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Nonstandard arithmetic of Hilbert subsets

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Abstract

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Let $f(X, Y) \in \mathbb{Z}[X, Y]$ be irreducible. We give a condition that there are only finitely many integers $n \in \mathbb{Z}$ such that $f(n, Y)$ is reducible and we give a bound for such integers. We prove a similar result for polynomials with coefficients in polynomial rings. Both results are proved by, so-called, nonstandard arithmetic.

For each irreducible polynomial $f(X, Y) \in R[X, Y]$, we denote by $J(f)$ the set of all $r \in R$ that $f(r, Y)$ is reducible in $R[Y]$. In case of $R = \mathbb{Z}$, $\mathbb{Z} - J(f)$ (such a set of integers is called a Hilbert subset) is infinite (Hilbert's irreduciblility theorem), moreover it is known [1] that $J(f)$ is very thin. In Section 1, we give a sufficient condition that $J(f)$ is finite and give its bound. Let F be a function field over $\mathbb{Q}$ of an algebraic curve Γ defined by the equation $f(X, Y) = 0$, in other words, $F = \mathbb{Q}(x, y)$ where x is transcendental over $\mathbb{Q}$ and $f(x, y) = 0$. By a functional prime divisor of F, we mean an equivalence class of nontrivial valuations of F which are trivial on $\mathbb{Q}$. For a functional prime divisor P, we denote by v_P the normalized valuation (i.e., its valuation group is $\mathbb{Z}$) belonging to P. A functional prime P is called a pole of $z \in \mathbb{Q}(x, y)$ if $v_P(z) < 0$. For each $f(X, Y) \in \mathbb{Z}[X, Y]$, its height denoted by $H(f)$ is defined to be the maximum of absolute values of coefficients of $f(X, Y)$. In Section 1, we prove

Theorem 1. *Let $f(X, Y)$ be an irreducible polynomial with integer coefficients and $F = \mathbb{Q}(x, y)$ its function field. Assume there are more than $\deg_Y(f)/2$ poles of x. Then there are only finitely many integers $n \in \mathbb{Z}$ such that $f(n, Y)$ is reducible.*

Moreover, if $f(n, Y)$ is reducible, then

$$|n| < (H(f)+1)^C$$

where C is a constant determined by the degree of $f(X, Y)$.

Let us give an example. Let

$$f(X, Y) = X^4 - Y^4 + g(X, Y)$$

be an irreducible polynomial where $\deg(g(X, Y)) \leq 3$. Let $F = \mathbb{Q}(x, y)$ be its function field. There are 3 poles of x corresponding to irreducible factors of $X^4 - Y^4$. Hence the assumption of Theorem 1 is satisfied. So there are only finitely many integers n such that $n^4 - Y^4 + g(n, Y)$ is reducible and there is a constant C such that $n < (H(g)+1)^C$ for any integer n with $n^4 - Y^4 + g(n, Y)$ reducible. Next we consider examples not satisfying the conclusion of Theorem 1. For each natural number d, we let

$$f_d(x, y) = (x+y)^2(x+2y)^2 \cdots (x+dy)^2 - 2x^2 + 1$$

and let $F_d = \mathbb{Q}(x, y)$ be its function field. It is well known that there are infinitely many integers n such that $2n^2 - 1$ is a square, say k_n^2 for $k_n \in \mathbb{Z}$. Then for such integers n, $f_d(n, y) = (n+y)^2(n+2y)^2 \cdots (n+dy)^2 - k_n^2$ is reducible. Since x has $d = \deg_Y(f_d)/2$ poles, these examples mean that as far as degrees are concerned, the assumption in Theorem 1 that there are more than $\deg_Y(f)/2$ poles of x is best possible.

In order to prove Theorem 1, we use the fact that $^*\mathbb{Q}$ has a unique internal archimedean absolute value, so Theorem 1 cannot be generalized for algebraic number fields of finite degree. Next we consider the case that $R = K[t]$ is a polynomial ring where K is a field and t is transcendental over K. Let $f(X, Y, t) \in K[X, Y, t]$ be an irreducible polynomial. As before, let $F = K(x, y, t)$ where x is transcendental over $K(t)$ and $f(x, y, t) = 0$. We consider F as an algebraic function field of one variable over $K(t)$. So a functional prime divisor of F is an equivalence class of nontrivial valuations of F which are trivial on $K(t)$. In Section 2, we prove

Theorem 2. *Let $f(X, Y, t)$ be an irreducible polynomial with coefficients in a field K. Let F be its function field of one variable over $K(t)$. Assume there are more than $\deg_Y(f)/2$ poles of x. Let $\varphi(t) \in K[t]$. If $f(\varphi(t), Y, t)$ is reducible, then*

$$\deg(\varphi(t)) < C(\deg_t(f(X, Y, t)) + 1)$$

where C is a constant determined by $\deg_X(f(X, Y, t))$ and $\deg_Y(f(X, Y, t))$.

Both Theorems 1 and 2 are proved by a nonstandard method, so we assume the reader is familiar with nonstandard arithmetic on the rational number field [2] and on rational function fields [3].

1.

Let $^*\mathbb{Q}$ and $^*\mathbb{Z}$ denote enlargements of the rational number field $\mathbb{Q}$ and the integer ring $\mathbb{Z}$ respectively where by an enlargement, we mean an elementary extension which satisfies the ω_1-saturation property. Let H be the height function of $^*\mathbb{Q}$; i.e. $H(\alpha/\beta) = \max(|\alpha|, |\beta|)$ where α and β are mutually prime nonstandard integers. A subfield Q_1 of $^*\mathbb{Q}$ is called H-convex if $x \in Q_1$ and $H(x) > H(y)$ imply $y \in Q_1$. In the rest of this section, Q_1 always denotes an H-convex subfield of $^*\mathbb{Q}$ and let $Z_1 = Q_1 \cap {}^*\mathbb{Z}$. Let x be a nonstandard integer not contained in Q_1. Then x is transcendental over Q_1 [4, Lemma 1]. Let F be a finite algebraic extension of $Q_1(x)$. (F is not necessary included in $^*\mathbb{Q}$.) Since $^*\mathbb{Q}F$ is a finite algebraic extension of $^*\mathbb{Q}$, $^*\mathbb{Q}F$ is internal. Let $\mathcal{O}$ be the ring of all algebraic integers in $^*\mathbb{Q}F$. Let K_1 denote the algebraic closure of Q_1 in $^*\mathbb{Q}F$. Then F is an algebraic function field of one variable over K_1. By a functional prime of F, we mean, as before, an equivalence class of nontrivial valuations of F which are trivial on K_1. Let $|x|_1, \ldots, |x|_s$ be all internal archimedean absolute values of $^*\mathbb{Q}F$ which induce in $^*\mathbb{Q}$ the ordinary absolute value. Since $s \leqslant [^*\mathbb{Q}F : {}^*\mathbb{Q}]$, s is finite. For each $z \in {}^*\mathbb{Q}F$, we define

$$I_\infty(z) = \{t \in \mathcal{O} \mid tz \in \mathcal{O}\}.$$

Lemma 1. *Let $z \notin K_1$. If for all $i \leqslant s$, there is $\gamma \in Z_1$ such that $|z|_i < \gamma$, then $I_\infty(z) \cap Z_1 = \{0\}$.*

Proof. Assume there exists a nonzero $t \in I_\infty(z) \cap Z_1$. Then $tz \in \mathcal{O}$. Since $|tz|_i < |t|\,\gamma$ for all $i \leqslant s$, tz is algebraic over Z_1, so $tz \in \mathcal{O} \cap K_1$, hence $z \in K_1$, a contradiction. □

For each $i \leqslant s$, let $R_i = \{z \in {}^*\mathbb{Q}F \mid |z|_i < \gamma$ for some $\gamma \in Z_1\}$. Then R_i is a valuation ring whose maximal ideal is $\{z \in {}^*\mathbb{Q}F \mid |z|_i < 1/|\gamma|$ for all $\gamma \in Z_1\}$. If $F \cap R_i$ is not trivial, namely $F \not\subset R_i$, then $F \cap R_i$ is a valuation ring. Since $F \cap R_i \supset K_1$, this valuation ring yields a functional prime P of F. We say that P is induced by an archimedean absolute value.

Let $R = \{z \in {}^*\mathbb{Q}F \mid \gamma z$ is an algebraic integer for some $\gamma \in Z_1\}$ and I a maximal ideal of R. Let R_I denote the local ring of R by I. If $F \cap R_I$ is not trivial, then $F \cap R_I$ is a valuation ring, hence it also yields a functional prime P of F. We say that P is induced by I.

Lemma 2 (cf. [4, Lemma 2], [2, Lemma 4.1]). *Every functional prime P of F is induced by an archimedean prime or a maximal ideal I of R.*

Proof. By the theorem of Riemann–Roch, there exists $z \in F$ which admits P as its only pole. If there is $i \leqslant s$ such that $|z|_i > \gamma$ for all $\gamma \in Z_1$, then $z \notin R_i$. Hence $z \notin F \cap R_i$. Then $F \cap R_i$ yields a functional prime which is a pole of z. Since P is the only functional pole of z, P is induced by an archimedean absolute value. Next assume for all $i \leq s$ there is $\gamma \in Z_1$ such that $|z|_i < \gamma$. By Lemma 1, $I_\infty(z) \cap Z_1 = \{0\}$. Hence $I_\infty(z)R$ is a proper ideal of R. Let I be a maximal ideal of R which includes $I_\infty(z)R$. Then the local ring of I does not contain z, so $z \notin F \cap R_I$. Hence $F \cap R_I$ is not trivial. By the same arguments as above P is induced by I. □

Proof of Theorem 1. Suppose Theorem 1 is false. Let $d \in \mathbb{N}$. For any natural number N, there exists an integer α and an irreducible polynomial $f(X, Y) \in \mathbb{Z}[X, Y]$ of degree d which satisfies the assumption of the theorem such that $f(\alpha, Y)$ is reducible and

$$|\alpha| > (H(f) + 1)^N. \tag{1}$$

By a nonstandard principle, the above assertion holds for any enlargement. We take $N \in {}^*\mathbb{N} - \mathbb{N}$. Then $f(X, Y) \in {}^*(\mathbb{Z}[X, Y])$, but since the degree of $f(X, Y)$ is $d \in \mathbb{N}$, $f(X, Y) \in {}^*\mathbb{Z}[X, Y]$, i.e., $f(X, Y)$ is a polynomial with coefficients in ${}^*\mathbb{Z}$. Let Q_1 be the smallest H-convex subfield of ${}^*\mathbb{Q}$ which contains all coefficients of $f(X, Y)$, i.e.,

$$Q_1 = \{z \in {}^*\mathbb{Q} \mid H(z) \leq (H(f) + 1)^n \text{ for some } n \in \mathbb{N}\}.$$

By (1), $\alpha \notin Q_1$. Since Q_1 is algebraically closed in ${}^*\mathbb{Q}$, α is transcendental over Q_1. Let $f(\alpha, Y) = f_1(\alpha, Y)f_2(\alpha, Y)$ where $f_1(X, Y)$, $f_2(X, Y) \in {}^*\mathbb{Z}[X, Y]$ and $1 \leq \deg_Y(f_1) \leq \deg_Y(f_2)$. Let $F = Q_1(\alpha, \beta)$ where β satisfies $f_1(\alpha, \beta) = 0$. Then

$$s \leq [{}^*\mathbb{Q}F : {}^*\mathbb{Q}] \leq \deg_Y(f_1) \leq \tfrac{1}{2}\deg_Y(f). \tag{2}$$

Since α is a nonstandard integer, by Lemma 2 every functional pole of α in F is induced by an archimedean absolute value in ${}^*\mathbb{Q}F$, so the number of functional poles of α is not more than s, hence by (2) not more than $\deg_Y(f)/2$. Let x be transcendental over ${}^*\mathbb{Q}$ and let y satisfy $f(x, y) = 0$. Then the number of functional poles of x in ${}^*\mathbb{Q}(x, y)$ is, by the assumption of the theorem, larger than $\deg_Y(f)/2$. But there is an embedding

$$\pi : F = Q_1(\alpha, \beta) \to {}^*\mathbb{Q}(x, y)$$

where $\pi(\alpha) = x$, $\pi(\beta) = y$ and for all $z \in Q_1$, $\pi(z) = z$. Since Q_1 is algebraically closed in ${}^*\mathbb{Q}$, the number of poles of α and x must be the same, this is a contradiction and it completes the proof of Theorem 1. □

Lemma 2 is very useful and has many applications in number theory other than the above. In the rest of this section we give one of them. Let $\varphi(X) \in \mathbb{Q}(X)$. It is easily proved that if for a sufficiently large $n \in \mathbb{N}$, $\varphi(n)$ is an integer, then $\varphi(X) \in \mathbb{Q}[X]$. We generalize this fact for algbraic function fields of one variable. Let $f(X, Y) \in \mathbb{Q}[X, Y]$ be irreducible and $\mathbb{Q}(x, y)$ its function field over $\mathbb{Q}$. For each integer n, let β_n be an algebraic number satisfying $f(n, \beta_n) = 0$. Let $\varphi(X, Y) = g(X, Y)/h(X, Y) \in \mathbb{Q}(X, Y)$ where $g(X, Y)$, $h(X, Y) \in \mathbb{Z}[X, Y]$ are coprime polynomials. We define $H(\varphi)=\max(H(g), H(h))$ and $D(\varphi)=\max(\deg(g), \deg(h))$. We prove

Theorem 3. *If there exists an integer $n > H(\varphi)^C$ such that $\varphi(n, \beta_n)$ is an algebraic integer, then $\varphi(x, y)$ is integral over $\mathbb{Q}[x]$ where $C = C(f, d)$ is a constant determined by $f(X, Y)$ and $d = D(\varphi)$.*

Proof. Assume otherwise. Let $d \in \mathbb{N}$. For any natural number N, there exists a $\varphi(X, Y) \in \mathbb{Q}(X, Y)$ with $d = D(\varphi)$ and an integer $n > H(\varphi)$ such that $\varphi(n, \beta_n)$ is an algebraic integer but $\varphi(x, y)$ is not integral over $\mathbb{Q}[x]$. The above statement is also valid in the enlargement, in other words, for any $N \in {}^*\mathbb{N}$, there exists a $\varphi(X, Y) \in {}^*\mathbb{Q}(X, Y)$ with $d = D(\varphi)$ and an integer $\alpha > H(\varphi)^N$ such that $\varphi(\alpha, \beta_\alpha)$ is an algebraic integer but $\varphi(x, y)$ is not integral over ${}^*(\mathbb{Q}[x])$. We take $N \in {}^*\mathbb{N} - \mathbb{N}$. Since $d = D(\varphi)$ is finite, $\varphi(X, Y) \in {}^*\mathbb{Q}(X, Y)$, i.e., $\varphi(X, Y)$ is a rational function with coefficients in ${}^*\mathbb{Q}$. Let Q_1 be the smallest H-convex subfield of ${}^*\mathbb{Q}$ which contains all coefficients of $\varphi(X, Y)$, i.e.,

$$Q_1 = \{z \in {}^*\mathbb{Q} \mid H(z) \leq (H(\varphi)+1)^n \text{ for some } n \in \mathbb{N}\}.$$

Since N is infinite and $\alpha > H(\varphi+1)^N$, $\alpha \notin Q_1$. Since Q_1 is algebraically closed in ${}^*\mathbb{Q}$, α is transcendental over Q_1. Let $F = Q_1(\alpha, \beta_\alpha)$. Then there is an embedding

$$\pi : F = Q_1(\alpha, \beta_\alpha) \to {}^*\mathbb{Q}(x, y)$$

where $\pi(\alpha) = x$, $\pi(\beta_\alpha) = y$ and for all $z \in Q_1$, $\pi(z) = z$. Since $\varphi(x, y)$ is not integral over ${}^*(\mathbb{Q}[x])$, $\varphi(\alpha, \beta_\alpha)$ is not integral over $Q_1[\alpha]$. Hence there is a functional prime P of F which is a pole of $\varphi(\alpha, \beta_\alpha)$ but is not a pole of α. Since $\varphi(\alpha, \beta_\alpha)$ is an algebraic integer, P cannot be induced by an maximal ideal I of R. Hence by Lemma 2, P is induced by archimedean primes only. Since $\alpha \in {}^*\mathbb{Z} - Z_1$, any functional prime which is induced by an archimedean prime is a pole of α, this is a contradiction. □

2.

Let K be a field and t transcendental over K. For each $x = g(t)/h(t) \in K(t)$, we define $D(x) = \max(\deg(g), \deg(h))$ where $g(X)$, $h(X) \in K[X]$ are coprime polynomials. As before, ${}^*(K(t))$ denotes an enlargment of the rational function field $K(t)$. The following lemma is well known.

Lemma 3 ([3], [5]). *${}^*K(t)$ is algebraically closed in ${}^*(K(t))$.*

${}^*K(t)$ is the set of all x with $D(x) \in \mathbb{N}$. A subfield L of ${}^*(K(t))$ is called D-convex if $x \in L$ and $D(x) > D(y)$ imply $y \in L$. Remark that $K(t)$ is not D-convex in ${}^*(K(t))$. In the following, L always denotes a D-convex subfield of

$^*(K(t))$ and we define $L_1 = L \cap {}^*(K[t]))$ and $N_1 = \{D(x) \in {}^*\mathbb{N} \mid x \in L\}$. The following lemma is a generalization of Lemma 3.

Lemma 4. *If L is a D-convex subfield of $^*(K(t))$, then L is algebraically closed in $^*(K(t))$.*

Proof. Let $\alpha(t)/\beta(t) \in {}^*(K(t)) - L$ where $\alpha(t)$, $\beta(t) \in {}^*(K[t])$ and $\gcd(\alpha(t), \beta(t)) = 1$. Since $\alpha(t) \notin L$, $\alpha(t) \notin L_1$ or $\beta(t) \notin L_1$. We may assume $\alpha(t) \notin L_1$. Assume $\alpha(t)/\beta(t)$ is algebraic over K. Then there are $\gamma_0(t), \gamma_1(t), \ldots, \gamma_n(t) \in L_1$ such that $\gamma_n(t) \neq 0$ and

$$\gamma_0(t)\left(\frac{\alpha(t)}{\beta(t)}\right)^n + \gamma_1(t)\left(\frac{\alpha(t)}{\beta(t)}\right)^{n-1} + \cdots + \gamma_n(t) = 0.$$

Then

$$\gamma_0(t)\alpha(t)^n + \gamma_1(t)\alpha(t)^{n-1}\beta(t) + \cdots + \gamma_n(t)\beta(t)^n = 0.$$

Hence

$$\gamma_n(t)\beta(t)^n \equiv 0 \quad \mod \alpha(t).$$

Since $\gcd(\alpha(t), \beta(t)) = 1$,

$$\gamma_n(t) \equiv 0 \quad \mod \alpha(t).$$

L is D-convex and $\alpha(t) \notin L_1 = L \cap {}^*(K[t])$, therefore $\deg(\gamma_n(t)) < \deg(\alpha(t))$. Hence

$$\gamma_n(t) = 0,$$

this is a contradiction. □

Let $x \in {}^*(K([t]) - L$ and F a finite algebraic extension of $L(x)$. Since $^*(K(t))F$ is a finite algebraic extension of $^*(K(t))$, $^*(K(t))F$ is internal. Let M denote the algebraic closure of L in $^*(K(t))F$. Then F is an algebraic function field of one variable over M. By a functional prime of F, we mean, as before, an equivalence class of nontrivial valuations of F which are trivial on M. Let $\mathcal{O}$ be the integral closure of $^*(K[t])$ in $^*K(t)F$. Let v_∞ be the valuation on $^*(K(t))$ such that $v_\infty(\alpha(t)/\beta(t)) = \deg(\beta(t)) - \deg(\alpha(t))$. Let $v_1, v_2, \ldots, v_s$ be all internal valuations of $^*(K(T))F$ which extend v_∞. Since $s \leqslant [{}^*(K(t))F : {}^*(K(t))]$, s is finite. For each $z \in {}^*(K(t))F$, we define

$$I_\infty(z) = \{\gamma(t) \in \mathcal{O} \mid \gamma(t)z \in \mathcal{O}\}.$$

Lemma 5. *Let $z \in {}^*(K(t))F - M$. If for all $i \leqslant s$, there is $x \in M$ such that $v_i(z) < v_i(x)$, then $I_\infty(z) \cap M = \{0\}$.*

Proof. For each $i \leqslant s$, let $R_i = \{z \in {}^*(K(t))F \mid v_i(z) > v_i(x)$ for some $x \in M\}$. Then R_i is a valuation ring whose maximal ideal is $\{z \in {}^*(K(t))F \mid v_i(z) > v_i(x)$

for all $x \in M\}$. If $F \cap R_i$ is not trivial, namely $F \not\subset R_i$, then $F \cap R_i$ is a valuation ring. Since $F \cap R_i \supset M$, this valuation ring yields a functional prime P of F. We say P is induced by an infinite prime of t.

Let $R = \{z \in {}^*(K(t))F \mid \gamma z \in \mathcal{O}$ for some $\gamma \in M\}$ and I a maximal ideal of R. Let R_I denote, as before, the local ring of R by I. If $F \cap R_I$ is not trivial, then $F \cap R_I$ is a valuation ring, hence it also yields a functional prime P of F. We say that P is induced by I. □

Lemma 6 (cf. Lemma 2). *Every functional prime P of F is induced by an infinite prime of t or a maximal ideal I of R.*

Proof. By the theorem of Riemann–Roch, there exists $z \in F$ which admits P as its only pole. If there is $i \leqslant s$ such that $v_i(z) \leqslant v_i(x)$ for all $x \in M$, then $z \neq R_i$, hence $F \not\subset R_i$, i.e., $F \cap R_i$ is not trivial, so it yields a functional prime which is a pole of z. Since P is the only functional pole of z, P is induced by an infinite prime of t. Next assume for all $i \leqslant s$ there is $x \in M$ such that $v_i(z) < v_i(x)$. Then by Lemma 5, $I_\infty(z)R$ is a proper ideal of R. Let I be a maximal ideal including $I_\infty(z)R$. Then $z \notin R_I$, hence $F \cap R_I$ is not trivial, so it yields a functional prime which is a pole of z because $z \notin F \cap R_I$. But P is the only pole of z, so P is induced by I. □

Proof of Theorem 2. The proof of Theorem 2 is essentially the same as that of Theorem 1. Suppose Theorem 2 is false. Let $d \in \mathbb{N}$. For any natural number N, there exist a polynomial $\varphi \in K[t]$ and an irreducible polynomial $f(X, Y, t) \in K[X, Y, t]$ with $\deg_X(f)$, $\operatorname{def}_Y(f) \leqslant d$ which satisfies the assumption of the theorem such that $f(\varphi(t), Y, t)$ is reducible and

$$\deg(\varphi(t)) > N(\deg_t((X, Y, t)) + 1). \tag{3}$$

By nonstandard principle, the above assertion holds for any enlargement. We take $N \in {}^*\mathbb{N} - \mathbb{N}$. Then $f(X, Y, t) \in {}^*(K[X, Y, t])$, but since the X-degree and the Y-degree of $f(X, Y, t)$ are at most $d \in \mathbb{N}$, $f(X, Y, t) \in {}^*(K[t])[X, Y]$, i.e., $f(X, Y, t)$ is a polynomial with coefficients in ${}^*(K[t])$. We define

$$L = \{z \in {}^*(K(t)) \mid D(z) < n(\deg_t(f(X, Y, t)) \text{ for some } n \in \mathbb{N}\}.$$

By (3), $\varphi(t) \notin L$. Since L is D-convex, L is algebraically closed in ${}^*(K(t))$, hence $\varphi(t)$ is transcendental over L. Let $f(\varphi(t), Y, t) = f_1(\varphi(t), Y, t)f_2(\varphi(t), Y, t)$ where $f_1(X, Y, t)$, $f_2(X, Y, t) \in {}^*(K[t])[X, Y]$ and $1 \leqslant \deg_Y(f_1) \leqslant \deg_Y(f_2)$. Let $F = L(\varphi(t), \psi)$ where ψ satisfies $f_1(\varphi(t), \psi, t) = 0$. Then

$$s \leqslant [{}^*(K(t))F : {}^*(K(t))] \leqslant \deg_Y(f_1) \leqslant \tfrac{1}{2}\deg_Y(f). \tag{4}$$

Since $\varphi(t)$ is a nonstandard polynomial, by Lemma 6 every functional pole of $\varphi(t)$ in F is induced by an infinite prime of t, so the number of functional poles of $\varphi(t)$ in F is not more than s, hence by (4) not more than $\deg_Y(f)/2$. Let x be transcendental over ${}^*(K(t))$ and let y satisfy $f(x, y, t) = 0$. Then the number of functional poles of x in $^*(K(t))$ is larger than $\deg_Y(f)/2$. But there is an embedding

$$\pi : F = L(\varphi, \psi) \rightarrow {}^*(K(t))(x, y)$$

where $\pi(\varphi) = x$, $\pi(\psi) = y$ and for all $z \in L$, $\pi(z) = z$. Since L is algebraically closed in $^*(K(t))$, the number of poles of φ and x must be same, this is a contradiction and it completes the proof of Theorem 2. □

References

[1] S.D. Cohen, The distribution of galois groups and Hilbert's irreducibility theorem, Proc. London Math. Soc. (3) 43 (1981) 227–250.
[2] A. Robinson and P. Roquette, On the finiteness theorem of Siegel and Mahler concerning diophantine equations, J. Number Theory 7 (1975) 121–176.
[3] P. Roquette, Nonstandard aspects of Hilbert's irreducibility theorem, Lecture Notes in Math. 498 (Springer, Berlin, 1975) 231–275.
[4] M. Yasumoto, Nonstandard arithmetic of function fields over H-convex subfields of $^*\mathbb{Q}$, J. Reine Angew. Math. 342 (1983) 1–11.
[5] M. Yasumoto, Nonstandard arithmetic of polynomial rings, Nayoya Math. J. 105 (1987) 33–37.
[6] M. Yasumoto, Algebraic extensions in nonstandard models and Hilbert's irreducibility theorem, J. Symbolic Logic 53 (1988) 470–480.