JOURNAL OF DIFFERENTIAL EQUATIONS 122, 114–136 (1995)

Dynamics of Almost Periodic Scalar Parabolic Equations

WENXIAN SHEN

Department of Mathematics, Auburn University, Auburn University, Alabama 36849

AND

YINGFEI YI*

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332

Received October 25, 1993; revised May 4, 1994

1. INTRODUCTION

The current paper is devoted to study of the asymptotic behavior of bounded solutions for the following type of parabolic equation:

$$u_t = u_{xx} + f(t, x, u, u_x), \quad t > 0, \quad 0 < x < 1, \tag{1.1}$$

with the boundary conditions:

$$\beta u(t, 0) + (1-\beta)\, u_x(t, 0) = 0, \ \beta u(t, 1) + (1-\beta)\, u_x(t, 1) = 0, \ t > 0, \tag{1.2}$$

where $\beta = 0$ or 1, $f\colon \mathbb{R}^1 \times [0, 1] \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$ is C^2, and $f(t, x, u, p)$ with all its partial derivatives (up to order 2) are almost periodic in t uniformly for (x, u, p) in compact subsets.

To carry out our study for the nonautonomous equation (1.1)–(1.2), we define a dynamical system associated to it in the following way. Let $C = C(\mathbb{R}^1 \times [0, 1] \times \mathbb{R}^1 \times \mathbb{R}^1, \mathbb{R}^1)$ be the space of continuous functions $F\colon \mathbb{R}^1 \times [0, 1] \times \mathbb{R}^1 \times \mathbb{R}^1 \to \mathbb{R}^1$. Give C the compact open topology, that is, the topology of uniform convergence on compact subsets. It follows from classical topological dynamical system theory ([26]) that the time translation $(F, t) \to F_t : F_t(s, x, u, p) = F(t + s, x, u, p)$ defines a flow on C, and the *hull of* f, $H(f) = cl\{f_t \mid t \in \mathbb{R}^1\}$ is an *almost periodic minimal set* (that is, $H(f)$ is minimal and each motion in $H(f)$ is almost periodic). Furthermore, each $g \in H(f)$ is also a C^2 function (see [17]). By introducing the

* Partially supported by NSF Grant DMS 9207069.

0022-0396/95 $12.00

hull $H(f)$, (1.1)–(1.2) gives rise to a family of equations associated to each $g \in H(f)$,

$$u_t = u_{xx} + g(t, x, u, u_x), \qquad t > 0, \qquad 0 < x < 1,$$
$$\beta u(t, 0) + (1-\beta)\, u_x(t, 0) = 0, \qquad \beta u(t, 1) + (1-\beta)\, u_x(t, 1) = 0, \qquad t > 0. \tag{1.3$_g$}$$

Let X be a fractional power space associated with the operator $u \to -u_{xx} : \mathscr{D} \to L^2(0, 1)$ that satisfies $X \subsetneq C^1[0, 1]$, where $\mathscr{D} = \{u \mid u \in H^2(0, 1),\ u \text{ satisfies } (1.2)\}$. Then $F: \mathbb{R}^1 \times X \to L^2(0, 1)$, $F(t, u)(x) = f(t, x, u, u_x)$, is well defined, and for any $U \in X$, equation $(1.3)_g$ admits (locally) a unique solution $u(t, \cdot, U, g)$ in X with $u(0, \cdot, U, g) = U(\cdot)$. This solution also continuously depends on $g \in H(f)$ and $U \in X$ ([16]). Therefore, $(1.3)_g$ defines a (local) skew product semiflow $\prod_t$ on $X \times H(f)$:

$$\textstyle\prod_t (U, g) = (u(t, \cdot, U, g), g \cdot t),\ t > 0, \tag{1.4}$$

where $g \cdot t$ is the flow on $H(f)$ defined by time translations.

In the terminology of the (local) skew product semiflow (1.4), the study of asymptotic behavior for a bounded solution $u(t, x)$ of (1.1)–(1.2) then gives rise to the problem of understanding the ω-limit set $\omega(U_0, f)$ of the bounded motion $\prod_t (U_0, f)$ in $X \times H(f)$, where $U_0(x) = u(0, x)$. Following from the work in [16] and the standard a priori estimates for parabolic equations, we know that if $u(t, \cdot, U, g)$ $(U \in X)$ is bounded in X for t in the existence interval of the solution, then u is a globally defined classical solution; moreover, for any $\delta > 0$, $\{u(t, \cdot, U, g) \mid t \geqslant \delta\}$ is relatively compact both in X and in $H^2(0, 1)$. Therefore $\omega(U, g)$ is a nonempty connected compact subset of $X \times H(f)$. Furthermore, since $\prod_t$ on the ω-limit set $\omega(U, g)$ has a unique continuous backwards time extension ([15]), it defines a usual skew product (two sided) flow on $\omega(U, g)$.

In the case that f is time periodic with period T, it is well known that each bounded solution $u(t, x, U_0, f)$ of (1.1)–(1.2) approaches a periodic solution with period T (see [4], [7], and references therein). In the language of skew product semiflow (1.4), this is to say that each ω-limit set $\omega(U, g)$ $(g \in H(f) \sim S^1)$ is a *periodic minimal set* in $X \times H(f)$ with period T (that is, $\omega(U, g)$ is minimal and each motion in $\omega(U, g)$ is periodic with period T) (in the autonomous case, each ω-limit set is an equilibrium, see [5], [21], and references therein). Nevertheless, similar results are false in general for time almost periodic equation (1.1)–(1.2), namely, one does not always expect an ω-limit set $\omega(U, g)$ to be an almost periodic minimal set in $X \times H(f)$. There are examples in scalar ODEs which suggest that the ω-limit sets of (1.4) may not be minimal (see [24]), and the ω-limit set may not be almost periodic minimal even if it is minimal (see [13], [19]).

Two natural questions then arise in the study of (1.1)–(1.2): (1) what kind of structure can one expect for an ω-limit set $\omega(U, g)$ of (1.4) if it is not minimal? (2) Does an ω-limit set still carry over some "oscillation" properties of the original system (1.1)–(1.2) if it is not an almost periodic minimal set? The current paper gives partial answers to these questions. We shall prove that for the (local) skew product semiflow (1.4), each ω-limit set $\omega(U, g)$ contains at most two (obviously at least one) minimal invariant sets, and each minimal invariant set contained in $\omega(U, g)$ is a *proximal extension* of $H(f)$ (see definition in section 3). In the case where two minimal invariant sets appear in the ω-limit set $\omega(U, g)$, both are *almost automorphic extensions* of $H(f)$ (see definition in section 3). If $\omega(U, g)$ is distal (see definition in section 3) or almost periodic minimal, then it must be an *almost periodic extension* of $H(f)$ (see definition in section 3), and therefore the frequency module of any almost periodic solution of (1.1)–(1.2) (if exists) is contained in that of f.

There is an example (see section 4) showing that an ω-limit set of (1.4) may contain two minimal sets, and at least one of them is an elmost automorphic but not an almost periodic extension of $H(f)$. In the case that an ω-limit set $\omega(U, g)$ contains precisely one minimal set E, we have shown in some special situation that E is actually an almost automorphic extension of $H(f)$ (see section 3). However, we conjecture that any minimal set E of (1.4) is an almost automorphic extension of $H(f)$.

For time almost periodic equation (1.1)–(1.2), it is important to know the existence of almost periodic solutions. This issue has been studied for both PDEs and ODEs by various authors (see [13], [24], [26], [27], [30], [31], [32], and references therein). In this paper, we shall also discuss cases in which (1.4) admits almost periodic minimal ω-limit sets.

We remark that our results hold true for more general equations:

$$\begin{gathered} u_t = a(x, t)\, u_{xx} + f(t, x, u, u_x), \qquad t > 0, \qquad x \in (0, 1) \\ \alpha_0 u(t, 0) + \beta_0 u_x(t, 0) = f_0(t),\ \alpha_1 u(t, 1) + \beta_1 u_x(t, 1) = f_1(t),\ t > 0, \end{gathered} \tag{1.5}$$

where $\alpha_i^2 + \beta_i^2 \neq 0$ $(i = 0, 1)$, f is as in (1.1)–(1.2), $a \geqslant \delta > 0$ is smooth and almost periodic in t uniformly for $x \in [0, 1]$, f_i $(i = 0, 1)$ are almost periodic functions. For the case of (1.1) with periodic boundary conditions, relevant results should hold following arguments in the current paper. We shall discuss this issue separately.

As in [4], [7], [9], [10], [21], the zero number properties developed in [1], [20] play important roles in our current studies. For other dynamic studies of scalar parabolic equations, we refer readers to [2], [6], [8], [12], [22], [25], etc.

2. Structure of ω-Limit Sets

For a given C^1 function u: $[0, 1] \to \mathbb{R}^1$, the zero number of u is defined as

$$Z(u(\cdot)) = \#\{x \in (0, 1) \mid u(x) = 0\}.$$

We first summarize zero number properties from [1], [20].

Lemma 2.1. *Consider the following scalar linear parabolic equation*:

$$\begin{gathered} u_t = a(t, x)\, u_{xx} + b(t, x)\, u_x + c(t, x) u, \qquad t > 0, \qquad x \in (0, 1), \\ \beta u(t, 0) + (1-\beta)\, u_x(t, 0) = 0,\ \beta u(t, 1) + (1-\beta)\, u_x(t, 1) = 0,\ t > 0, \end{gathered} \tag{2.1}$$

where a, a_t, a_x, a_{xx}, b, b_t, b_x *and* c *are bounded continuous functions,* $a \geqslant \delta > 0$. *Let* $u(t, x)$ *be a classical nontrivial solution of* (2.1). *Then the following holds*:

(1) $Z(u(t, \cdot))$ *is finite for* $t > 0$ *and is nonincreasing in* t;

(2) $Z(u(t, \cdot))$ *can drop only at* t_0 *such that* $u(t_0, \cdot)$ *has a multiple zero in* $[0, 1]$;

(3) $Z(u(t, \cdot))$ *can drop only finite many times, and there exists a* $t^* > 0$ *such that* $u(t, \cdot)$ *has only simple zeros in* $[0, 1]$ *as* $t \geqslant t^*$ (*hence* $Z(u(t, \cdot)) =$ *constant as* $t \geqslant t^*$).

Next, consider the (local) skew product semiflow $\prod_t$: $X \times H(f) \to X \times H(f)$ defined in (1.4). Recall $\prod_t (U, g) = (u(t, \cdot, U, g), g \cdot t)$, where $u(t, x, U, g)$ is the solution of $(1.3)_g$ with $u(0, x, U, g) = U(x)$. Let P: $X \times H(f) \to H(f)$, $(u, g) \mapsto g$ be the natural projection.

Lemma 2.2. *Consider* (1.4) *and fix* $g, g^* \in H(f)$. *Let* $(U_i, g) \in P^{-1}(g)$, $(U_i^*, g^*) \in P^{-1}(g^*)$ $(i = 1, 2,\ U_1 \neq U_2,\ U_1^* \neq U_2^*)$ *be such that* $\prod_t (U_i, g)$ *is defined on* $\mathbb{R}^+$ *or* $\mathbb{R}^-$ *and* $\prod_t (U_i, g^*)$ *is globally defined on* $\mathbb{R}^1$. *If there exists a sequence* $\{t_n\}$, $t_n \to +\infty$ *or* $-\infty$ *as* $n \to \infty$, *such that* $\prod_{t_n} (U_i, g) \to (U_i^*, g^*)$ *as* $n \to \infty$ $(i = 1, 2)$, *then* $Z(u(t, \cdot, U_1^*, g^*) - u(t, \cdot, U_2^*, g^*)) =$ *constant for all* $t \in \mathbb{R}^1$.

Proof. Denote $u_i(t, x) = u(t, x, U_i, g)$, $u_i^*(t, x) = u(t, x, U_i^*, g^*)$ $(i = 1, 2)$. Then $V(t, x) \equiv u_1(t, x) - u_2(t, x)$ satisfies the following linear parabolic equation:

$$\begin{gathered} V_t = V_{xx} + b(t, x)\, V_x + c(t, x)\, V, \qquad t > 0, \qquad 0 < x < 1, \\ \beta V(t, 0) + (1-\beta)\, V_x(t, 0) = 0,\ \beta V(t, 1) + (1-\beta)\, V_x(t, 1) = 0, \quad t > 0, \end{gathered} \tag{2.2}$$

where

$$b(t, x) = \int_0^1 g_p(t, x, u_1(t, x), su_{1x}(t, x) + (1-s)\, u_{2x}(t, x))\, ds,$$

$$c(t, x) = \int_0^1 g_u(t, x, su_1(t, x) + (1-s)\, u_2(t, x), u_{2x}(t, x))\, ds.$$

Similarly, $V^*(t, x) \equiv u_1^*(t, x) - u_2^*(t, x)$ also satisfies a linear parabolic equation of form (2.2). By Lemma 2.1, there is $T > 0$ such that $Z(V(t, \cdot)) =$ constant, $Z(V^*(t, \cdot)) =$ constant, for $t \geqslant T$ (note $V(t, \cdot)$, $V^*(t, \cdot)$ have only simple zeros in $[0, 1]$ as $t \geqslant T$).

Case 1. Suppose that $t_n \to \infty$ as $n \to \infty$. Let $t_0 \in \mathbb{R}^1$ be such that $V^*(t_0, \cdot)$ has only simple zeros in $[0, 1]$. It then follows that

$$Z(V(t_n + t_0, \cdot)) = Z(V^*(t_0, \cdot)) \tag{2.3}$$

as $n \gg 1$. Since $Z(V(t_n + t_0, \cdot)) = Z(V(T, \cdot))$ for $n \gg 1$, one has

$$Z(V^*(t_0, \cdot)) = Z(V(T, \cdot)). \tag{2.4}$$

Note that (2.4) holds for any t_0 such that $V^*(t_0, \cdot)$ has only simple zeros. By Lemma 2.1, we conclude that $Z(V^*(t, \cdot)) =$ constant for all $t \in \mathbb{R}^1$.

Case 2. Suppose that $t_n \to -\infty$ as $n \to \infty$. Let $T_0 > 0$ be such that $Z(V^*(t, \cdot)) =$ constant as $t \geqslant T_0$. Since $V(s_n + T_0, \cdot) \to V^*(T_0, \cdot)$ as $n \to \infty$, there is an integer $N > 0$ such that $Z(V(s_n + T_0, \cdot)) = Z(V^*(T_0, \cdot))$ as $n \geqslant N$. It follows that $Z(V(t, \cdot)) \equiv Z(V^*(T_0, \cdot))$ as $t \leqslant t_N + T_0$. Let $T = -t_N - T_0$, and without loss of generality, assume that $T \geqslant T_0$. Then

$$Z(V(-t, \cdot)) \equiv Z(V^*(T_0, \cdot)) \qquad \text{for} \quad t \geqslant t_N + T_0. \tag{2.5}$$

If $t_0 \in \mathbb{R}^1$ is such that $V^*(t_0, \cdot)$ has only simple zeros in $[0, 1]$, then $Z(V(-T, \cdot)) = Z(V(t_n + t_0, \cdot)) = Z(V^*(t_0, \cdot))$ as $n \gg 1$. This implies that $Z(V^*(T_0, \cdot)) = Z(V^*(t_0, \cdot))$. Since such a t_0 is arbitrary chosen, by Lemma 2.1, $Z(V^*(t, \cdot)) =$ constant for all $t \in \mathbb{R}^1$. ∎

LEMMA 2.3. *Let $E \subset X \times H(f)$ be a minimal invariant set of (1.4). Then for any $g \in H(f)$ and any two points (U_1, g), $(U_2, g) \in E \cap P^{-1}(g)$, there are sequences $\{t_n\}$, $\{s_n\}$ with $t_n \to \infty$, $s_n \to -\infty$ as $n \to \infty$ such that*

$$u(t_n, \cdot, U_1, g) - u(t_n, U_2, g) \to 0,$$

and

$$u(s_n, \cdot, U_1, g) - u(s_n, \cdot, U_2, g) \to 0,$$

as $n \to \infty$.

Proof. We only prove the existence of $\{t_n\}$ with $t_n \to \infty$ and $u(t_n, \cdot, U_1, g) - u(t_n, \cdot, U_2, g) \to 0$ as $n \to \infty$. Denote $V(t, x) = u(t, x, U_1, g) - u(t, x, U_2, g)$. If such $\{t_n\}$ does not exist, then there is a $\delta > 0$ such that $\|V(t, \cdot)\| \geq \delta$ for all $t > 0$. Let $\{t_n\}$ be a sequence such that $t_n \to \infty$ and $\prod_{t_n}(U_i, g)$ converge to some $(U_i^*, g^*) \in E$ as $n \to \infty$ $(i = 1, 2)$. Denote $V^*(t, x) = u(t, x, U_1^*, g^*) - u(t, x, U_2^*, g^*)$. Then $\|V^*(t, \cdot)\| \geq \delta$ for all $t \in \mathbb{R}^1$. It follows from Lemma 2.2 that $Z(V^*(t, \cdot)) = \text{constant}$ for all $t \in \mathbb{R}^1$. Thus, $(1-\beta)\, V^*(t, 0) + \beta V_x^*(t, 0)$ has constant sign for all $t \in \mathbb{R}^1$. Without loss of generality, we assume that

$$(1-\beta)\, V^*(t, 0) + \beta V_x^*(t, 0) > 0 \qquad \text{for} \quad t \in \mathbb{R}^1. \tag{2.6}$$

Define $u_{\min} = \min\{(1-\beta)\, U(0) + \beta U_x(0) \mid (U, g^*) \in E \cap P^{-1}(g^*)\}$. Let $(\tilde{U}, g^*) \in E \cap P^{-1}(g^*)$ be such that $(1-\beta)\, \tilde{U}(0) + \beta \tilde{U}_x(0) = u_{\min}$. Since E is minimal, there exists a sequence $\{\tilde{t}_n\}$ with $\tilde{t}_n \to \infty$ such that $\prod_{\tilde{t}_n}(U_1^*, g^*) \to (\tilde{U}, g^*)$ as $n \to \infty$. Take subsequence if necessary, we assume that $\prod_{\tilde{t}_n}(U_2^*, g^*)$ converges to some $(\hat{U}, g^*) \in E \cap P^{-1}(g^*)$ as $n \to \infty$. Denote $\tilde{V}(t, x) = u(t, x, \tilde{U}, g^*) - u(t, x, \hat{U}, g^*)$. Again, one has $\|\tilde{V}(t, \cdot)\| \geq \delta$ for $t \in \mathbb{R}^1$, and

$$Z(\tilde{V}(t, \cdot)) = \text{constant}, \qquad \text{for} \quad t \in \mathbb{R}^1. \tag{2.7}$$

By (2.6), it is clear that $(1-\beta)\, \hat{U}(0) + \beta \hat{U}_x(0) = u_{\min}$. Therefore, $(1-\beta)\, \tilde{V}(0, 0) + \beta \tilde{V}_x(0, 0) = 0$. Combinning the above with the boundary condition (1.2), one has that $\tilde{V}(0, 0) = 0$, $\tilde{V}_x(0, 0) = 0$, that is, $\tilde{V}(0, \cdot)$ has a multiple zero at $x = 0$. This contradicts with (2.7). ∎

LEMMA 2.4. *Let $E_1, E_2 \subset X \times H(f)$ be two minimal invariant sets of (1.4) (hence the flows on E_1, E_2 are two-sided). Then there is an integer $N > 0$ such that for any $g \in H(f)$ and any $(U_i, g) \in E_i \cap P^{-1}(g)$ $(i = 1, 2)$, one has $Z(U_1(\cdot) - U_2(\cdot)) = N$.*

Proof. *Claim* 1. For any $g \in H(f)$, and any $(U_i, g) \in E_i \cap P^{-1}(g)$ $(i = 1, 2)$, there is a $T > 0$ such that $Z(u(t, \cdot, U_1, g) - u(t, \cdot, U_2, g)) = \text{constant}$ for $t \leq -T$.

To prove the claim, we take a sequence $\{t_n\}$ such that $t_n \to -\infty$ and $\prod_{t_n}(U_i, g)$ $(i = 1, 2)$ converges to some $(U_i^*, g^*) \in E_i \cap P^{-1}(g^*)$ $(i = 1, 2)$ as $n \to \infty$. The claim then follows from arguments in Lemma 2.2.

Claim 2. For any $g \in H(f)$, and any $(U_1, g) \in E_1 \cap P^{-1}(g)$, there is a $(U_2, g) \in E_2 \cap P^{-1}(g)$ such that $Z(u(t, \cdot, U_1, g) - u(t, \cdot, U_2, g)) =$ constant for all $t \in \mathbb{R}^1$.

To show the claim, we fix a $(\tilde{U}_{2,g}) \in E_2 \cap P^{-1}(g)$. By minimality of E_i $(i = 1, 2)$, there is a sequence $\{t_n\}$ with $t_n \to \infty$ such that $\prod_{t_n}(U_1, g) \to (U_1, g)$, and $\prod_{t_n}(\tilde{U}_2, g)$ converges to some point $(U_2, g) \in E_2 \cap P^{-1}(g)$ as $n \to \infty$. Again, the claim follows from Lemma 2.2.

We are now ready to prove the lemma. First, for given $g \in H(f)$, let $(U_i, g) \in E_i \cap P^{-1}(g)$ $(i = 1, 2)$ be chosen. By claim 1, there is a $T > 0$ and integers N_1, N_2 such that

$$Z(u(t, \cdot, U_1, g) - u(t, \cdot, U_2, g)) = \begin{cases} N_1, & t \geqslant T, \\ N_2, & t \leqslant -T. \end{cases} \tag{2.8}$$

It follows from claim 2 that there is a $(\tilde{U}_2, g) \in E_2 \cap P^{-1}(g)$ and an integer N such that

$$Z(u(t, \cdot, U_1, g) - u(t, \cdot, \tilde{U}_2, g)) = N \qquad \text{for} \quad t \in \mathbb{R}^1. \tag{2.9}$$

By Lemma 2.3, there are sequences $\{t_n\}$, $\{s_n\}$ with $t_n \to \infty$ and $s_n \to -\infty$ such that $u(t_n, \cdot, U_2, g) - u(t_n, \cdot, \tilde{U}_2, g) \to 0$, $u(s_n, \cdot, U_2, g) - u(s_n, \cdot, \tilde{U}_2, g) \to 0$ as $n \to \infty$. Without of loss generality, we assume that there are g^*, $g^{**} \in H(f)$ and points $(U_i^*, g^*) \in E_i \cap P^{-1}(g^*)$, $(U_i^{**}, g^{**}) \in E_i \cap P^{-1}(g^{**})$ $(i = 1, 2)$ such that $\prod_{t_n}(U_i, g) \to (U_i^*, g^*)$, $\prod_{s_n}(U_i, g) \to (U_i^{**}, g^{**})$ $(i = 1, 2)$ as $n \to \infty$. By Lemma 2.2 and (2.8), one has

$$Z(u(t, \cdot, U_1^*, g) - u(t, \cdot, U_2^*, g)) = N_1, \qquad t \in \mathbb{R}^1, \tag{2.10}$$

and

$$Z(u(t, \cdot, U_1^{**}, g^{**}) - u(t, \cdot, U_2^{**}, g^{**})) = N_2, \qquad t \in \mathbb{R}^1. \tag{2.11}$$

Since $\prod_{t_n}(\tilde{U}_2, g) \to (U_2^*, g^*)$, $\prod_{s_n}(\tilde{U}_2, g) \to (U_2^{**}, g^{**})$ as $n \to \infty$, it follows from (2.9) and (2.10) that $N = N_1$ and from (2.9) and (2.11) that $N = N_2$. Thus $N_1 = N_2$. By (2.8), we see that

$$Z(u(t, \cdot, U_1, g) - u(t, \cdot, U_2, g)) = N \qquad \text{for} \quad t \in \mathbb{R}^1. \tag{2.12}$$

Next, for given $g \in H(f)$, take any (U_i, g), $(\tilde{U}_i, g) \in E_i \cap P^{-1}(g)$ $(i = 1, 2)$. By (2.12), we know that there are integers N_1, N_2, such that

$$Z(u(t, \cdot, U_1, g) - u(t, \cdot, U_2, g)) = N_1, \quad t \in \mathbb{R}^1, \tag*{(2.13)$_1$}$$

and

$$Z(u(t, \cdot, \tilde{U}_1, g) - u(t, \cdot, \tilde{U}_2, g)) = N_2, \quad t \in \mathbb{R}^1. \tag{2.13$_2$}$$

Applying Lemma 2.3, and using the above arguments, one finds

$$\begin{aligned} N_1 &= Z(u(t, \cdot, U_1, g) - u(t, \cdot, \tilde{U}_2, g)) \\ &= Z(u(t, \cdot, \tilde{U}_1, g) - u(t, \cdot, \tilde{U}_2, g)) = N_2. \end{aligned}$$

Finally, take any g^*, $g^{**} \in H(f)$, and $(U_i^*, g^*) \in E_i \cap P^{-1}(g^*)$, $(U_i^{**}, g^{**}) \in E_i \cap P^{-1}(g^{**})$ $(i = 1, 2)$. By the minimality of E_i $(i = 1, 2)$, there exist $(\tilde{U}_i^*, g^*) \in E_i$ $(i = 1, 2)$ and sequence $\{t_n\}$ with $t_n \to \infty$ such that $\prod_{t_n}(\tilde{U}_i, g^*) \to (U_i^{**}, g^{**})$ $(i = 1, 2)$ as $n \to \infty$. By the above argument and Lemma 2.1, one has

$$\begin{aligned} Z(u(t, \cdot, U_1^*, g^*) - u(t, \cdot, U_2^*, g^*)) &= Z(u(t, \cdot, \tilde{U}_1^*, g^*) - u(t, \cdot, \tilde{U}_2^*, g^*)) \\ &= Z(u(t, \cdot, U_1^{**}, g^{**}) - u(t, \cdot, U_2^{**}, g^{**})) \\ &= \text{constant} \end{aligned}$$

for $t \in \mathbb{R}^1$. This proves the lemma. ∎

LEMMA 2.5. *Let E_1, $E_2 \subset X \times H(f)$ be two minimal invariant sets of (1.4). Define*

$$\begin{aligned} a_i(g) &= \min\{(1-\beta)\, U(0) + \beta U_x(0) \mid (U, g) \in E_i \cap P^{-1}(g)\} \\ b_i(g) &= \max\{(1-\beta)\, U(0) + \beta U_x(0) \mid (U, g) \in E_i \cap P^{-1}(g)\}. \end{aligned} \tag{2.14}$$

Then E_1, E_2 are separated in the following sense:

(1) *$[a_1(g), b_1(g)] \cap [a_2(g), b_2(g)] = \varnothing$ for all $g \in H(f)$;*

(2) *Without loss of generality, assume that $a_1(g_0) - b_2(g_0) > 0$ for some $g_0 \in H(f)$. Then there is a $\delta > 0$ such that $a_1(g) - b_2(g) \geqslant \delta$ for all $g \in H(f)$.*

Proof. (1) Let

$$A_i(g) = \{(1-\beta)\, U(0) + \beta U_x(0) \mid (U, g) \in E_i \cap P^{-1}(g)\}, \; i = 1, 2.$$

We first claim that there is a $g_0 \in H(f)$ such that either $a_1(g_0) > b_2(g_0)$ or $a_2(g_0) > b_1(g_0)$. If not, then $a_1(g) \leqslant b_2(g)$, $a_2(g) \leqslant b_1(g)$ for all $g \in H(f)$. Now, fix g_1, $g_2 \in H(f)$. Let $(U_1, g_1) \in E_1$ be such that $(1-\beta)\, u_1(0) + \beta u_{1x}(0) = a_1(g_1)$, $(U_2, g_1) \in E_2$ be such that $(1-\beta)\, U_2(0) + \beta U_{2x}(0) = b_2(g_1)$. By Lemma 2.4, $(1-\beta)\; u(t, 0, U_1, g_1) + \beta u_x(t, 0, U_1, g) < (1-\beta)\; u(t, 0, U_2, g_1) + \beta u_x(t, 0, U_2, g_1)$ for all $t \in \mathbb{R}^1$. By minimality of E_1, there is a sequence

$\{t_n\}$ with $t_n \to \infty$ such that $(1-\beta)\, u(t_n, 0, U_1, g_1) + \beta u_x(t_n, 0, U_1, g_1) \to b_1(g_2)$ as $n \to \infty$. Without loss of generality, we assume $(1-\beta)\, u(t_n, 0, U_2, g_1) + \beta u_x(t_n, 0, U_1, g_1)$ converges to some $b(g_2) \in A_2(g_2)$ as $n \to \infty$. Hence $b_1(g_2) \leqslant b(g_2) \leqslant b_2(g_2)$. By Lemma 2.4, $b_1(g_2) \neq b_2(g_2)$. Thus $b_1(g_2) < b_2(g_2)$. Similarly, we have $b_2(g_2) < b_1(g_2)$. This is a contradiction.

Without loss of generality, we now assume that $b_2(g_0) < a_1(g_0)$ for some $g_0 \in H(f)$. We need to show that $b_2(g) < a_1(g)$ for all $g \in H(f)$. If this is not true, then there is a $g_* \in H(f)$ such that $b_2(g_*) \geqslant a_1(g_*)$. Let $(U_2, g_*) \in E_2$ be such that $(1-\beta)\, U_2(0) + \beta U_{2x}(0) = b_2(g_*)$, $(U_1, g_*) \in E_1$ be such that $(1-\beta)\, U_1(0) + \beta U_{1x}(0) = a_1(g_*)$. By minimality of E_1, one can find a sequence $\{t_n\}$ with $t_n \to \infty$ such that $(1-\beta)\, u(t_n, 0, U_2, g_*) + \beta u_x(t_n, 0, U_2, g_*) \to b_2(g_0)$, and, $(1-\beta)\, u(t_n, 0, U_1, g_*) + \beta u_x(t_n, 0, U_1, g_*) \to a(g_0)$ as $n \to \infty$, where $a(g_0)$ is some point in $A_1(g_0)$. Thus $b_2(g_0) \geqslant a(g_0) \geqslant a_1(g_0)$, a contradiction.

(2) We know by (1) that $b_2(g) < a_1(g)$ for all $g \in H(f)$. Now, suppose by contradiction that there is a sequence $\{g_n\} \subset H(f)$ such that $a_1(g_n) - b_2(g_n) \to 0$ as $n \to \infty$. Without loss of generality, we assume that $\{g_n\}$ converges to some $g^* \in H(f)$, $\{a_1(g_n)\}$ and $\{b_2(g_n)\}$ converge to some number c as $n \to \infty$. Since E_1, E_2 are both compact, then $c \in A_1(g^*) \cap A_2(g^*)$. This contradicts with (1). ∎

We now state our main theorem in this section.

THEOREM 2.6. *Let $(U_0, g_0) \in X \times H(f)$ be such that the motion $\prod_t (U_0, g_0)$ $(t > 0)$ is bounded. Then the ω-limit set $\omega(U_0, g_0)$ contains at most two minimal sets. More precisely, one of the following is true:*

(1) *$\omega(U_0, g_0) = E_1 \cup E_2 \cup E_{12}$, where E_1, E_2 are minimal sets, $E_{12} \neq \varnothing$. E_{12} connects E_1, E_2 in the sense that if $(U_{12}, g) \in E_{12}$, then $\omega(U_{12}, g) \cap (E_1 \cup E_2) \neq \varnothing$, $\alpha(U_{12}, g) \cap (E_1 \cup E_2) \neq \varnothing$ (where α is referred to as the α-limit set).*

(2) *$\omega(U_0, g_0) = E_1 \cup E_{11}$, where E_1 is minimal, $E_{11} \neq \varnothing$, E_{11} connects E_1 in the sense that if $(U_{11}, g) \in E_{11}$, then $\omega(U_{11}, g) \cap E_1 \neq \varnothing$, $\alpha(U_{11}, g) \cap E_1 \neq \varnothing$.*

(3) *$\omega(U_0, g_0)$ is a minimal invariant set.*

Proof. Suppose that $\omega(U_0, g_0)$ contains three minimal sets E_1, E_2 and E_3. Define

$$\begin{aligned} A_i(g) &= \{(1-\beta)\, U(0) + \beta U_x(0) \mid (U, g) \in E_i \cap P^{-1}(g)\}, \\ a_i(g) &= \min A_i(g), \\ b_i(g) &= \max A_i(g), \end{aligned} \tag{2.15}$$

($i=1, 2, 3$). By Lemma 2.5, without loss of generality, we assume that there is $\delta > 0$ such that

$$b_1(g)+\delta \leqslant a_2(g), \quad b_2(g)+\delta \leqslant a_3(g), \quad \forall g \in H(f). \tag{2.16}$$

Take $(U_i, g_0) \in E_i$ ($i=1, 2, 3$). Define $u_i(t)=(1-\beta)\, u(t, 0, U_i, g_0)+\beta u_x(t, 0, U_i, g_0)$ ($i=1, 2, 3$), and $u(t)=(1-\beta)\, u(t, 0, U_0, g_0)+\beta u_x(t, 0, U_0, g_0)$ for $t \geqslant 0$. By Lemma 2.1, there is a $T>0$ such that $u(t)-u_i(t)$ ($i=1, 2, 3$) has constant sign as $t \geqslant T$. Without loss of generality, we assume that $u(t)<u_2(t)$ as $t \geqslant T$. Next, fix a $g^* \in H(f)$ and let $\{t_n\}$ be a sequence such that $t_n \to \infty$, $g_0 \cdot t_n \to g^*$, $u(t_n) \to a_3(g^*)$ as $n \to \infty$. For such sequence $\{t_n\}$, we further assume that $u_2(t_n)$ converges to some $a(g^*) \in A_2(g^*)$ as $n \to \infty$. It then follows that $a_3(g^*) \leqslant a(g^*) \leqslant b_2(g^*) < b_2(g^*)+\delta$. This contradicts with (2.16). Thus, $\omega(U_0, g_0)$ contains at most two minimal sets. Now, write $\omega(U_0, g_0)=E_1 \cup E_2 \cup E_{12}$, where E_1, E_2 are minimal sets. If $E_1 \neq E_2$, since $\omega(U_0, g_0)$ is connected, $E_{12} \neq \emptyset$. Now, take $(U_{12}, g) \in E_{12}$. It is clear that $\omega(U_{12}, g) \cap (E_1 \cup E_2) \neq \emptyset$, and $\alpha(U_{12}, g) \cap (E_1 \cup E_2) \neq \emptyset$, for otherwise, either $\omega(U_{12}, g)$ or $\alpha(U_{12}, g)$ would contain a minimal set and therefore $\omega(U_0, g_0)$ would have three minimal sets. In the case $\omega(U_0, g_0)$ contains only one minimal set, that is, $E_1=E_2$. If $E_{11} \equiv E_{12} \neq \emptyset$, then a similar argument shows that $\omega(U_{11}, g) \cap E_1 \neq \emptyset$, $\alpha(U_{11}, g) \cap E_1 \neq \emptyset$ for any $(U_{11}, g) \in E_{11}$. ∎

We remark here that the above theorem is true for the α-limit set $\alpha(U_0, g_0)$ if it can be defined. As we mentioned in section 1, there is an example ([24]) in scalar ODEs which shows that certain ω-limit set in the corresponding skew-product flow is not minimal, and it contains only one minimal set. An example exhibiting the appearance of two minimal sets in an ω-limit set is provided in section 4. The following lemma can also be found in [29]. We give a different proof here since more detailed information in the proof is needed later on.

LEMMA 2.7. *Let $E \subset X \times H(f)$ be a minimal invariant subset. Then there is a residual subset $A_0 \subset H(f)$ which satisfies the following properties*:

(1) *For any $g_* \in A_0$, $g \in H(f)$ and any $(U_*, g_*) \in E \cap P^{-1}(g_*)$, if $\{t_n\}$ is a sequence with $t_n \to +\infty$ or $-\infty$ such that $g \cdot t_n \to g_*$ as $n \to \infty$, then there is a sequence $\{(U_n, g)\} \subset E \cap P^{-1}(g)$ such that $\prod_{t_n}(U_n, g) \to (U_*, g_*)$ as $n \to \infty$.*

(2) *Let 2^E be the set of all closed subset of E furnished with Hausdroff metric σ. Then $A_0=\{g \in H(f) \mid q : H(f) \to 2^E,\ g \mapsto E \cap P^{-1}(g)$ is continuous at $g\}$.*

Proof. Let 2^E be the set of all closed subset of E furnished with Hausdroff metric σ. Recall, for any $E_1, E_2 \in 2^E$, $\sigma(E_1, E_2) = \max\{\mu(E_1, E_2), \mu(E_2, E_1)\}$, where $\mu(E_1, E_2) = \max_{x \in E_1} \min_{x' \in E_2} d_E(x, x')$, d_E is the metric on E (note that the compact open topology on $H(f)$ is metrizable, see [26] or [33]). Now, consider the function $q: H(f) \to 2^E$, $g \mapsto E \cap P^{-1}(g)$. It is clear that q as a set valued map is upper semi-continuous. Let $A_0 \subset H(f)$ be the set of continuous points of q. Then $A_0 \subset H(f)$ is a resiudal subset ([3]). Take $g_* \in A_0$, $g \in H(f)$ and $(U_*, g_*) \in E \cap P^{-1}(g_*)$. Let $\{t_n\}$ be a sequence with $t_n \to +\infty$ or $-\infty$ such that $g \cdot t_n \to g_*$ as $n \to \infty$. Now, by lower semicontinuity of q as a set valued map, there is a sequence $\{(\tilde{U}_n, g \cdot t_n)\} \subset E$ such that $d_E((\tilde{U}_n, g \cdot t_n), (U_*, g_*)) \to 0$ as $n \to \infty$. Let $(U_n, g) = \prod_{-t_n} (\tilde{U}_n, g \cdot t_n)$ $(n = 1, 2, ...)$. (1) is proved.

(2) follows from (1) and the definition of Hausdroff metric. ∎

PROPOSITION 2.8. *Consider* (1.1)–(1.2). *Let* $(U_0, g_0) \in X \times H(f)$ *be such that the motion* $\prod_t (U_0, g_0)$ $(t > 0)$ *is bounded. If* $U_0(x) - U(x) \geqslant 0$ $(\leqslant 0)$ *for any* $x \in [0, 1]$, *and any* $(U, g_0) \in \omega(U_0, g_0) \cap P^{-1}(g_0)$, *then* $\omega(U_0, g_0)$ *contains only one minimal invariant set.*

Proof. Suppose that $\omega(U_0, g_0)$ has two minimal invariant sets E_1, E_2 $(E_1 \neq E_2)$. For each $g \in H(f)$, and $x \in [0, 1]$, define

$$\begin{aligned} a_i(g, x) &= \min\{U(x) \mid (U, g) \in E_i \cap P^{-1}(g)\}, \\ b_i(g, x) &= \max\{U(x) \mid (U, g) \in E_i \cap P^{-1}(g)\}, \end{aligned} \tag{2.17}$$

$(i = 1, 2)$. Suppose that $U_0(x) - U(x) \geqslant 0$ for any $(U, g_0) \in \omega(U_0, g_0) \cap P^{-1}(g_0)$. Then by standard strong maximal principal for parabolic equations ([14], [23]), we have $a_i(g_0 \cdot t, x) \leqslant u(t, x, U_0, g_0)$ and $b_i(g_0 \cdot t, x) \leqslant u(t, x, U_0, g_0)$ for any $x \in [0, 1]$ and $t \geqslant 0$ $(i = 1, 2)$. Let $A_i \subset E_i$ $(i = 1, 2)$ be the residual subset of $H(f)$ satisfying the property in Lemma 2.7, that is, for any $g_* \in A_i$, $g \in H(f)$ and any $(U_*, g_*) \in E_i \cap P^{-1}(g_*)$, if $\{t_n\}$ with $t_n \to +\infty$ or $-\infty$ is a sequence such that $g \cdot t_n \to g_*$ as $n \to \infty$, then there is a sequence $\{(U_n, g)\} \subset E \cap P^{-1}(g)$ such that $\prod_{t_n} (U_n, g) \to (U_*, g_*)$ as $n \to \infty$ $(i = 1, 2)$. Let $A_0 = A_1 \cap A_2$. Then $A_0 \subset H(f)$ is also a residual subset of $H(f)$. Fix any $g^* \in A_0$ and take any $x_0 \in [0, 1]$. Without loss of generality, we assume that $a_2(g^*, x_0) \leqslant b_1(g^*, x_0)$. Let $(U_2^*, g^*) \in E_2$ be such that $U_2^*(x_0) = a_2(g^*, x_0)$, and let $(U_1^*, g^*) \in E_1$ be such that $U_1^*(x_0) = b_1(g^*, x_0)$. Let $\{t_n\}$ with $t_n \to \infty$ be such that $\prod_{t_n} (U_0, g_0) \to (U_2^*, g^*)$ as $n \to \infty$. Then $u(t_n, x_0, U_0, g_0) \to U_2^*(x_0) = a_2(g^*, x_0)$ as $n \to \infty$. By Lemma 2.7, there are $(U_1^n, g_0) \in E_1 \cap P^{-1}(g_0)$, $n = 1, 2, ...,$ such that $\prod_{t_n} (U_1^n, g_0) \to (U_1^*, g^*)$ as $n \to \infty$. Hence $u(t_n, x_0, U_1^n, g_0) \to U_1^*(x_0) = b_1(g^*, x_0)$ as $n \to \infty$. By the above assumptions, $u(t_n, x_0, U_0, g_0) \geqslant u(t_n, x_0, U_1^n, g_0)$ for $t_n \geqslant 0$. This implies $b_1(g^*, x_0) \leqslant a_2(g^*, x_0)$. Therefore,

$a_2(g^*, x_0) = b_1(g^*, x_0)$. Similarly, we have $a_1(g^*, x_0) = b_2(g^*, x_0)$. Hence $U_1(x_0) = U_2(x_0)$ for any $(U_i, g^*) \in E_i$ $(i = 1, 2)$. Since $x_0 \in [0, 1]$ is arbitrary chosen, we have $U_1 \equiv U_2$ for any $(U_1, g^*) \in E_1$ and $(U_2, g^*) \in E_2$. This is a contradiction. ∎

Remark 2.1. By the arguments in the above proof, we actually have $\operatorname{card}(E \cap P^{-1}(g)) = 1$ for any $g \in A_0$, where E is the minimal invariant set in $\omega(U_0, g_0)$ in Proposition 2.8.

3. Lifting Properties of ω-Limit Sets

Definition 3.1. Consider the local skew product semiflow (1.4) and let E be an invariant set. For any $g \in H(f)$, a pair (U_1, g), $(U_2, g) \in E \cap P^{-1}(g)$ is said to be (*positively, negatively*) *proximal* if

$$\inf_{(t \in \mathbb{R}^+, \mathbb{R}^-)\, t \in \mathbb{R}^1} \|u(t, \cdot, U_1, g) - u(t, \cdot, U_2, g)\| = 0. \tag{3.1}$$

The pair (U_1, g), (U_2, g) is said to be *positively* (*negatively*) *distal* if it is not positively (negatively) proximal. It is said to be *distal* if it is neither positively nor negatively proximal.

Definition 3.2. Consider (1.4) and let $E \subset X \times H(f)$ be a compact invariant set.

(1) E is said to be an *almost periodic extension* of $H(f)$ if $\operatorname{card}(E \cap P^{-1}(g)) = 1$ for all $g \in H(f)$;

(2) E is an *almost automorphic extension* of $H(f)$ if there is a $g_0 \in H(f)$ such that $\operatorname{card}(E \cap P^{-1}(g_0)) = 1$;

(3) E is a *proximal extension* of $H(f)$ if any (U_1, g), $(U_2, g) \in E$ are either positively or negatively proximal;

(4) E is said to be (*negatively, positively*) *distal* if any (U_1, g), $(U_2, g) \in E (U_1 \neq U_2)$ forms a (negatively positively) distal pair.

Remark 3.1. It is clear that if E is an almost periodic extension (1-cover) of $H(f)$ (this implies that E is minimal), then for any $(U, g) \in E$, $u(t, \cdot, U, g)$ is an almost periodic solution of $(1.3)_g$. If E is an almost automorphic extension of $H(f)$ and is minimal, then it follows from [28], [29] that $H_0(f) = \{ g \in H(f) \mid \operatorname{card}(E \cap P^{-1}(g)) = 1\}$ is actually a residual subset of $H(f)$ (hence E is almost a 1-cover of $H(f)$). Points in $E \cap P^{-1}(g)$ $(g \in H_0(f))$ are called *almost automorphic points*. Let (U_0, g_0) be an almost automorphic point. Then it is easy to verify that $u(t, \cdot, U_0, g_0)$ is a (Bochner) *almost automorphic solution* of $(1.3)_{g_0}$ in the following sense: For any sequence $\{\alpha'_n\} \subset \mathbb{R}^1$, there exists a subsequence $\{\alpha_n\} \subset \{\alpha'_n\}$ and a function $v(t, x)$ $(v(t, \cdot) \in X)$ such that $u(t+\alpha_n, \cdot, U_0, g_0) \to v(t, \cdot)$, $v(t-\alpha_n, \cdot) \to u(t, \cdot, U_0, g_0)$ as $n \to \infty$.

DEFINITION 3.3. A motion $\prod_t (U_0, g_0)$ of (1.4) is said to be uniformly stable if for any $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that if

$$\|u(\tau, \cdot, U_0, g_0) - u(\tau, \cdot, U_1, g_0)\| < \delta(\varepsilon)$$

for some $(U_1, g_0) \in X \times H(f)$, and some $\tau \in \mathbb{R}^+$, then

$$\|u(t+\tau, \cdot, U_0, g_0) - u(t+\tau, \cdot, U_1, g_0)\| < \varepsilon$$

for all $t \in \mathbb{R}^+$.

Suppose that $E \subset X \times H(f)$ is a minimal invariant set. Define

$$\begin{aligned} a(g) &= \min\{(1-\beta)\, U(0) + \beta U_x(0) \mid (U, g) \in E \cap P^{-1}(g)\}, \\ b(g) &= \max\{(1-\beta)\, U(0) + \beta U_x(0) \mid (U, g) \in E \cap P^{-1}(g)\}, \end{aligned} \tag{3.3}$$

and

$$\tilde{E} = \{((1-\beta)\, U(0) + \beta U_x(0), g) \mid (U, g) \in E\}. \tag{3.4}$$

LEMMA 3.1. *Let $E \subset X \times H(f)$ be a minimal invariant set of* (1.4) *and $A_0 \subset H(f)$ be as in Lemma* 2.7. *Then $a(\cdot)$, $b(\cdot)$: $H(f) \to \mathbb{R}^1$ are continuous at $g \in A_0$.*

Proof. Denote the Hausdorff metric on $2^{\tilde{E}}$ by $\tilde{\sigma}$. Then it is easy to see that $\tilde{\sigma}(\tilde{E} \cap P^{-1}(g_1), \tilde{E} \cap P^{-1}(g_2)) \leqslant K \cdot \sigma(E \cap P^{-1}(g_1), E \cap P^{-1}(g_2))$ for any $g_1, g_2 \in H(f)$ and some $K > 0$ $(X \subsetneq C^1[0, 1])$. This implies that the function $\tilde{q}$: $H(f) \to 2^{\tilde{E}}$, $g \mapsto P^{-1}(g) \cap \tilde{E}$ is continuous on A_0. Therefore, functions $a(g)$, $b(g)$ are continuous on A_0. ▌

LEMMA 3.2. *Consider a $(U_0, g_0) \in X \times H(f)$ such that the motion $\prod_t (U_0, g_0)$ $(t > 0)$ of* (1.4) *is bounded. Let $E \subset \omega(U_0, g_0)$ be a minimal set. If there is a $T > 0$ such that $u(t, \cdot, U_0, g_0) - u(t, \cdot, U, g_0)$ has only simple zeros in* $[0, 1]$ *as $t \geqslant T$ for any $(U, g_0) \in E \cap P^{-1}(g_0)$, then E is an almost automorphic extension of $H(f)$.*

Proof. Let $A_0 \subset H(f)$ be as in Lemma 2.7. Since A_0 is the set of continuous points of q: $H(f) \to 2^E$, $g \mapsto E \cap P^{-1}(g)$, it is clear that $E_0 = \bigcup_{g \in A_0} E \cap P^{-1}(g) \subset E$ is $\prod_t$-invariant. We want to show that $\text{card}(E \cap P^{-1}(g)) = 1$ for any $g \in A_0$. Suppose this is not true. Then there is a $g_* \in A_0$ such that $\text{card}(E \cap P^{-1}(g_*)) > 1$. Now, take any two points

(U_1, g_*), $(U_2, g_*) \in E \cap P^{-1}(g_*)$. Let $\{t_n\}$ with $t_n \to \infty$ be such that $\prod_{t_n}(U_0, g_0) \to (U_1, g_*)$ as $n \to \infty$. By Lemma 2.7, there is a sequence $\{(U_n, g_0)\} \subset E \cap P^{-1}(g_0)$ such that $\prod_{t_n}(U_n, g_0) \to (U_2, g_*)$ as $n \to \infty$. By Lemma 2.1, we may assume without loss of generality that $U_1(\cdot) - U_2(\cdot)$ has only simple zeros in $[0, 1]$. Let $Z(U_1(\cdot) - U_2(\cdot)) = N$. Then $Z(u(t_n, \cdot, U_0, g_0) - u(t_n, \cdot, U_n, g_0)) = N$ as $n \gg 1$, and therefore $Z(u(t, \cdot, U_0, g_0) - u(t, \cdot, U_n, g_0)) = N$ for $t \geqslant T$. Now, let $t_0 \in \mathbb{R}^1$ be any number that $u(t_0, \cdot, U_1, g_*) - u(t_0, \cdot, U_2, g_*)$ has only simple zeros in $[0, 1]$. Since $\prod_{t_n + t_0}(U_0, g_0) \to \prod_{t_0}(U_1, g_*)$, $\prod_{t_n + t_0}(U_n, g_0) \to \prod_{t_0}(U_2, g_*)$ as $n \to \infty$, $Z(u(t_0, \cdot, U_1, g_*) - u(t_0, \cdot, U_2, g_*)) = N$. By Lemma 2.1, this implies that $Z(u(t, \cdot, U_1, g_*) - u(t, \cdot, U_2, g_*)) = N$ for all $t \in \mathbb{R}^1$.

Let $a(\cdot)$, $b(\cdot)$ be as in (3.3). By Lemma 3.1, functions $a(g)$, $b(g)$ are continuous on A_0. Note that for any $g \in A_0$ and any two points (U_1, g), $(U_2, g) \in E \cap P^{-1}(g)$, $U_1(\cdot) - U_2(\cdot)$ has only simple zeros in $[0, 1]$. This implies that for any $g \in A_0$, there is a unique $(U_1, g) \in E \cap P^{-1}(g)$ and a unique $(U_2, g) \in E \cap P^{-1}(g)$ such that $a(g) = (1 - \beta)\, U_1(0) + \beta U_{1x}(0)$ and $b(g) = (1 - \beta)\, U_2(0) + \beta U_{2x}(0)$, moreover, $a(g \cdot t) = (1 - \beta)\, u(t, 0, U_1, g) + \beta u_x(t, 0, U_1, g)$, $b(g \cdot t) = (1 - \beta)\, u(t, 0, U_2, g) + \beta u_x(t, 0, U_1, g)$ for all $t \in \mathbb{R}^1$. Now, fix a $g \in A_0$. By minimality of E, one can take a sequence $\{t_n\}$ with $t_n \to \infty$ such that $g \cdot t_n \to g_*$, $(1 - \beta)\, u(t_n, 0, U_1, g) + \beta u_x(t_n, 0, U_1, g) \to b(g_*)$, that is, $a(g \cdot t_n) \to b(g_*)$, as $n \to \infty$. Since $a(\cdot)$: $H(f) \to \mathbb{R}^1$ is continuous at g_*, one also has $a(g \cdot t_n) \to a(g_*)$ an $n \to \infty$. Hence $a(g_*) = b(g_*)$. Thus, $\text{card}(E \cap P^{-1}(g_*)) = 1$, a contradiction. Therefore, we must have $\text{card}(E \cap P^{-1}(g)) = 1$ for all $g \in A_0$. This proves the Lemma. ∎

THEOREM 3.3. *Let* $(U_0, g_0) \in X \times H(f)$ *be such that* $\prod_t (U_0, g_0)$ $(t > 0)$ *of* (1.4) *is bounded. The following holds*:

(1) *Any minimal invariant set* $E \subset \omega(U_0, g_0)$ *is a proximal extension of* $H(f)$.

(2) *If* $\omega(U_0, g_0)$ *contains two minimal sets* E_1, E_2, *then both* E_1 *and* E_2 *are almost automorphic extensions of* $H(f)$.

Proof. (1) Follows immediately from Lemma 2.3. We now prove (2). First, we claim that there is a $\delta > 0$ such that for any $g \in H(f)$ and any $(U^i, g) \in E_i \cap P^{-1}(g)$ $(i = 1, 2)$,

$$|U^1(x) - U^2(x)| + |U^1_x(x) - U^2_x(x)| \geqslant \delta, \qquad \text{for all} \quad x \in [0, 1]. \tag{3.5}$$

If this is not true, then there are sequences $\{(U^i_n, g_n)\} \subset E_i$ $(i = 1, 2)$, $\{x_n\} \subset [0, 1]$ such that $|U^1_n(x_n) - U^2_n(x_n)| + |U^1_{nx}(x_n) - U^2_{nx}(x_n)| \to 0$ as $n \to \infty$. Take subsequences if it is necessary, one has that (U^i_n, g_n) $(i = 1, 2)$ converge to some $(U^i_*, g_*) \in E_i$ $(i = 1, 2)$, $\{x_n\}$ converges to a $x_* \in [0, 1]$ as $n \to \infty$. It turns out that x_* is a multiple zero of $U^1_*(\cdot) - U^2_*(\cdot)$, a contradiction to Lemma 2.4. Now, fix a $(U^2, g_0) \in E_2 \cap P^{-1}(g_0)$, similar to the argument in Lemma 2.3, there is a sequence $\{t_n\}$ with $t_n \to \infty$ such that

$$\|u(t_n, \cdot, U_0, g_0) - u(t_n, \cdot, U^2, g_0)\| \to 0, \tag{3.6}$$

as $n \to \infty$. Thus, there is a $N > 0$ such that $\|u(t_n, \cdot, U_0, g_0) - u(t_n, \cdot, U^2, g_0)\|_{C^1[0,1]} \leqslant \delta/2$, as $n \geqslant N$. Now, for any $(U, g_0) \in E_1 \cap P^{-1}(g_0)$, since

$$\begin{aligned}
&|u(t_n, x, U_0, g_0) - u(t_n, x, U, g_0)| + |u_x(t_n, x, U_0, g_0) - u_x(t_n, x, U, g_0)| \\
&\geqslant |u(t_n, x, U, g_0) - u(t_n, x, U^2, g_0)| + |u_x(t_n, x, U, g_0) - u_x(t_n, x, U^2, g_0)| \\
&\quad - \|u(t_n, \cdot, U_0, g_0) - u(t_n, \cdot, U^2, g_0)\|_{C^1[0,1]} \\
&\geqslant \delta - \delta/2 = \delta/2 > 0
\end{aligned}$$

as $n \geqslant N$, $u(t_n, \cdot, U_0, g_0) - u(t_n, \cdot, U, g_0)$ has only simple zeros as $n \geqslant N$. This implies that $u(t, \cdot, U_0, g_0) - u(t, \cdot, U, g_0)$ has only simple zeros as $t \geqslant t_N$ for any $(U, g_0) \in E_1 \cap P^{-1}(g_0)$. By Lemma 3.2, E_1 is an almost automorphic extension of $H(f)$. Similarly, E_2 is an almost automorphic extension of $H(f)$. ∎

PROPOSITION 3.4. *Let conditions in Proposition 2.8 be satisfied. Then the unique minimal set E in $\omega(U_0, g_0)$ (for some $(U_0, g_0) \in X \times H(f)$) is in fact an almost automorphic extension of $H(f)$.*

Proof. It follows from the proof of Proposition 2.8 (see Remark 2.1). ∎

Remark 3.2. By Proposition 3.4, if (1.4) has a bounded global attractor, then it has at least one minimal invariant set which is an alomost automorphic extension of $H(f)$.

We now discuss a situation in which an ω-limit set $\omega(U_0, g_0)$ of (1.4) can be an almost periodic extension of $H(f)$.

LEMMA 3.5. *If the ω-limit set $\omega(U, g)$ is distal, then $\omega(U, g)$ is an almost periodical extension of $H(f)$.*

Proof. This is a consequence of Theorem 2.6 and Theorem 3.3. ∎

It is known that, for a skew-product flow generated from a scalar time almost periodic ODE, the ω-limit set of a uniformly stable motion is an almost periodic extension of $H(f)$ (see [24]). We now claim that this is also true for the equation (1.1)–(1.2).

THEOREM 3.6. *Consider* (1.4). *Let* $(U_0, g_0) \in X \times H(f)$ *be such that the motion* $\prod_t (U_0, g_0)$ $(t > 0)$ *is bounded and uniformly stable. Then* $\omega(U_0, g_0)$ *is an almost periodic extension of* $H(f)$.

Proof. By [24], $\omega(U_0, g_0)$ is minimal, and flow $\prod_t$ on $\omega(U_0, g_0)$ is distal. The theorem then follows from Lemma 3.5. ▮

THEOREM 3.7. *Assume system* (1.1) *is monotone, that is,* $f_u(t, x, u, p) \leqslant 0$ *for all* $(t, x, u, p) \in \mathbb{R}^1 \times [0, 1] \times \mathbb{R}^1 \times \mathbb{R}^1$ *in* (1.1). *Then for any bounded motion* $\prod_t (U_0, g_0)$ $(t > 0)$, $\omega(U_0, g_0)$ *is an almost periodic extension of* $H(f)$.

Proof. We first claim that $\omega(U_0, g_0)$ is distal. Take any (U_1, g), $(U_2, g) \in \omega(U_0, g_0) \cap P^{-1}(g)$, by strong maximal principal (see [14], [23]),

$$\max_{x \in [0,1]} |u(t, x, U_1, g) - u(t, x, U_2, g)| \geqslant \max_{x \in [0,1]} |U_1(x) - U_2(x)|,$$

for any $t \leqslant 0$. This implies that

$$\|u(t, \cdot, U_1, g) - u(t, \cdot, U_2, g)\| \geqslant K \cdot \max_{x \in [0,1]} |U_1(x) - U_2(x)|$$

for some $K > 0$ and any $t \leqslant 0$. Hence $\omega(U_0, g_0)$ is negatively distal. It follows from [11], [24] that $\omega(U_0, g_0)$ is distal. By Lemma 3.5, $\omega(U_0, g_0)$ is an almost periodic extension of $H(f)$. ▮

4. Comments and Remarks

4.1. Consider a scalar time almost periodic ODE:

$$u' = f(t, u). \tag{4.1}$$

Equation (4.1) generates a skew product flow $\prod_t$ on $\mathbb{R}^1 \times H(f)$,

$$\prod_t (u_0, g) = (u(t, u_0, g), g \cdot t), \tag{4.2}$$

where $g \cdot t$ is the flow on $H(f)$ defined by time translations, $u(t, u_0, g)$ is the solution of (4.1) with $u(0, u_0, g) = u_0$. Let $E \subset \mathbb{R}^1 \times H((f)$ be a minimal subset of $\prod_t$. Then, by using precisely the same arguments in the proofs of Lemma 3.1 and Lemma 3.2, one shows that E is actually an almost automorphic extension of $H(f)$. In fact, from arguments in the proofs of Lemma 3.1 and Lemma 3.2, we see that once zero number plays a role, the PDE solutions of (1.1) preserve some properties of scalar ODE solutions (for example, "order" between solutions).

4.2. Let $E \subset X \times H(f)$ be a minimal set of (1.4). The lifting properties (say, almost periodic, almost automorphic extensions of $H(f)$) of E naturally reflect the oscillations of the solutions $u(t, x, U, g)$ with $(U, g) \in E$ in the time variable t which are carried over from the original system (1.1) (that is, from the function f). For example, let $E \subset X \times H(f)$ be a minimal set of (1.4). If E is an almost automorphic (almost periodic) extension of $H(f)$, then there is residual set $H_0(f) \subset H(f)$ such that for any $(U, g) \in E \cap P^{-1}(g)$, $g \in H_0(f)$ $(g \in H(f))$, $\prod_t (U, g)$ is an almost automorphic (almost periodic) motion (see Remark 3.1), in other words, $u(t, x, U, g)$ is an almost automorphic (almost periodic) solution of $(1.3)_g$. We now ask the inverse question: If there is an almost automorphic (almost periodic) motion lying in E, is then E necessarily an almost automorphic (almost periodic) extension of $H(f)$? The answer is yes. That is, the oscillation properties of the motions lying in a minimal invariant set E also reflect the lifting properties of E.

THEOREM 4.1. *Let $E \subset X \times H(f)$ be a minimal set of (1.4). Then the following holds:*

(1) *If there is an almost automorphic motion $\prod_t (U_0, g_0)$ lying in E, then* $\mathrm{card}(E \cap P^{-1}(g_0)) = 1$, *that is, E is an almost automorphic extension of $H(f)$.*

(2) *If there is an almost periodic motion lying in E, then E is an almost periodic extension of $H(f)$.*

Proof. (1) Suppose that $\prod_t (U_0, g_0)$ is an almost automorphic motion in E. We claim that $\mathrm{card}(E \cap P^{-1}(g_0)) = 1$. Otherwise, let $(U, g_0) \in E \cap P^{-1}(g_0)$ be any point which differs from (U_0, g_0). Let $\{t_n\}$ be a sequence such that $t_n \to \infty$ and $\prod_{t_n \to \infty} (U, g_0) \to (U, g_0)$ as $n \to \infty$. Let $(U^*, g_0) \in E \cap P^{-1}(g_0)$ be such that $\prod_{-t_n} (U_0, g_0) \to (U^*, g_0)$ as $n \to \infty$ (take subsequence if necessary). Since $\prod_{t_n} (U^*, g_0) \to (U_0, g_0)$ as $n \to \infty$, $(U^*, g_0) \neq (U, g_0)$. By Lemma 2.2, $Z(u(t, \cdot, U_0, g_0) - u(t, \cdot, U, g_0)) =$ constant for all $t \in \mathbb{R}^1$. Applying Lemma 3.2 for $T = 0$, E is an almost automorphic extension of $H(f)$. Let $g^* \in H(f)$ be such that $\mathrm{card}(E \cap P^{-1}(g^*)) = 1$, that is, $E \cap P^{-1}(g^*) = \{(U^*, g^*)\}$. Fix a $(U, g_0) \in E \cap P^{-1}(g_0)$ such that $(U, g_0) \neq (U_0, g_0)$. Let $\{t_n\}$ with $t_n \to \infty$ be a sequence such that $\prod_{t_n} (U^*, g^*) \to (U, g_0)$ as $n \to \infty$. Since $\prod_{-t_n} (U_0, g_0) \to (U^*, g^*)$ as $n \to \infty$, then $\prod_{t_n} (U^*, g^*) \to (U_0, g_0)$ as $n \to \infty$. This contradicts with the fact that $(U, g_0) \neq (U_0, g_0)$. Hence $\mathrm{card}(E \cap P^{-1}(g_0)) = 1$.

(2) Is a corollary of (1). ∎

Note that, given E is minimal invariant and $(U_0, g_0) \in E$, then by the above theorem, $\prod_t (U_0, g_0)$ is almost automorphic if and only if $\mathrm{card}(E \cap P^{-1}(g_0)) = 1$.

4.3. We remark here that by Theorem 4.1 and the following proposition, $u(t, x, U_0, g_0)$ is asymptotically almost periodic if and only if $\omega(U_0, g_0)$ is an almost periodic extension of $H(f)$.

PROPOSITION 4.2. *Consider* $(U_0, g_0) \in X \times H(f)$ *such that* $\prod_t(U_0, g_0)$ $(t>0)$ *is bounded. Suppose that* $\omega(U_0, g_0)$ *is an almost periodic extension of* $H(f)$ (*hence for each* $(U, g) \in \omega(U_0, g_0)$, $\prod_t(U, g)$ *is almost periodic*). *Then*

$$\prod_t(U_0, g_0) - \prod_t(U, g) \to 0, \qquad \text{as} \quad t \to \infty,$$

where $(U, g_0) = \omega(U_0, g_0) \cap P^{-1}(g_0)$.

Proof. If not, there is a $\delta > 0$ and a sequence $\{t_n\}$ with $t_n \to \infty$ such that

$$\|u(t_n, \cdot, U_0, g_0) - u(t_n, \cdot, U, g_0)\| \geqslant \delta,$$

for all n.

Without loss of generality, we assume that $\prod_{t_n}(U_0, g_0)$, $\prod_{t_n}(U, g_0)$ converge to (U_0^*, g^*), $(U^*, g^*) \in \omega(U_0, g_0)$ respectively as $n \to \infty$. It follows that $\|U_0^* - U^*\| \geqslant \delta$. But $\text{card}(\omega(U_0, g_0) \cap P^{-1}(g^*)) = 1$. This is a contradiction. ∎

4.4. Suppose that the flow on a compact invariant set $E \subset X \times H(f)$ is distal. It follows from classical topological dynamical system theory (see [11]) that E laminates into minimal distal flows. It then follows from Lemma 3.5 that E is a union of almost periodic extensions of $H(f)$.

4.5. We have seen from previous sections that the zero number plays an important role in describing the oscillations of solutions in time t variable. In the theory of scalar one dimensional parabolic equations, oscillation properties of a solution $u(x, t)$ in the space variable x are often described by the so called *Lap number* ([20]).

Let $u(x) \in C^1[0, 1]$. The Lap number of u is defined as

$$\begin{aligned} l(u) = \sup\{k \mid & \text{ there are points } 0 = x_0 < x_1 < \cdots < x_k = 1 \text{ such that} \\ & (u(x_{i+1}) - u(x_i))(u(x_i) - u(x_{i-1})) < 0, \quad i = 1, 2, ..., k-1\}. \end{aligned} \tag{4.3}$$

An essential requirement to consider the Lap number $l(u(\cdot, t))$ for a classical solution $u(x, t)$ of (1.1) is that the function f in (1.1) does not depend on x explicitly (see [20]). If this assumption is made, then the Lap number of the solutions in a minimal invariant set is a constant.

PROPOSITION 4.3. *Let $E \subset X \times H(f)$ be a minimal set of* (1.4) *and suppose that for any* $(U, g) \in E$, $l(u(t, \cdot, U, g))$ *is nonincreasing (it is always true if f in* (1.1) *does not depends on x explicitly and $\beta = 0$ in* (1.2)). *Then there is an integer $N \geqslant 0$ such that $l(U) = N$ for all $(U, g) \in E$.*

Proof. Fix $(U_0, g_0) \in E$. Since $u_x(t, x, U_0, g_0)$ satisfies a linear parabolic equation, $Z(u_x(t, \cdot, U_0, g_0)) < \infty$ for all $t \in \mathbb{R}^1$. Thus, $l(u(t, \cdot, U_0, g_0)) \leqslant Z(u_x(t, \cdot, U_0\, g_0)) + 1 < \infty$ for all $t \in \mathbb{R}^1$. By minimality of E, there is a sequence $\{t_n\}$ with $t_n \to \infty$ such that $\prod_{t_n}(U_0, g_0) \to (U_0, g_0)$ as $n \to \infty$. By lower semi-continuity of Lap number ([20]), one has

$$l(U_0) \leqslant \liminf_{n \to \infty} l(u(t_n, \cdot, U_0, g_0)) \leqslant l(u(t_n, \cdot, U_0, g_0)) \tag{4.4}$$

for $n \gg 1$.

But $l(u(t_n, \cdot, U_0, g_0) \leqslant l(U_0)$ for all n. It follows that $l(u(t, \cdot, U_0, g_0)) = l(U_0)$ for $t \geqslant 0$. This implies that $l(u(t, \cdot, U_0, g_0)) = l(U_0(\cdot))$ for any $t \in \mathbb{R}^1$.

Now, for any $(U, g) \in E$, there are sequences $\{t_n\}$, $\{s_n\}$ with $t_n \to \infty$, $s_n \to -\infty$ such that

$$\textstyle\prod_{t_n}(U_0, g_0) \to (U, g), \quad \prod_{s_n}(U, g) \to (U_0, g_0),$$

as $n \to \infty$. Therefore,

$$l(U) \leqslant \liminf_{n \to \infty} l(u(t_n, \cdot, U_0, g_0) = l(U_0),$$

$$l(U_0) \leqslant \liminf_{n \to \infty} l(u(s_n, \cdot, U, g) = l(U),$$

that is, $l(U) = l(U_0)$. ∎

This proposition simply states that by means of Lap number all motions $\{\prod_t(U, g)\} \subset E$ have similar oscillations in the space variable x.

4.6. We now give an example adopted from Johnson [19] in which an ω-limit set of (1.4) contains precisely two minimal sets.

Consider the scalar parabolic equation:

$$\begin{aligned} &u_t = u_{xx} - (a(t) \cos u + b(t) \sin u) \sin u, \quad t > 0, \quad 0 < x < 1, \\ &u_x(t, 0) = u_x(t, 1) = 0, \quad t > 0, \end{aligned} \tag{4.5}$$

where $f(t) = (a(t), b(t))$ is almost periodic such that the scalar ODE

$$y' = a(t)\, y + b(t) \tag{4.6}$$

admits no almost periodic solutions but the solution $y_0(t)$ with $y_0(0)=0$ is bounded (see [19]).

In what follows, we will consider only the ω-limit sets of bounded solutions which are space homogeneous, that is, the solutions of the scalar ODE:

$$u' = -(a(t)\cos u + b(t)\sin u)\sin u. \tag{4.7}$$

For $(U, g) \equiv (U, a_g, b_g) \in \mathbb{R}^1 \times H(f)$, denote by $u(t, U, g)$ the solution of

$$u' = -(a_g(t)\cos u + b_g(t)\sin u)\sin u \tag{4.8}$$

with $u(0, U, g) = U$. Then

$$\prod_t (U, g) = (u(t, U, g), g\cdot t) \tag{4.9}$$

is the skew product flow on $\mathbb{R}^1 \times H(f)$ generated by (4.7).

Clearly, $E_1 = \{0\} \times H(f)$ is an invariant set of $\prod_t$, and it is in fact an almost periodic extension of $H(f)$. Next, consider transformation $y(t) = \cot u(t)$ to (4.7). A simple calculation shows that $y(t)$ satisfies (4.6). Let $M = \mathrm{cl}\{(y_0(t), f\cdot t) \mid t \in \mathbb{R}^1\}$. Then M contains a minimal set $M_2 \subset \mathbb{R}^1 \times H(f)$ which is necessary an almost automorphic but not almost periodic extension of $H(f)$ (see 4.1). Hence, $E = \mathrm{cl}\{\prod_t(\pi/2, f) \mid t \in \mathbb{R}^1\}$ contains a minimal set $E_2 \subset (0, \pi) \times H(f)$ which is an almost automorphic (not almost periodic) extension of $H(f)$. Define $u(g) = \min\{U \mid (U, g) \in E_2\}$. We shall show that there are $g_0 \in H(f)$ and $U_0 \in (0, u(g_0))$ such that

$$E_1 \cup E_2 \subset \omega(U_0, g_0).$$

To do so, for each $(U, g) = (U, a_g, b_g) \in E_2$, consider the transformation

$$\cot u = \frac{\cot \tilde{u}}{\sin u(t, U, g)} + \cot u(t, U, g) \tag{4.10}$$

to (4.8). Then the equation for $\tilde{u}$ reads

$$\tilde{u}' = \beta((U, g)\cdot t)\sin \tilde{u}\cos \tilde{u}, \tag{4.11}$$

here $\beta((U, g)\cdot t) = -a_g(t)\sin^2 u(t, U, g) + b_g(t)\sin u(t, U, g)\cos u(t, U, g)$.

Let $\tilde{\prod}_t$ be denoted as the skew product flow on $\mathbb{R}^1 \times E_2$ generated by (4.11). Then minimal sets $\tilde{E}_1 = \{0\} \times E_2$, $\tilde{E}_2 = \{\pi/2\} \times E_2$ of $\tilde{\prod}_t$ correspond to E_1 and E_2 respectively by means of transformation (4.10). By arguments in [18], one has that

$$\lim_{|t|\to\infty} \frac{1}{t}\int_0^t \beta((U, g)\cdot s)\,ds = 0. \tag{4.12}$$

Since E_1 and E_2 are only two minimal sets of $\prod_t$ in $[0, \pi)\times H(f)$ (see [19]), it follows also that $\tilde{E}_1$ and $\tilde{E}_2$ are only minimal sets of $\tilde{\prod}_t$ in $[0, \pi/2]\times E_2$. Note that $V \equiv \cot\tilde{u}$ satisfies

$$V_t = -\beta((U, g)\cdot t)V, \tag{4.13}$$

that is, $\cot\tilde{u} = \cot\tilde{U}e^{\int_0^t \beta((U,g)\cdot s)\,ds}$ $(\tilde{u}(0) = \tilde{U})$. Now take $(\tilde{U}, U, g)\in(0, \pi/2)\times E_2$. Since $\tilde{E}_1$, $\tilde{E}_2$ are only minimal sets of $\tilde{\prod}_t$ in $[0, \pi/2]\times E_2$, there is a sequence $\{t_n\}$ with $t_n\to\infty$ such that if $\tilde{u}(t)\equiv\tilde{u}(t, \tilde{U}, U, g)$ is the solution of (4.11) with $\tilde{u}(0) = \tilde{U}$, then $\tilde{u}(t_n)$ converges to either 0 or $\pi/2$ as $n\to\infty$, that is, $\cot\tilde{u}(t_n)$ converges to either $+\infty$ or 0 as $n\to\infty$. Hence $\int_0^{t_n}\beta((U, g)\cdot s)\,ds$ converges to either $+\infty$ or $-\infty$ as $n\to\infty$. In any case, $\int_0^t\beta((U, g)\cdot s)\,ds$ is unbounded. Using this fact and (4.12), one has by [18] that the set

$$E_0 = \left\{(U, g)\in E_2 \mid \limsup_{t\to\infty}\int_0^t \beta((U, g)\cdot s)\,ds = \infty,\right.$$
$$\left.\liminf_{t\to\infty}\int_0^t \beta((U, g)\cdot s)\,ds = -\infty\right\} \tag{4.14}$$

is a residual subset of E_2. Now take $(\tilde{U}, U, g_0)\in(0, \pi/2)\times E_0$. It follows from (4.14) that $\tilde{E}_1\cup\tilde{E}_2\subset\omega(\tilde{U}, U, g_0)$. Let $U_0 = \cot^{-1}(\cot\tilde{U}/\sin U + \cot U)$. Then $E_1\cup E_2\subset\omega(U_0, g_0)$, with E_1 being an almost periodic extension of $H(f)$ and E_2 being an almost automorphic extension of $H(f)$.

Acknowledgments

We thank X.-Y. Chen, J. K. Hale, R. A. Johnson, K. Lu, J. Mallet-Paret for comments and discussions. We are grateful to the referee for careful reading of the current paper and many valuable comments which led to the improvement of our original manuscript.

References

1. S. B. Angenent, The zero set of a solution of a parabolic equation, *J. Reine Angew. Math.* **390** (1988).
2. S. Angenent and B. Fiedler, The dynamics of rotating waves in scalar reaction diffusion equations, *Trans. Amer. Math. Soc.* **307** (1988).

3. J. P. Aubin and I. Ekeland, "Applied Nonlinear Analysis," Wiley, New York, 1984.
4. P. Brunovský, P. Poláčik and B. Sandstede, Convergence in general periodic parabolic equations in one space dimension, *Nonlinear Anal. Theory, Meth. Appl.* **18** (1992).
5. N. Chafee, Asymptotic behavior for solutions of a one-dimensional parabolic equation with homogeneous Neumann boundary conditions, *J. Diff. Eq.* **18** (1975).
6. M. Chen, X. Y. Chen and J. K. Hale, Structural stability for time-periodic one-dimensional parabolic equations, *J. Diff. Eq.* **96** (1992).
7. X.-Y. Chen and H. Matano, Convergence, asymptotic periodicity, and finite-point blow-up in one-dimensional semilinear heat equations, *J. Diff. Eq.* **78** (1989).
8. X. Y. Chen and P. Poláčik, Gradient-like structure and morse decompositions for time-periodic one-dimensional parabolic equations, *J. Dyn. and Diff. Eq.* **7** (1995).
9. S.-N. Chow, K. Lu and J. Mallet-Paret, Floquet theory for parabolic differential equations I: the time periodic case, preprint.
10. S.-N. Chow, K. Lu and J. Mallet-Paret, Floquet bundles for scalar parabolic equations, preprint.
11. R. Ellis, "Lectures on Topological Dynamics," Benjamin, New York, 1969.
12. B. Fiedler and J. Mallet-Paret, A Poincaré–Bendixson theorem for scalar reaction diffusion equations, *Arch. Rational Mech. Anal.* **107** (1989).
13. A. M. Fink, "Almost Periodic Differential Equations," Lecture Notes in Mathematics, Vol. 377, Springer-Verlag, Berlin/Heidelberg/New York, 1974.
14. A. Friedman, Partial differential equations of parabolic type, Prentice–Hall, Englewood Cliffs, NJ, 1964.
15. J. K. Hale, Asymptotic behavior of dissipative systems, *in* "Mathematical Surveys and Monographs," Amer. Math. Soc., Providence, RI, 1988.
16. D. Henry, "Geometric Theory of Semilinear Parabolic Equations," Lecture Notes in Math., Vol. 840, Springer-Verlag, Berlin, 1981.
17. R. A. Johnson, Concerning a theorem of sell, *J. Diff. Eq.* **30** (1978).
18. R. A. Johnson, On a floquet theory for almost periodic two-dimensional linear systems, *J. Diff. Eq.* **37** (1980).
19. R. A. Johnson, A linear almost periodic equations with an almost automorphic solution, *Proc. Amer. Math. Soc.* **82** (1981).
20. H. Matano, Nonincrease of the lap-number of a solution for a one-dimensional semilinear parabolic equation, *J. Fac. Sci. Univ. Tokyo Sect. IA* **29** (1982).
21. H. Matano, Convergence of solutions of one-dimensional semilinear parabolic equations, *J. Math. Kyoto Univ.* **18-2** (1978).
22. H. Matano, Asymptotic behavior of solutions of semilinear heat equations on the circle, *in* "Nonlinear Diffusion Equations and Their Equilibrium States" (W. M. Ni, L. A. Peletier, and J. Serrin, Eds.), Proc. Berkeley, Springer-Verlag, New York, 1986.
23. M. H. Protter and H. F. Weinberger, Maximum principles of equations, Prentice–Hall, Englewood Cliffs, NJ, 1967.
24. R. J. Sacker and G. R. Sell, Lifting properties in skew-product flows with applications to differential equations, *Mem. Amer. Math. Soc.* **11** (1977).
25. B. Sandstede and B. Fiedler, Dynamics of periodically forced parabolic equations on the circle, *Ergodic Theory, Dynam. Systems* **12** (1992).
26. G. R. Sell, Topological dynamics and ordinary differential equations, Van Nostrand Reinhold, 1971.
27. K. Schmitt and J. R. Ward, Jr., Periodic and almost periodic solutions of nonlinear evolution equations, preprint.
28. W. A. Veech, Almost automorphic functions on groups, *Amer. J. Math.* **87** (1965).
29. W. A. Veech, Point-Distal Flows, *Amer. J. Math.* **92** (1970).

30. P. Vuillermot, Almost-Periodic attractors for a class of nonautonomous reaction-diffusion equations on $\mathbb{R}^N$, I. Global stabilization process, *J. Diff. Eq.* **94** (1991).
31. P. Vuillermot, Almost-periodic attractors for a class of non autonomous reaction-diffusion equations on $\mathbb{R}^N$, II. Codimension-one stable manifolds, preprint, III. Center curves and Liapounov stability, preprint.
32. J. R. Ward, Jr., Bounded and almost periodic solutions of semi-linear parabolic equations, *Rocky Mountain J. Math.* **18** (1988).
33. Y. Yi, A generalized integral manifold theorem, *J. Diff. Eq.* **102** (1993).