimension $k = 2\sum_{j=1}^{m}(d_j + 1)$. This vector space is closed under differentiation. Hence the first $(k+1)$ derivatives of f are related by $f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_0 f = 0$, where each a_j can be found in polynomial time. By Cauchy's theorem for ordinary differential equations, a function f is completely determined by the given relation and the initial conditions $f^{(k-1)}(0), f^{(k-2)}(0), \ldots, f(0)$.

(iv) ⇒ (i): The characteristic equation of the linear homogeneous differential equation is given by $z^k + z^{k-1}a_{k-1} + \cdots + a_0 = 0$. It is well known that we can form the companion matrix of the equation in order to convert the problem into an instance of the Continuous Skolem Problem. The initial values are then present in the initial vector $x(0)$. □

Lemma 8. *Let $A \in \mathbb{R}^{n\times n}$ and $c, x(0) \in \mathbb{R}^n$ form an instance of the* Continuous Skolem Problem. *For any $\lambda \in \mathbb{C}$ we may form a system $f_\lambda(t) = u^T \exp(t(A + \lambda I))v$, where $u, v \in \mathbb{C}^n$, $\sigma(A + \lambda I) = \sigma(A) + \lambda$ and $f(t) = 0$ if and only if $f_\lambda(t) = 0$.*

Proof. Let $\lambda \in \mathbb{C}$ and define $y(t) = \mathrm{e}^{\lambda t}x(t)$, thus:

$$\begin{aligned}\frac{\mathrm{d}y(t)}{\mathrm{d}t} &= \lambda \mathrm{e}^{\lambda t}x(t) + \mathrm{e}^{\lambda t}\frac{\mathrm{d}x(t)}{\mathrm{d}t}\\ &= \mathrm{e}^{\lambda t}(\lambda I + A)x(t)\\ &= (\lambda I + A)y(t).\end{aligned}$$

Define $A_\lambda = \lambda I + A$, thus:

$$y(t) = \exp(tA_\lambda)y(0).$$

Note that there exists $t \ge 0$ such that $c^T x(t) = 0$ if and only if there exists $t \ge 0$ such that $c^T y(t) = 0$. □

As an example, which will be useful later, let us set $\lambda = -\max\{\Re(\theta) | \theta \in \sigma(A)\}$, so that all eigenvalues are shifted to the left complex half-plane or the imaginary axis. This means that we have, in effect, split the set of eigenvalues into two sets, one which decays exponentially with time and one which consists of purely imaginary values.

We now remark that any nontrivial solution to the problem will in fact be *transcendental*.

Theorem 9. *Given an instance of the* Continuous Skolem Problem, *all solutions, if any exist, are transcendental unless the polynomials $P_j(t)$ share a common positive real root.*

Proof. The corresponding exponential polynomial formed as in Theorem 7 will be in the form:

$$f(t) = \sum_{j=1}^{m} P_j(t)\mathrm{e}^{\theta_j t} = 0.$$

We may assume no $P_j \in \mathbb{C}[X]$ is zero, otherwise simply remove it from the sum, and that each θ_j is distinct, otherwise group them together. Thus, according to Theorem 3 (the Lindemann–Weierstrass theorem), this exponential polynomial only has solutions for transcendental times t where $t \in \mathbb{R}_{\ge 0}$. □

4. Decidable cases

We shall now investigate some classes of instances for which the Continuous Skolem Problem is decidable.

Theorem 10. *The* Continuous Skolem Problem *for depth* 2 *is decidable.*

Proof. Assume we have an instance of the Continuous Skolem Problem given by $f(t) = (c_1, c_2)\exp(At)(x_1, x_2)^T$ with $A \in \mathbb{R}^{2\times 2}$. Let $S \in GL(\mathbb{C}, 2)$ put A into Jordan canonical form. We can rewrite $f(t) = (\alpha_1, \alpha_2)\exp(Jt)(\beta_1, \beta_2)^T$, where $J = S^{-1}AS$ is a Jordan matrix.

If A has one eigenvalue θ, with algebraic multiplicity 2, then $\theta \in \mathbb{R}$. If θ has geometric multiplicity 1 then Jordan matrix J is not diagonal and thus

$$f(t) = (\alpha_1, \alpha_2)\begin{pmatrix}\exp(t\theta) & t\cdot\exp(t\theta)\\ 0 & \exp(t\theta)\end{pmatrix}(\beta_1, \beta_2)^T = \exp(t\theta)(\alpha_1\beta_1 + t\alpha_1\beta_2 + \alpha_2\beta_2),$$

which has a solution if and only if $\frac{-\alpha_1\beta_1 - \alpha_2\beta_2}{\alpha_1\beta_2} \in \mathbb{R}_{\ge 0}$ (if α_1 or β_2 equals zero, finding solutions is trivial). If θ has geometric multiplicity 2 then J is diagonal and we must solve $\mathrm{e}^{\theta t}(\alpha_1\beta_1 + \alpha_2\beta_2) = 0$, which has a solution if and only if $\alpha_1\beta_1 + \alpha_2\beta_2 = 0$.

Otherwise, J has two different eigenvalues and thus it is diagonal. We must determine if there exists a $t \in \mathbb{R}_{\geq 0}$ such that $e^{t\theta_1} + \frac{\alpha_2\beta_2}{\alpha_1\beta_1}e^{t\theta_2} = 0$ (in the case that $\alpha_1\beta_1 = 0$ or $\alpha_2\beta_2 = 0$, a solution is trivial to determine). Either $\theta_1, \theta_2 \in \mathbb{R}$ or $\theta_1 = \overline{\theta_2} \in \mathbb{C}$.

If $\theta_1, \theta_2 \in \mathbb{R}$ then $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, and by taking logarithms of $e^{t\theta_1} = -\frac{\alpha_2\beta_2}{\alpha_1\beta_1}e^{t\theta_2}$ we see $t = \frac{\ln(-\frac{\alpha_2\beta_2}{\alpha_1\beta_1})}{\theta_1-\theta_2}$ is a solution of $f(t) = 0$, and thus there exists a solution if and only if this value lies inside $\mathbb{R}_{\geq 0}$.

In the other case $\theta_1 = \overline{\theta_2} \in \mathbb{C}$. Since we may therefore shift the real part, as allowed by Lemma 8, assume that $\theta_1, \theta_2 \in i\mathbb{R}$. We have

$$e^{t\theta_1} + \frac{\alpha_2\beta_2}{\alpha_1\beta_1}e^{t\theta_2} = x\cos(\Im(\theta_1)t) + y\sin(\Im(\theta_1)t),$$

for some real numbers x, y. This expression takes positive, as well as negative values when $t > 0$, and will then vanish for some $t > 0$. □

The following theorem shows that the class of instances where all elements of the input are defined over nonnegative real algebraic numbers in the continuous setting is trivially decidable in polynomial time, whereas in the classical Skolem problem, the problem is NP-hard, as shown in [4]. In fact, using Lemma 8, we see that in the continuous setting the Skolem-Pisot problem is polynomial-time decidable even where the matrix given is a *Metzler matrix*, meaning only off-diagonal elements need be nonnegative.

Theorem 11. *For an instance of the* Continuous Skolem Problem *given by $A \in \mathbb{R}^{n\times n}$ and $c, x(0) \in \mathbb{R}^n_{\geq 0}$ where A is a Metzler matrix (thus all off-diagonal elements are nonnegative) and $f(t) = c^T \exp(At)x(0)$, then we may decide if there exists a solution in polynomial time.*

Proof. Let λ be the minimal diagonal element of A. If $\lambda < 0$, then by Lemma 8, we may form an equivalent instance $A' = A + \lambda I$, where $A' \in \mathbb{R}^{n\times n}_{\geq 0}$. Thus assume without loss of generality that A is a nonnegative matrix and $c, x(0)$ are nonnegative vectors.

Note that $\exp(t_2A) > \exp(t_1A)$ for any $t_2 > t_1 \in \mathbb{R}_{\geq 0}$, which is a consequence of the power series representation of $\exp(At) = \sum_{j=0}^{\infty} \frac{t^j}{j!}A^j$ and the fact that $A \in \mathbb{R}^{n\times n}_{\geq 0}$. We see that $f(0) = c^T x(0) \in \mathbb{R}_{\geq 0}$. Now, if $f(0) = 0$ then this is a solution, otherwise, since the matrix exponential increases monotonically componentwise with time for a nonnegative matrix, there exists no solution. □

In some special cases, some eigenvalues of A do not influence the function $f(t)$. This is easily seen when A is put in its Jordan form $J = P^{-1}AP$:

$$f(t) = u^T \exp(tJ)v \tag{3}$$

where $u, v \in \mathbb{C}^n$ are defined by $u^T = c^T P$ and $v = P^{-1}x(0)$. Obviously, if the entries of c or $x(0)$ corresponding to a particular Jordan block are zero, this block does not play any role and one may remove it without changing the function $f(t)$. More generally, from Eq. (3) it is easy, as shown in Theorem 7, to write the function f as follows:

$$f(t) = \sum_{j=1}^{m} P_j(t)e^{\theta_j t}, \tag{4}$$

where the θ_j are the distinct eigenvalues of A and the P_j are complex polynomials. If no polynomial $P_j(t)$ is identically zero, then we say that the triple $(A, c, x(0))$ is *reduced*. If some of the P_j are zero, we can remove the corresponding terms from Eq. (4), since it does not change the value of f.

Theorem 7 shows how to build an equivalent instance of the form $(A', c', x'(0))$ from an instance of the form of Eq. (4). One would then obtain a reduced, but equivalent, instance of the Continuous Skolem Problem. This can be done in a preprocessing phase.

Theorem 12. *Let $\frac{dx(t)}{dt} = Ax(t)$ for $A \in \mathbb{R}^{n\times n}$ and $x(t) \in \mathbb{R}^n$ define an instance of the* Continuous Skolem Problem *given by $f(t) = c^T \exp(At)x(0) = 0$. If $(A, c, x(0))$ is reduced and none of the dominant eigenvalues of A are real then the problem is decidable.*

Proof. By Lemma 8, let us assume all eigenvalues have real part less than or equal to 0. Then, using Theorem 7, we may consider the system as being represented by

$$f(t) = \sum_{j=1}^{m} P_j(t)e^{\gamma_j t}$$

where each γ_j is a shifted eigenvalue of A. We may split this exponential polynomial in two (reordering as necessary) and write $f(t) = f_1(t) + f_2(t)$, where

$$f_1(t) = \sum_{j=1}^{k} P_j(t)\exp(i\lambda_j t)$$

are those terms corresponding to shifted eigenvalues with 0 real part ($\lambda_j \in \mathbb{R}$) and $f_2(t)$ is the summation of the remaining terms (corresponding to eigenvalues with *negative* real part). Since $(A, c, x(0))$ is reduced, $f_1(t)$ is not identically zero. Note that $f_2(t)$ tends to zero exponentially fast as t increases. Let $d \in \mathbb{Z}^+$ denote the maximal degree of any polynomial of $f_1(t)$, i.e., the maximum degree of a polynomial in $\{P_j | 1 \leq j \leq k\}$. We now also separate $f_1(t)$ and write

$$f_1(t) = t^d \sum_{j=1}^{k} c_j \exp(\mathrm{i}\lambda_j t) + \sum_{j=1}^{k} O(t^{d-1}) \exp(\mathrm{i}\lambda_j t)$$

for some (not all zero) constants $c_1, \ldots, c_k \in \mathbb{C}$. Let $g(t) = \sum_{j=1}^{k} c_j \exp(\mathrm{i}\lambda_j t)$.

We now note $f(t)/t^d = g(t) + h(t)$, and prove that $f(t)/t^d$ takes the value zero for arbitrarily large times t, because we show below that there is some $c > 0$ such that $g(t)$ takes both values c and $-c$ for arbitrarily large times, while $h(t)$ tends to zero when t tends to infinity.

It is well known that a sum of imaginary exponentials such as $g(t)$, and its integral $\int_0^t g(s)\mathrm{d}s$, is an *almost periodic function* (see page 30 of [6]). If the function $g(t)$ is nonnegative, then $\int_0^t g(s)\mathrm{d}s$ is nondecreasing. This is impossible, since $\int_0^t g(s)\mathrm{d}s$ is non constant and almost periodic. Similarly, $g(t)$ cannot be nonpositive. Hence there are nonnegative times t_a and t_b such that $g(t_a)g(t_b) < 0$. From almost periodicity, this implies that there is a value c such that $g(t_i) = c$ for arbitrarily large times $t_1, t_3, t_5, \ldots, t_i, \ldots$ and $g(t_j) = -c$ for arbitrarily large times $t_2, t_4, t_6, \ldots, t_j, \ldots$ $\square$

We now use Theorem 2 (Baker's theorem) to provide bounds on sums of exponential polynomials. We first start with some lemmata.

Lemma 13. *Let ω_1 and ω_2 be different algebraic numbers, linearly independent over $\mathbb{Q}$, and $\mathrm{e}^{\mathrm{i}\phi_1}$, $\mathrm{e}^{\mathrm{i}\phi_2}$ be algebraic numbers on the unit circle. There exist effective constants $C, N, T > 0$ such that at any time instant $t > T$, either $1 - \cos(\omega_1 t + \phi_1) > C/t^N$ or $1 - \cos(\omega_2 t + \phi_2) > C/t^N$.*

Proof. First note that for any $\alpha \in [-\pi, \pi[$, $\cos\alpha \leq 1 - \alpha^2/10$. Around 0, this is a consequence of Taylor approximation $\cos\alpha \approx 1 - \alpha^2/2$. The denominator 10 is chosen to make the inequality hold for all $\alpha \in [-\pi, \pi[$. Therefore, for any α, if $1 - \cos\alpha \leq \delta$, then there exists an integer k such that $|\alpha - 2k\pi| \leq \sqrt{10\delta}$.

For a time $t \geq 0$, denote $M_t = \max(1 - \cos(\omega_1 t + \phi_1), 1 - \cos(\omega_2 t + \phi_2))$. From the above, this means that for some k, l depending on t, both $|\omega_1 t + \phi_1 - 2k\pi| \leq \sqrt{10M_t}$ and $|\omega_2 t + \phi_2 - 2l\pi| \leq \sqrt{10M_t}$ hold. Then $|t + \phi_1/\omega_1 - 2k\pi/\omega_1| \leq \sqrt{10M_t}/|\omega_1|$ and $|t + \phi_2/\omega_2 - 2l\pi/\omega_2| \leq \sqrt{10M_t}/|\omega_2|$. By difference we find $|\phi_1/\omega_1 - \phi_2/\omega_2 + 2l\pi/\omega_2 - 2k\pi/\omega_1| \leq \sqrt{10M_t}(\frac{1}{|\omega_1|} + \frac{1}{|\omega_2|})$. Let us introduce $\omega = (\frac{1}{|\omega_1|} + \frac{1}{|\omega_2|})^{-1}$. Then $|\frac{\omega}{\mathrm{i}\omega_1}\mathrm{i}\phi_1 - \frac{\omega}{\mathrm{i}\omega_2}\mathrm{i}\phi_2 - k\frac{\omega}{\mathrm{i}\omega_1}2\pi\mathrm{i} + l\frac{\omega}{\mathrm{i}\omega_2}2\pi\mathrm{i}| \leq \sqrt{10M_t}$.

Observing that $2\pi\mathrm{i}$, $\mathrm{i}\phi_1$, $\mathrm{i}\phi_2$ are logarithms of algebraic numbers, we will now apply Baker's theorem to get a lower bound on $|\frac{\omega}{\mathrm{i}\omega_1}\mathrm{i}\phi_1 - \frac{\omega}{\mathrm{i}\omega_2}\mathrm{i}\phi_2 - k\frac{\omega}{\mathrm{i}\omega_1}2\pi\mathrm{i} + l\frac{\omega}{\mathrm{i}\omega_2}2\pi\mathrm{i}|$, therefore on M_t, as required.

Let us first prove that $\frac{\omega}{\mathrm{i}\omega_1}\mathrm{i}\phi_1 - \frac{\omega}{\mathrm{i}\omega_2}\mathrm{i}\phi_2 - k\frac{\omega}{\mathrm{i}\omega_1}2\pi\mathrm{i} + l\frac{\omega}{\mathrm{i}\omega_2}2\pi\mathrm{i}$ is not zero, for large enough times. First observe that since $M_t \leq 2$, it is clear that $|\omega_1 t + \phi_1 - 2k\pi| \leq \sqrt{10M_t}$ implies $c_3 t < |k| < C_3 t$ (for some C_3), and similarly for $|\omega_2 t + \phi_2 - 2l\pi|$. If $\mathrm{e}^{\mathrm{i}\phi_1} = \mathrm{e}^{\mathrm{i}\phi_2} = 1$, then choose a time T' such that $|k| > \phi_1/|\omega_1|$ and $|l| > \phi_2/|\omega_2|$; for all times larger than T', the quantity $\frac{\omega}{\mathrm{i}\omega_1}\mathrm{i}\phi_1 - \frac{\omega}{\mathrm{i}\omega_2}\mathrm{i}\phi_2 - k\frac{\omega}{\mathrm{i}\omega_1}2\pi\mathrm{i} + l\frac{\omega}{\mathrm{i}\omega_2}2\pi\mathrm{i}$ is nonzero, otherwise the ratio ω_1/ω_2 would be rational. If $\mathrm{e}^{\mathrm{i}\phi_1} \neq 1$, then $\frac{\omega}{\mathrm{i}\omega_1}\mathrm{i}\phi_1 - \frac{\omega}{\mathrm{i}\omega_2}\mathrm{i}\phi_2 - k\frac{\omega}{\mathrm{i}\omega_1}2\pi\mathrm{i} + l\frac{\omega}{\mathrm{i}\omega_2}2\pi\mathrm{i} = 0$ would imply $\mathrm{e}^{\mathrm{i}\phi_2} = (\mathrm{e}^{\mathrm{i}\phi_1})^{\omega_1/\omega_2}$, which by Theorem 4 (Gelfond–Schneider) would imply that $\mathrm{e}^{\mathrm{i}\phi_2}$ is transcendental, in contradiction with the assumption. In this case we may set $T' = 0$. The case $\mathrm{e}^{\mathrm{i}\phi_2} \neq 1$ is similar.

Let us find the bound given by Baker's theorem. Note that the height of $k\alpha$, for any algebraic number α of degree d, is at most $|k|^d$ times the height of α, from the definition of height. Baker's theorem then yields $\sqrt{10M_t} > \max(C_1C_3^d t^d, C_2C_3^d t^d, C_0)^{-N_0}$ if $t > T'$. This can be written as $\sqrt{10M_t} > \min(C_4, C_5 t^{-N_1})$ for some C_4, C_5, N_1. Therefore, $M_t > \min(C_6, Ct^{-N})$, for some C_6, C, N.

Choosing $T > T'$ such that $Ct^{-N} < C_6$, we have found $C, N, T > 0$ such that for every $t > T$, either $1 - \cos(\omega_1 t + \phi_1) > Ct^{-N}$ or $1 - \cos(\omega_2 t + \phi_2) > Ct^{-N}$. $\square$

We say that a property *T-eventually* holds for a function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ if it holds for all time instants $t \geq T$. We say that a property *eventually* holds if it T-eventually holds for some T. For instance, g is *eventually positive* if there is a threshold T such that $g(t) > 0$ for all $t \geq T$. Clearly, if f is the solution of a linear differential equation, then it has finitely many zeros if and only if it is eventually positive or eventually negative.

We say that g_1 is (T, r)-*exponentially dominated* by g_2 if $|g_1(t)| < \mathrm{e}^{-rt}|g_2(t)|$ for $r > 0$ and all $t \geq T$.

Lemma 14. *Let us consider T-eventually nonzero continuous functions $g_1, \ldots, g_k$ and the function $f(t) = g_0(t) + \sum_{j=1}^{k} g_j(t)\cos(\omega_j t + \phi_j)$, where ω_1, ω_2 are linearly independent positive algebraic numbers, g_0 is (T, r)-exponentially dominated by g_1 and g_2, and ϕ_1, ϕ_2 are angles such that $\mathrm{e}^{\mathrm{i}\phi_1}$, $\mathrm{e}^{\mathrm{i}\phi_2}$ are algebraic. Then the following T'-eventually holds, for some T':*

$$-\sum_{j=1}^{k} |g_j| < f < \sum_{j=1}^{k} |g_j|.$$

Moreover, T' can be effectively computed as a function of $T, r, \omega_1, \omega_2, \phi_1, \phi_2$.

If all ω_j are linearly independent over the rationals, then for any $\epsilon > 0$, there exist arbitrarily large times t such that

$$f(t) > (1-\epsilon)\sum_{j=1}^{k} |g_j(t)|$$

and arbitrarily large times t such that

$$f(t) < -(1-\epsilon)\sum_{j=1}^{k} |g_j(t)|.$$

Proof. It is obvious that $-|g_0| - \sum_{j=1}^{k} |g_j| < f < |g_0| + \sum_{j=1}^{k} |g_j|$.

To prove that we can get rid of g_0, we exploit the fact that the cosines $\cos(\omega_1 t + \phi_1)$ and $\cos(\omega_2 t + \phi_2)$ never take the value ± 1 exactly at the same time, except possibly once; this is a consequence of the linear independence of ω_1 and ω_2. Due to Lemma 13, one of the cosines is Ct^{-N} away from 1, for some C, N and all large enough times. Then for all large enough times t, either $|g_0 + g_1\cos(\omega_1 t + \phi_1) + g_2\cos(\omega_2 t + \phi_2)| < |g_0| + |g_1|(1 - Ct^{-N}) + |g_2|$ or $|g_0 + g_1\cos(\omega_1 t + \phi_1) + g_2\cos(\omega_2 t + \phi_2)| < |g_0| + |g_1| + |g_2|(1 - Ct^{-N})$. In any case, since g_0 is exponentially dominated by both g_1 and g_2, we have that $|g_0 + g_1\cos(\omega_1 t + \phi_1) + g_2\cos(\omega_2 t + \phi_2)| < |g_1| + |g_2|$ is T'-eventually true for some T', computable as a function of T, r, C, N. Adding all the terms $g_j\cos(\omega_j t + \phi_j)$ proves the first claim of the theorem.

We now prove the second claim. From Kronecker's theorem and the linear independence of frequencies, we have that the set $\Gamma = \{(\cos(\omega_j t + \phi_j))_{1\le j\le k} | t \ge 0\}$ is dense in $[-1, 1]^k$. Hence, Γ will approach all the vertices of $[-1, 1]^k$ by less than any $\epsilon > 0$ for arbitrarily large times. For those times such that for all j, $\cos(\omega_j t + \phi_j)$ is close within $\epsilon/2$ to sign g_j, and $|g_0(t)/g_1| < \epsilon/2$, the first part of the second claim holds. The second part is similar. □

We now prove the main theorem of this section, which says that in some circumstances, one can reduce the search for a solution to an instance of the Continuous Skolem Problem to a *finite* time interval. Recall that an eigenvalue is *nondefective* if its algebraic and geometric multiplicities coincide. The *frequency* of an eigenvalue is the absolute value of the imaginary part. Recall that for a real matrix, complex eigenvalues come in conjugate pairs, determining one equal frequency.

Theorem 15. *Given an instance of the* Continuous Skolem Problem *where all dominant eigenvalues of the input matrix A are nondefective, at least four in number and such that the set of their distinct nonzero frequencies is linearly independent over the rationals.*

Then

- *The existence of infinitely many solutions is decidable;*
- *If there are finitely many solutions, then those solutions are in $[0, T]$, where T is computable.*

Note that multiple dominant eigenvalues are allowed.

Proof. As allowed by Lemma 8, we can suppose without loss of generality that the dominant eigenvalues are on the imaginary axis.

Then we are looking for real zeros of a function $f(t) = \gamma_0 + f_1(t) + f_2(t)$, where γ_0 is the contribution of the dominant zero eigenvalue (if any),

$$\begin{aligned} f_1(t) &= \sum_{j=1}^{k} z_j \exp(\mathrm{i}\lambda_j t) \\ &= \sum_{j=1}^{k} \alpha_j \cos(\lambda_j t) + \beta_j \sin(\lambda_j t) \end{aligned}$$

collects the dominant terms corresponding to dominant complex eigenvalues $\theta_j = \mathrm{i}\lambda_j$ and $f_2(t)$ is exponentially decreasing. By elementary trigonometric manipulations, f_1 can be converted into

$$f_1(t) = \sum_{j=1}^{k} \gamma_j \cos(\lambda_j t + \phi_j),$$

for some $\phi_j = 0$ or $\phi_j = \pi/2$. Hence f_1 is a linear combination of shifted cosines.

Since there are at least four distinct dominant eigenvalues, f_1 contains at least two different frequencies. We apply Lemma 14, with $g_i = \gamma_i$, $g_0 = f_2$ to obtain that the following eventually holds:

$$-\sum_{j=1}^{k} |\gamma_j| < f - \gamma_0 < \sum_{j=1}^{k} |\gamma_j|.$$

Moreover, the same lemma tells us that for any $\epsilon > 0$ there are arbitrarily large times t such that:

$$-(1-\epsilon)\sum_{j=1}^{k}|\gamma_j| \geq f(t) - \gamma_0$$

and arbitrarily large times t such that:

$$f(t) - \gamma_0 \geq (1-\epsilon)\sum_{j=1}^{k}|\gamma_j|.$$

As mentioned above, f has finitely many zeros if and only if f is eventually positive or eventually negative. It results from the above that $f < 0$ (T-eventually, for some T) if and only if $\gamma_0 + \sum_{j=1}^{k}|\gamma_j| \leq 0$. Moreover, when $f < 0$ (T-eventually, for some T), such a T can be computed. A similar argument holds for $f > 0$. This proves the claim. □

Note that in discrete time, checking the existence of a zero in a finite time interval is a trivial task, while in continuous time we do not know how to decide the existence of a zero between time 0 and T.

5. NP-Hardness of the nonnegativity problem

We now prove a continuous version of Blondel–Portier's result [4].

Theorem 16. *The nonnegativity problem for instances of the* Continuous Skolem Problem *given by a skew-symmetric matrix is NP-hard and decidable in exponential time. In particular, the general nonnegativity problem is NP-hard.*

Proof. A skew symmetric matrix has only imaginary eigenvalues and Jordan blocks of size one. Using the methods of Theorem 7, we are lead to check the nonnegativity of a function of the form

$$f(t) = \sum_i \alpha_i \cos(\lambda_i t) + \beta_i \sin(\lambda_i t).$$

We can, in polynomial time, find a basis $\xi_1, \ldots, \xi_m$ over the rationals for the family $\lambda_1, \ldots, \lambda_k$, such that every λ_i is an integral combination of $\xi_1, \ldots, \xi_m$. For every ξ_i we introduce two variables $x_i = \cos(\xi_i t)$ and $y_i = \sin(\xi_i t)$, which satisfy $x_i^2 + y_i^2 = 1$. Hence $f(t)$ is a polynomial P in x_i, y_i (by elementary trigonometry). From Theorem 5 (Kronecker's theorem), the trajectory $(\xi_1 t, \ldots, \xi_k t)$ is dense in $[0, 2\pi]^k$, from which $\overline{\{f(t) | t \in \mathbb{R}\}} = \{P(x_1, y_1, \ldots, x_m, y_m) | x_i, y_i \in \mathbb{R} \text{ s.t. } x_i^2 + y_i^2 = 1\}$ follows. Hence, f is nonnegative if and only if P is, when taken over the set $\{x_1, y_1, \ldots, x_m, y_m | x_j, y_j \in \mathbb{R} \text{ and } x_j^2 + y_j^2 = 1 \text{ for } 1 \leq j \leq m\}$. This problem is solvable in time exponential in m by Tarski's procedure (see for example [2]).

Suppose we are given a polynomial $P(x_1, \ldots, x_k)$. Choose k real algebraic numbers $\xi_1, \ldots, \xi_k$ which are linearly independent over the rationals. Substitute x_i with $\cos(\xi_i t)$ for every $1 \leq i \leq k$. Every monomial of P can therefore be substituted with a linear combination of cosines by elementary trigonometry. For instance, $x_1 x_2$ becomes $\cos \xi_1 t \cos \xi_2 t = \frac{\cos(\xi_1 - \xi_2)t + \cos(\xi_1 + \xi_2)t}{2}$, and so on. In this way, the polynomial $P(x_1, \ldots, x_k)$ is associated to a function $f(t) = \sum_i \alpha_i \cos(\lambda_i t)$, such that $\overline{\{f(t) | t \in \mathbb{R}\}} = \{P(x_1, \ldots, x_k) | x_i \in [-1, 1]\}$. Hence f is nonnegative if and only if P is nonnegative on $[-1, 1]^k$. Since checking the nonnegativity of a polynomial on $[-1, 1]^k$ is NP-hard (which is easily proved via an encoding of the 3-SAT problem, see, e.g., [10]), then the nonnegativity problem for instances of the Continuous Skolem Problem is also NP-hard. □

Note that physical linear systems that preserve energy can often be modelled by differential equations with a skew-symmetric matrix, because these are precisely, up to a change of variables, the systems for which the energy $1/2x^T x$ (where x is the state) is constant along the trajectories, (see, e.g., [29]). This case is therefore particularly relevant.

6. Conclusion

In studying this problem, we are not so much interested in exactly describing the solutions to the problem, as determining the *existence* of solutions. For example, if we have algebraic times $t_1, t_2 \in \mathbb{R}_{\geq 0}$ with $t_1 < t_2$ such that $f(t_1)$ and $f(t_2)$ have different signs then there exists $t \in [t_1, t_2]$ such that $f(t) = 0$ by the intermediate value theorem.

The main problem encountered in solving the Continuous Skolem Problem however appears to be that $f(t)$ can reach 0 tangentially, i.e. we may have a solution $f(t) = 0$ where there exists $\varepsilon > 0$ such that $f(\tau) \geq 0$ for all $\tau \in [t - \varepsilon, t + \varepsilon]$. Since, by Lemma 9, the solution will, for non trivial cases, be transcendental, it is difficult to determine when such a situation arises. Indeed, given a real valued exponential polynomial, if we take its square then it is positive real valued and reaches zero tangentially if and only if the first exponential polynomial had a zero.

We have therefore attempted to show several instances in which the problem is decidable, but the general problem remains open. The equivalent problem of determining if an exponential polynomial has real zeros seems equally interesting. It is surprising that the problem is open even for a finite time interval. Solving Skolem's problem in the discrete case for finite time is obviously decidable since we can simply compute the values in the interval.

Open Problem 17. Is the Bounded Continuous Skolem Problem decidable? I.e. Given a fixed $T \in \mathbb{R}_{\geq 0}$, and an instance of the Continuous Skolem Problem, $f(t) = c^T \exp(At)x(0)$, does there exist $t \leq T$ such that $f(t) = 0$?

We also showed that the nonnegativity problem is NP-hard in the continuous case. It is not clear if a similar technique can be used to show that the Continuous Skolem Problem is also NP-hard. In the discrete Skolem's problem it turns out that determining the nonnegativity and positivity of a linear recurrent sequence are equivalent in terms of complexity, however this is not clear in the continuous case.

Open Problem 18. Are the Continuous Skolem Problem and the Continuous Nonnegativity Problem computationally equivalent?

Acknowledgements

This article presents research results of the Belgian Programme on Interuniversity Attraction Poles, initiated by the Belgian Federal Science Policy Office. This research has been also supported by the ARC (Concerted Research Action) ''Large Graphs and Networks'', of the French Community of Belgium. The scientific responsibility rests with the authors. J.-C. Delvenne and R. Jungers hold FNRS fellowships (Belgian Fund for Scientific Research). We would like to thank Alexandre Megretski for a useful discussion on this problem and the anonymous reviewers for their careful checking and helpful comments on this manuscript.

References

[1] A. Baker, Linear forms in the logarithms of algebraic numbers I–IV, Mathematika 13 (1966) 204–216; 14 (1967), 102–107 et 220–224; 15 (1968), 204–216.
[2] S. Basu, R. Pollack, M. Roy, On the combinatorial and algebraic complexity of quantifier elimination, J. ACM 43 (6) (1996) 1002–1045.
[3] J. Berstel, M. Mignotte, Deux Propriétés Décidables des Suites Récurrentes Linéaires, Bull. Soc. Math. France 104 (1976) 175–184.
[4] V. Blondel, N. Portier, The presence of a zero in an integer linear recurrent sequence is NP-hard to decide, Linear Algebra Appl. 351–352 (2002) 91–98.
[5] V.D. Blondel, J.N. Tsitsiklis, A survey of computational complexity results in systems and control, Automatica 36 (9) (2000) 1249–1274.
[6] H. Bohr, Almost Periodic Functions (H. Cohn, Trans.), Chelsea Publishing Company, 1951.
[7] J. Cassels, An Introduction to Diophantine Approximation, Cambridge Univ. Press, 1957.
[8] J. Cai, R.J. Lipton, Y. Zalcstein, The complexity of the membership problem for 2-generated commutative semigroups of rational matrices, in: Proceedings of the 35th Annual Symposium on Foundations of Computer Science, IEEE Computer Society, 1994, pp. 135–142.
[9] G. Everest, A. van der Poorten, I. Shparlinski, T. Ward, Recurrence Sequences, in: AMS Surveys and Monographs, vol. 104, ISBN: 0-8218-3387-1, 2003.
[10] M.R. Garey, D.S. Johnson, Computers and Intractability: A Guide to the Theory of NP-Completeness, W.H. Freeman, New York, ISBN: 0-7167-1045-5, 1979.
[11] E. Hainry, Reachability in linear dynamical systems, in: Computability in Europe 2008, in: Lecture Notes in Computer Science, vol. 5028, Springer, 2008.
[12] V. Halava, T. Harju, M. Hirvensalo, Positivity of second order linear recurrent sequences, Discrete Appl. Math. 154 (2006) 447–451.
[13] V. Halava, T. Harju, M. Hirvensalo, J. Karhumäki, Skolem's problem — on the border between decidability and undecidability, TUCS Technical Report Number 683, 2005.
[14] V. Halava, M. Hirvensalo, Improved matrix pair undecidability results, Acta Inform. 44 (3) (2007) 191–205.
[15] M. Kac, On the distribution of values of trigonometric polynomials with linearly independent frequencies, Amer. J. Math. 65 (1943) 609–615.
[16] R. Kannan, R. Lipton, Polynomial-time algorithm for the orbit problem, J. Assoc. Comput. Mach. 33 (4) (1986) 808–821.
[17] C. Lech, A note on recurring sequences, Ark. Mat. 2 (1953) 417–421.
[18] K. Mahler, Eine Arithmetische Eigenschaft der Taylor-Koeffizienten Rationaler Funktionen, Proc. Akad. Wet. Amsterdam 38 (1935) 50–60.
[19] C.J. Moreno, The zeros of exponential polynomials, Comp. Math. 26 (1973) 69–78.
[20] J.D. Nulton, K.B. Stolarsky, The first sign change of a cosine polynomial, Proc. Amer. Math. Soc. 84 (1) (1982) 55–59.
[21] G. Pólya, Geometrisches über die Verteilung der Nullstellen gewisser ganzer Transzendenter Funktionen, Münch. Sitzungsber. 50 (1920) 285–290.
[22] E. Schwengler, Geometrisches über die Verteilung der Nullstellen Spezieller Ganzer Funktionen, Dissertation, ETH Zürich, 1925.
[23] T. Skolem, Ein Verfahren zur Behandlung gewisser exponentialer Gleichungen und diophantischer Gleichungen, 8. Skand. Mat. Kongr. Stockholm, 163–188, 1934.
[24] J. Tamarkin, Some general problems of the theory of ordinary linear differential equations and expansion of an arbitrary function in series of fundamental functions, Math. Z. 27 (1) (1925) 1–54.
[25] A.J. van der Poorten, A note on the zeros of exponential polynomials, Compos. Math. 31 (2) (1975) 109–113.
[26] N. Vereshchagin, Occurrence of zero in a linear recursive sequence, Math. Notes 38 (2) (1985) 609–615.
[27] M. Voorhoeve, On the oscillation of exponential polynomials, Mat. Z. 151 (3) (1976) 217–293.
[28] C.E. Wilder, Expansion problems of ordinary linear differential equations with auxiliary conditions at more than two points, Trans. Math. Soc. 18 (1917) 415–442.
[29] J.C. Willems, Dissipative dynamical systems, Part II: linear systems with quadratic supply rates, Arch. Ration. Mech. Anal. 45 (1972) 352–393.