Model Fitting for Continuous-Time Stationary Processes from Discrete-Time Data

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Let \( X=\{X(t), -\infty < t < \infty \} \) be a continuous-time stationary process with spectral density \( \phi_X(\lambda; \Theta) \), where \( \Theta \) is a vector of unknown parameters. Let \( \{\tau_k\} \) be a stationary point process on the real line which is independent of \( X \). The identifiability and the estimation of \( \Theta \) from the discrete-time observation \( \{X(\tau_k), \tau_k\} \) are considered. The consistency of appropriate estimates \( \hat{\Theta}_T \) as the time \( T \to \infty \) is established and a central limit theorem for \( \hat{\Theta}_T \) is given.

1. INTRODUCTION

Let \( X=\{X(t), -\infty < t < \infty \} \) be a real-valued weakly stationary stochastic process with zero mean, continuous covariance function \( R_X(t) \in L_1 \) and spectral density \( \phi_X(\lambda) \). Suppose that the process \( X \) is sampled at instants \( \{\tau_k\} \) which constitute a stationary point process on the real line with mean rate \( \beta \). One seeks estimates of \( R_X(t) \) and \( \phi_X(\lambda) \) on the basis of a finite set of discrete-time observations \( \{X(\tau_k)\}_{k=1}^n \). If the sampling instants \( \{\tau_k\} \) are equally-spaced, \( \tau_k = k/\beta \), then aliasing is present and consistent estimates of \( R_X(t) \) and \( \phi_X(\lambda) \) from the discrete-time process

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\{X(k/\beta)\} do not exist unless the process \(X\) is bandlimited, \(\phi_X(\lambda) = 0\) for \(|\lambda| > W\), and \(\beta \geq W/\pi\). It is therefore desirable to assume that \(\\{\tau_k\}\) is an irregularly-spaced "alias-free" point process as discussed below.

The nonparametric estimation of \(R_X(t)\) and \(\phi_X(\lambda)\) has been studied in the literature. There are two approaches depending on whether the actual values of the sampling instants \(\\{\tau_k\}\) are known or not. In the first approach one is given the finite data sequence \(\{X(\tau_k)\}_{k=1}^n\) and the statistics of the point process \(\\{\tau_k\}\) without the actual values \(\\{\tau_k\}\). The feasibility of consistently estimating \(R_X(t)\) and \(\phi_X(\lambda)\) under these circumstances is studied in Shapiro and Silverman [15]. However, as shown in Masry [9], the quadratic-mean convergence rates for spectral estimation are only logarithmic in the sample size \(n\) and thus this approach is not useful practically. In the second approach one assumes that the observation is a realization of \(\{X(\tau_k), \tau_k\}_{k=1}^n\). The feasibility of consistently estimating \(R_X(t)\) and \(\phi_X(\lambda)\) is developed in Masry [7] and is implicit in the work of Brillinger [1]. The quadratic-mean convergence rates of spectral estimates are identical to those based on continuous-time data [1, 8]. These rates do not depend on the average sampling rate \(\beta\) (the asymptotic constant does). For covariance estimates see Masry [10].

While the statistical performance of nonparametric estimation of \(R_X(t)\) and \(\phi_X(\lambda)\) is well developed, surprisingly little is known for parametric models (e.g., continuous-time ARMA\((p, q)\) processes). An excellent overview of the problem of continuous-time model fitting from discrete data is given by Robinson [14]. The approach taken by Robinson is that of identifying and estimating the autoregressive and moving average coefficients \(\{\alpha_j\}_{j=1}^p\) and \(\{\beta_j\}_{j=0}^q, q < p\), of the continuous-time process \(X\) based on maximizing a functional of the form

\[ L = \frac{-n}{2} \log 2\pi - \frac{1}{2} \sum_j \left\{ \log s(\lambda_j) + \frac{I(\lambda_j)}{s(\lambda_j)} \right\}, \]  

(1.1)

where \(s(\lambda) = (1/2\pi) \sum_{k=-\infty}^{\infty} r_k e^{-ik\lambda}\) with \(r_k = E[X(\tau_{k+m})X(\tau_m)]\), \(I(\lambda) = (1/2\pi n) \left| \sum_{k=1}^n X(\tau_k) e^{-ik\lambda} \right|^2\), and \(\lambda_j = 2\pi j/n, \quad |j| < [n/2]\). Statistical analysis of such estimates are not given there. However, in view of the slow rate of quadratic-mean convergence for the nonparametric spectral estimate based on \(\{X(\tau_k)\}_{k=1}^n\), we expect that estimates of \(\{\alpha_j\}\) and \(\{\beta_j\}\) based on maximizing (1.1) will also have slow rates of convergence. A full likelihood approach for Gaussian \(X\), when the actual values (realization) of the sampling instants \(\tau_1, ..., \tau_n\) are known, leads to maximizing

\[ M = \frac{-n}{2} \log 2\pi - \frac{n}{2} \log |\Gamma| - \frac{1}{2} x' \Gamma^{-1} x, \]  

(1.2)

where \(\Gamma = [R_X(\tau_i - \tau_j)]_{i,j=1}^n\) and \(x' = (X(\tau_1), ..., X(\tau_n))\). The asymptotic
properties of estimates of \( \{ a_j \}_{j=1}^p \) and \( \{ \beta_j \}_{j=0}^p \) maximizing \( M \) are known only for the AR(1) case, \( p = 1 \), \( q = 0 \) [12], because of the complex structure of the matrix \( \Gamma \). Numerical methods for maximizing \( M \) are available [5].

The purpose of this paper is to develop the asymptotic statistical theory of estimates of the spectral parameters of a continuous-time processes from discrete-time observations \( \{ X(\tau_k), \tau_k \}_{k=1}^n \). We do not assume that the process \( X \) is Gaussian and we allow a broad class of "alias-free" point processes \( \{ \tau_k \} \). Since estimates based on the likelihood function of \( M (1.2) \) are not mathematically tractable, we base our estimates on a "continuous-discrete" version of the functional \( W_n \) suggested originally by Whittle [16] and used by Robinson [13] and others for discrete-time processes.

Applications for such models can be found in Roberts et al. [11] for laser anemometry, in Jones [6] for medical signal processing and atomic clock timing.

The organization of the paper is as follows: Section 2 introduces the estimation scheme and establishes consistency of the spectral parameters estimates of the continuous-time process. The asymptotic normality of these estimates is derived in Section 3.

2. Consistency

In this section, \( X = \{ X(t), -\infty < t < \infty \} \) is a stationary process with finite fourth-order moments with mean zero, continuous covariance function \( R_X(t) \in L_1 \), spectral density \( \phi_X(\lambda) \), and kth-order cumulant \( \phi_X^{(k)}(u_1, ..., u_{k-1}) \), \( k = 3, 4 \). The point process \( \{ \tau_k \}_{k=-\infty}^n \) is stationary and orderly, independent of \( X \), with finite fourth-order moments. Let \( N(\cdot) \) be the counting process associated with \( \{ \tau_k \} \) and \( \beta = E[N((0, 1))] \) be the main intensity of the point process \( \{ \tau_k \} \). Then [3]

\[
E[N((t, t + dt))] = \beta \, dt \tag{2.1}
\]

\[
\text{cov}\{N((t, t + dt)), N((t + u, t + u + du))\} = C_N(du) \, dt, \tag{2.2}
\]

where \( C_N \) is the reduced covariance measure which is a \( \sigma \)-finite measure on the Borel sets \( \mathcal{B} \) with an atom at the origin, \( C_N(\{0\}) = \beta \). We assume that outside of the origin \( C_N \) is absolutely continuous with covariance density function \( c_N(u) \), i.e.,

\[
C_N(B) = \beta \delta_0(B) + \int_B c_N(u) \, du, \quad B \in \mathcal{B}, \tag{2.3}
\]

where

\[
\delta_0(B) = \begin{cases} 1, & \text{if } 0 \in B \\ 0, & \text{otherwise.} \end{cases}
\]
In a differential notation \( dN(t) = N((0, t + dt]) - N((0, t]) \) and we can write, for \( k = 1, 2, 3, 4 \) \[ \text{cum} \{ dN(t_1), \ldots, dN(t_k) \} = c_{N}^{(k)}(t_2 - t_1, \ldots, t_k - t_1) \, dt_1 \cdots dt_k \] (2.4) for distinct \( t_j \)'s, where \( c_{N}^{(k)}(u_1, \ldots, u_{k-1}) \) is the \( k \)-th order cumulant density which is assumed to exist. Note that \( c_N(u) \equiv c_N^{(2)}(u) \) and \( E[dN(t)] = \beta \, dt \).

We define the sampled process by

\[
Z(B) = \sum_{\tau_i \in B} X(\tau_i), \quad B \in \mathcal{B}
\]

or, in differential form, \( dZ(t) = X(t) \, dN(t) \). The increment process \( Z \) has finite fourth-order moments and, in particular, \( E[dZ(t)] = 0 \) and

\[
\mu_Z(du) \, dt \triangleq E[dZ(t) \, dZ(t + u)] = R_X(u) \{ \beta^2 \, du + c_N(du) \} \, dt.
\]

If we define the \( \sigma \)-finite measure

\[
\nu_N(B) = \int_B \beta^2 \, du + c_N(du), \quad B \in \mathcal{B},
\]

then

\[
\mu_Z(B) = \int_B R_X(u) \, \nu_N(du) = \beta R_X(0) \, \delta_0(B) + \int_B R_X(u) \{ \beta^2 + c_N(u) \} \, du
\]

is a \( \sigma \)-finite signed measure on \( \mathcal{B} \). We define the spectral density \( \phi_Z(\lambda) \) of the increment process \( Z \) by

\[
\phi_Z(\lambda) \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ix\lambda} \mu_Z(du) = \beta^2 \phi_X(\lambda) + \frac{\beta R_X(0)}{2\pi} + \frac{1}{2\pi} \int_{-\infty}^{\infty} R_X(u) \, c_N(u) \, e^{-iux} \, du.
\]

(2.5) where \( \psi(\lambda) \triangleq (1/2\pi) \int_{-\infty}^{\infty} e^{-iux} c_N(u) \, du \) and \( c_N(u) \in L_1 \) is assumed for the last equality. Note that \( \phi_Z(\lambda) \) is bounded and uniformly continuous and by the Riemann–Lebesgue lemma \( \lim_{|\lambda| \to \infty} \phi_Z(\lambda) = \beta R_X(0)/2\pi > 0 \), so that \( \phi_Z(\lambda) \) is not integrable (in contrast to the spectral density \( \phi_X(\lambda) \) of stationary processes). Note that for a Poisson point process \( c_N(u) \equiv 0 \), in which case

\[
\phi_Z(\lambda) = \beta^2 \left[ \phi_X(\lambda) + \frac{R_X(0)}{2\pi \beta} \right].
\]
We say that the sampling scheme \( \{ \tau_k \} \) is alias-free relative to the family of all spectral densities \( \phi_X(\lambda) \) (with \( R_X(t) \in \mathcal{L}_1 \)) if no two distinct spectral densities \( \phi_X(\lambda) \) and \( \phi_X^\prime(\lambda) \) yield the same spectral density \( \phi_Z(\lambda) \) of the increment process \( Z \). A necessary and sufficient condition for \( \{ \tau_k \} \) to be alias-free is given in Masry [7, Theorem 1] in terms of the measure \( \mu_N \). The property of alias-free sampling does not guarantee by itself that \( \phi_X(\lambda) \) can be recovered from \( \phi_Z(\lambda) \) in a stable manner; additional conditions on the point process \( \{ \tau_k \} \) are required.

For the parametric case, we assume that the spectral density \( \phi_X(\lambda) \) of the continuous-time process \( X \) depends on a vector parameter \( \theta = (\theta_1, \ldots, \theta_p) \), excluding a multiplicative parameter \( \eta \) in \( \phi_X(\lambda) \). This would be the case, for example, when \( X \) is a continuous-time ARMA\((p, q)\) process with spectral density

\[
\phi_X(\lambda) = \frac{\eta}{2\pi} \left| \frac{\sum_{j=0}^{q} \beta_j(i\lambda)^j}{\sum_{j=0}^{p} \alpha_j(i\lambda)^j} \right|^2; \quad \alpha_p = 1, \beta_0 = 1, q < p.
\]

for which \( \theta = (\alpha_0, \ldots, \alpha_{p-1}, \beta_1, \ldots, \beta_q) \). We make this dependence explicit by writing \( \phi_X(\lambda; \theta) \) instead of \( \phi_X(\lambda) \) and suppressing the dependence on the parameter \( \eta \) which is taken to be fixed but unknown. The estimation of \( \theta \) is independent of the value of \( \eta \). At the end of this section we consider the separate estimation of the parameter \( \eta \). Correspondingly, the spectral density \( \phi_Z(\lambda) \) of the sampled process \( Z \) is written as \( \phi_Z(\lambda; \theta) \).

Put

\[
\sigma^2(\theta) = \int_{-\infty}^{\infty} \frac{\phi_Z(\lambda; \theta)}{1 + \lambda^2} d\lambda.
\]  

(2.7)

Note that the weight function \( 1/(1 + \lambda^2) \) is needed here since \( \phi_Z(\lambda; \theta) \) is bounded but is not integrable. Now set

\[
g(\lambda; \theta) = \frac{\phi_Z(\lambda; \theta)}{\sigma^2(\theta)}.
\]  

(2.8)

Note that \( g(\lambda; \theta) \) does not depend on the multiplicative parameter \( \eta \). We denote the true value of the parameter \( \theta \) by \( \theta^0 \). Define the functional

\[
K(\theta) = \int_{-\infty}^{\infty} \log g(\lambda; \theta) \frac{\phi_Z(\lambda; \theta^0)}{\sigma^2(\theta^0)} d\lambda.
\]  

(2.9)

Note that, since \( \log x \leq x \) and \( \phi_Z(\lambda; \theta^0) \leq D_1 < \infty \), we have

\[
K(\theta) \leq D_1 \int_{-\infty}^{\infty} \frac{g(\lambda; \theta)}{1 + \lambda^2} d\lambda = D_1,
\]

where \( D_1 \) is a finite constant so that \( K(\theta) \) can only diverge to \( -\infty \).
However, if $g(\lambda; \theta)$ is bounded away from zero, then $K(\theta)$ of (2.9) is clearly well defined. Poisson sampling is such an example (cf. (2.6)). Henceforth we make the assumption

$$
\int_{-\infty}^{\infty} \frac{|\log g(\lambda; \theta)|}{1 + \lambda^2} d\lambda < \infty
$$

(2.10)

under which $K(\theta)$ is well defined. Note that

$$
K(\theta) - K(\theta^0) = \int_{-\infty}^{\infty} \log \left[ \frac{g(\lambda; \theta)}{g(\lambda; \theta^0)} \right] \frac{\phi_{x}(\lambda; \theta^0)}{1 + \lambda^2} d\lambda
$$

and, since $\log x \leq x - 1$ with equality iff $x = 1$, then

$$
K(\theta) - K(\theta^0) \leq \int_{-\infty}^{\infty} \left[ \frac{g(\lambda; \theta)}{g(\lambda; \theta^0)} - 1 \right] \frac{\phi_{x}(\lambda; \theta^0)}{1 + \lambda^2} d\lambda
$$

$$
= \sigma^2(\theta^0) \int_{-\infty}^{\infty} \frac{g(\lambda; \theta)}{1 + \lambda^2} d\lambda - \int_{-\infty}^{\infty} \frac{\phi_{x}(\lambda; \theta^0)}{1 + \lambda^2} d\lambda = 0,
$$

so that $K(\theta) \leq K(\theta^0)$ with equality iff $g(\lambda; \theta) = g(\lambda; \theta^0)$. We now assume that

**Assumption 2.1.** (a) $\theta$ takes values in a closed bounded set $\Theta \subset \mathbb{R}^q$.

(b) $g(\lambda; \theta') \neq g(\lambda; \theta'')$ on a set of positive measure in $\lambda$ whenever $\theta' \neq \theta''$.

Clearly $g(\lambda; \theta)$ depends on $\phi_{x}(\lambda; \theta)$ and on $\psi(\lambda)$, (2.5) of the point process $\{\tau_k\}$. The identifiability condition is imposed on $g(\lambda; \theta)$ rather than on $\phi_{x}(\lambda; \theta)$ because of several reasons: (1) The estimate of $\theta$ utilizes $g(\lambda; \theta)$ directly; (2) In certain situations, condition (b) of Assumption 2.1 is satisfied even for non-alias-free sampling schemes, e.g., AR(1) process $X$ with equally-spaced sampling. For arbitrary alias-free sampling schemes and general spectral density $\phi_{x}(\lambda; \theta)$ it is difficult to recast condition (b) in terms of conditions on $\phi_{x}(\lambda; \theta)$ alone, in view of the convolution integral in (2.5). For the special case of Poisson sampling we show below that condition (b) is satisfied if $\phi_{x}(\lambda; \theta') \neq k\phi_{x}(\lambda; \theta'')$ for any constant $k$, whenever $\theta' \neq \theta''$. This would be the case if the multiplicative constant is excluded from $\theta$. The preceding assertion is seen as follows: For $\theta' \neq \theta''$, if $g(\lambda; \theta') = g(\lambda; \theta'')$ a.e. ($d\lambda$), then for Poisson sampling, (2.6) and (2.8) imply

$$
\frac{\phi_{x}(\lambda; \theta')}{\sigma^2(\theta')} - \frac{\phi_{x}(\lambda; \theta'')}{\sigma^2(\theta'')} = \frac{1}{2\pi\beta} \left[ \frac{R_{X}(0, \theta'')}{\sigma^2(\theta'')} - \frac{R_{X}(0, \theta')}{\sigma^2(\theta')} \right].
$$

The integrability of $\phi_{x}(\lambda; \theta)$ implies that the right-hand side of the
preceding equation is identically zero, which in turn implies \( \phi_x(\lambda; \theta') = k\phi_x(\lambda; \theta'') \) for a constant \( k \). This contradicts the assumption.

Assumption 2.1 ensures that the value of the parameter \( \theta^0 \) may be identified by the equation \( K(\theta^0) = \max_{\theta \in \Theta} K(\theta) \). It then follows that an estimate of \( \theta^0 \) may be based on an estimate of \( K(\theta^0) \). To this end, set

\[
\hat{K}_T(\theta) = \int_{-\infty}^{\infty} \frac{\log g(\lambda; \theta)}{1 + \lambda^2} I_{Z,T}(\lambda) d\lambda, \tag{2.11}
\]

where \( I_{Z,T}(\lambda) \) is the periodogram for \( \phi_Z(\lambda; \theta) \), based on the observations \( \{X(\tau_i), \tau_i\}_{i=1}^{N(T)} \), i.e.,

\[
I_{Z,T}(\lambda) = \frac{1}{2\pi T} \left| \int_0^T e^{-i\lambda t} dZ(t) \right|^2 = \frac{1}{2\pi T} |d_{Z,T}(\lambda)|^2
\]

\[
= \frac{1}{2\pi T} \left( \sum_{k=1}^{N(T)} e^{-\lambda i t_k} X(\tau_k) \right)^2 \tag{2.12}
\]

and the estimate \( \hat{\theta}_T \) of \( \theta^0 \) is defined by

\[
\hat{K}_T(\hat{\theta}_T) = \max_{\theta} \hat{K}_T(\theta). \tag{2.13}
\]

For continuous-time processes \( \{X(t), -\infty < t < \infty\} \) with spectral density \( \phi_X(\lambda; \theta), \hat{K}_T(\theta) \) of (2.11), with \( I_{Z,T}(\lambda) \) replaced by

\[
I_{X,T}(\lambda) = \frac{1}{2\pi T} \left| \int_0^T e^{-i\lambda t} X(t) dt \right|^2,
\]

was proposed by Ibragimov [4] when \( X(t), t \in (0, T) \), is available.

An estimate of \( \sigma^2(\theta^0) \) is given by

\[
\hat{\sigma}^2_T(\theta^0) = \int_{-\infty}^{\infty} \frac{I_{X,T}(\lambda)}{1 + \lambda^2} d\lambda. \tag{2.14}
\]

This will be used later to obtain an estimate of the multiplicative parameter \( \eta \) in the spectral density \( \phi_X(\lambda) \).

We now turn to the issue of consistency of \( \hat{\theta}_T \). We first show that the estimate \( \hat{K}_T(\theta) \) converges in quadratic-mean to \( K(\theta) \) as \( T \to \infty \) for each \( \theta \in \Theta \) and find the asymptotic expressions for its mean and variance. Similarly for the estimate \( \hat{\sigma}^2_T(\theta^0) \). Towards his end we make the following assumptions on the cumulants of the process \( X \) and the cumulant densities of the counting process \( N \).
MODEL FITTING

Assumption 2.2.

(i) \[ \int_{-\infty}^{\infty} (1 + |t|) |R_X(t)| \, dt < \infty \]

(ii) \[ \int_{R^{k-1}} (1 + |u_j|) |c_X^{(k)}(u_1, \ldots, u_{k-1})| \, du_1 \cdots du_{k-1} < \infty \]

\text{for } k = 3, 4, j = 1, \ldots, k - 1

(iii) \[ \int_{R^{k-1}} (1 + |u_j|) |c_N^{(k)}(u_1, \ldots, u_{k-1})| \, du_1 \cdots du_{k-1} < \infty \]

\text{for } k = 2, 3, 4, j = 1, \ldots, k - 1.

Note that, when \( X \) is Gaussian, condition (ii) is vacuous. Similarly, when \( N \) is Poisson counting process, condition (iii) is vacuous.

Theorem 2.1. Under Assumption 2.2 we have

(a) Assume \( A(C; \theta) \triangleq \log(g(C; \theta))/(1 + \lambda^2) \) is bounded, continuous in \( \lambda \), and \( E[\theta \in L_1(d\lambda)] \). We have for each \( \theta \in \Theta \),

\[ E[\hat{K}_T(\theta)] = K(\theta) + O(1/T) \]

and

\[ T \text{ var} [\hat{K}_T(\theta)] \rightarrow 2\pi \int_{R^2} A(\lambda_1; \theta) A(\lambda_2; \theta) \phi_Z^{(4)}(\lambda_1, -\lambda_1, \lambda_2; \theta^0) \, d\lambda_1 \, d\lambda_2 \]

\[ + 4\pi \int_{-\infty}^{\infty} A^2(\lambda; \theta) \phi_Z^2(\lambda; \theta^0) \, d\lambda, \]

(b) \[ E[\hat{\sigma}_T^2(\theta^0)] = \sigma^2(\theta^0) + O(1/T) \text{ and} \]

\[ T \text{ var}[\hat{\sigma}_T^2(\theta^0)] \rightarrow 2\pi \int_{R^2} \frac{\phi_Z^{(4)}(\lambda_1, -\lambda_1, \lambda_2; \theta^0)}{(1 + \lambda_1^2)(1 + \lambda_2^2)} \, d\lambda_1 \, d\lambda_2 + 4\pi \int_{-\infty}^{\infty} \frac{\phi_Z^2(\lambda; \theta^0)}{(1 + \lambda^2)^2} \, d\lambda \]

as \( T \to \infty \), where the \( O(1/T) \) term is uniform in \( \lambda \) and \( \phi_Z^{(4)}(\lambda_1, \lambda_2, \lambda_3; \theta) \) is the fourth-order cumulant spectra of the increment process \( Z \) defined in (2.17) below.

Proof. Define the cumulants \( C_Z^{(k)}(u_1, \ldots, u_{k-1}) \) of the increment process \( Z \) by

\[ \text{cum} \{ dZ(t_1), \ldots, dZ(t_k) \} = dC_Z^{(k)}(t_2 - t_1, \ldots, t_k - t_1) \, dt_1, \quad (2.15) \]
where $C^{(k)}_Z$ is of bounded variation over finite cubes. Then under Assumption 2.2 we have [1, pp. 485, 495]

$$\int_{R^{k-1}} (1 + |u_j|) \, d |C^{(k)}_Z(u_1, ..., u_{k-1})| < \infty; \quad k = 2, 3, 4, j = 1, ..., k - 1. \quad (2.16)$$

Moreover, define the $k$th-order cumulant spectrum $\phi^{(k)}_Z(\lambda_1, ..., \lambda_{k-1})$ by [1]

$$\phi^{(k)}_Z(\lambda_1, ..., \lambda_{k-1}) = \frac{1}{(2\pi)^{k-1}} \int_{R^{k-1}} \exp \left\{ -i \sum_{j=1}^{k-1} u_j \lambda_j \right\} dC^{(k)}_Z(u_1, ..., u_{k-1}) \quad (2.17)$$

and note that $\phi^{(k)}_Z$ is bounded, uniformly continuous, but not integrable in general. Also $\phi_Z(\lambda) \equiv \phi^{(2)}_Z(\lambda)$. By Theorem 4.1 of Brillinger [1], we have (see (2.12))

$$\text{cum}\{d_{Z,T}(\lambda_1), ..., d_{Z,T}(\lambda_k)\} = \frac{1}{(2\pi)^{k-1}} D_T \left( \sum_{j=1}^{k} \lambda_j \right) \phi^{(k)}_Z(\lambda_1, ..., \lambda_{k-1}) + O(1), \quad k = 2, 3, 4, \quad (2.18)$$

where

$$D_T(\lambda) = \int_0^T e^{-i \lambda t} \, dt = \frac{1 - e^{-i \lambda T}}{i \lambda}$$

and the $O(1)$ term is uniform in the $\lambda_j$'s. Hence,

$$E[d_{Z,T}(\lambda), d_{Z,T}(-\lambda)] = \phi_Z(\lambda; \theta^0) + O \left( \frac{1}{T} \right), \quad (2.19)$$

where $O(1/T)$ is uniform in $\lambda$. Thus by (2.11) and

$$\log(g(\lambda; \theta))/(1 + \lambda^2) \in L_1, \text{ by assumption,}$$

$$E[K_T(\theta)] = K(\theta) + O(1/T).$$

Let $L = (2\pi T)^2 \text{cov}\{d_{Z,T}(\lambda_1), d_{Z,T}(\lambda_2)\}$; then we have, by (2.18),

$$L = (2\pi)^3 T \phi^{(4)}_Z(\lambda_1, -\lambda_1, \lambda_2; \theta^0) + O(1)$$

$$+ \left[ (2\pi) D_T(\lambda_1 + \lambda_2) \phi_Z(\lambda_1; \theta^0) + O(1) \right] \times \left[ 2\pi D_T(-\lambda_1 - \lambda_2) \phi_Z(-\lambda_1; \theta^0) + O(1) \right]$$

$$+ \left[ (2\pi) D_T(\lambda_1 - \lambda_2) \phi_Z(\lambda_1; \theta^0) + O(1) \right] \times \left[ 2\pi D_T(-\lambda_1 + \lambda_2) \phi_Z(-\lambda_1; \theta^0) + O(1) \right].$$
With the notation $\Delta_T(\lambda) = (1/T) |D_T(\lambda)|^2$ being the Fejer kernel we have
\[
\text{cov}\{I_{Z,T}(\lambda_1), I_{Z,T}(\lambda_2)\} = \frac{1}{T} \left\{ 2\pi \phi_Z^{(4)}(\lambda_1, -\lambda_1, \lambda_2, \theta^0) + \phi_Z^2(\lambda_1; \theta^0) \left[ \Delta_T(\lambda_1 + \lambda_2) + \Delta_T(\lambda_1 - \lambda_2) \right] \right\}
\]
\[
+ \frac{O(1)}{T^2} \left\{ \phi_Z(\lambda_1; \theta^0) [D_T(\lambda_1 + \lambda_2) + D_T(\lambda_1 - \lambda_2)] + \phi_Z(-\lambda_1; \theta^0) [D_T(-\lambda_1 - \lambda_2) + D_T(-\lambda_1 + \lambda_2)] + 1 \right\},
\]
(2.20)
where the $O(1)$ term is uniform in $\lambda$'s. By (2.11) we then have
\[
\text{var}[\hat{K}_T(\theta)] = \int_{R^2} A(\lambda_1; \theta) A(\lambda_2; \theta) \text{cov}\{I_{Z,T}(\lambda_1), I_{Z,T}(\lambda_2)\} \ d\lambda_1 \ d\lambda_2
\]
(2.21)
and, substituting (2.20) in (2.21), we obtain
\[
T \text{var}[\hat{K}_T(\theta)] = 2\pi \int_{R^2} A(\lambda_1; \theta) A(\lambda_2; \theta) \phi_Z^{(4)}(\lambda_1, -\lambda_1, \lambda_2; \theta^0) \ d\lambda_1 \ d\lambda_2
\]
\[
+ \int_{R^2} \phi_Z^2(\lambda_1; \theta^0) [\Delta_T(\lambda_1 + \lambda_2) + \Delta_T(\lambda_1 - \lambda_2)] \ d\lambda_1 \ d\lambda_2
\]
\[
+ \Delta_T(\lambda_1 - \lambda_2) \right] A(\lambda_1; \theta) A(\lambda_2; \theta) \ d\lambda_1 \ d\lambda_2
\]
\[
+ \frac{O(1)}{T} \left\{ \int_{R^2} A(\lambda_1; \theta) A(\lambda_2; \theta) \phi_Z(\lambda_1; \theta^0) [D_T(\lambda_1 + \lambda_2) + D_T(\lambda_1 - \lambda_2)] \ d\lambda_1 \ d\lambda_2
\]
\[
+ \int_{R^2} A(\lambda_1; \theta) A(\lambda_2; \theta) \phi_Z(-\lambda_1; \theta^0)
\]
\[
\times [D_T(-\lambda_1 - \lambda_2) + D_T(-\lambda_1 + \lambda_2)] \ d\lambda_1 \ d\lambda_2 + 1 \right\}
\]
\[
= J_T + J_T^* + J_T^{**}.
\]
(2.22)
Now with $\Delta_T(\lambda) = TA(T\lambda)$, $A(\lambda) = (\sin(\lambda/2)/(\lambda/2))^2$ we have
\[
J_T \equiv \int_{R^2} \phi_Z^2(\lambda_1; \theta^0) A(\lambda_1; \theta) A(\lambda_2; \theta) \Delta_T(\lambda_1 - \lambda_2) \ d\lambda_1 \ d\lambda_2
\]
\[
= \int_{R^2} \phi_Z^2(\lambda_1; \theta^0) A(\lambda_1; \theta) A\left(\lambda_1 - \frac{v}{T}; \theta\right) A(v) \ d\lambda_1 \ dv.
\]
The integrand converges to \( \phi_Z(\lambda; \theta^0) A^2(\lambda; \theta) A(v) \) as \( T \to \infty \), since \( A(\lambda; \theta) \) is continuous in \( \lambda \) and is bounded by a constant multiple of \( A(\lambda_1; \theta) A(v) \in L_1(d\lambda_1 \times dv) \), since \( \phi_Z(\lambda_1; \theta^0) \) and \( A(\lambda_1 - v/T; \theta) \) are bounded. By dominated convergence we then have

\[
J_T \to 2\pi \int_{-\infty}^{\infty} \phi_Z(\lambda; \theta^0) A^2(\lambda; \theta) d\lambda \quad \text{as} \quad T \to \infty
\]

and, hence,

\[
J_T'' \to 2\pi \int_{-\infty}^{\infty} \phi_Z(\lambda; \theta^0) \left[ A^2(\lambda; \theta) + A(\lambda; \theta) A(-\lambda; \theta) \right] d\lambda.
\]

We next show that \( J'' \sim o(1) \) as \( T \to \infty \) from which the result follows. Now, since \( \phi_Z(\lambda; \theta^0) = \phi_Z(-\lambda; \theta^0) \), we have

\[
J'' = O\left(\frac{1}{T}\right) \left\{ 1 + \int_{\mathbb{R}^2} A(\lambda_1; \theta) A(\lambda_2; \theta) \phi_Z(\lambda_1; \theta^0) \right. \\
\left. \times \left[ \frac{\sin(\lambda_1 + \lambda_2) T}{(\lambda_1 + \lambda_2)} + \frac{\sin(\lambda_1 - \lambda_2) T}{(\lambda_1 - \lambda_2)} \right] d\lambda_1 d\lambda_2 \right\}.
\]

since \( \Re[D_T(\lambda)] = \sin \lambda T/\lambda \). Thus,

\[
J'' = O(1) \left\{ \frac{1}{T} + \int_{\mathbb{R}^2} A(\lambda_1; \theta) A(\lambda_2; \theta) \phi_Z(\lambda_1; \theta^0) \right. \\
\left. \times \left[ \frac{\sin(\lambda_1 \pm \lambda_2) T}{(\lambda_1 \pm \lambda_2) T} \right] d\lambda_1 d\lambda_2 \right\}.
\]

The integrand above tends to zero as \( T \to \infty \) except on the lines \( \lambda_1 = \pm \lambda_2 \) (which have zero measure on the plane). Also the integrand is bounded by a constant multiple of \( |A(\lambda_1; \theta) A(\lambda_2; \theta)| \in L_1(d\lambda_1 \times d\lambda_2) \), since \( \phi_Z(\lambda_1; \theta^0) \) is bounded. Hence by dominated convergence, \( J'' = O(1/T) + o(1) = o(1) \). The proof for \( \hat{\sigma}^2(\theta^0) \) is similar and is omitted.

It follows from Theorem 2.1 that \( \hat{K}_T(\theta) \) converges in quadratic-mean to \( K(\theta) \) as \( T \to \infty \) for every \( \theta \in \Theta \). It is therefore a consistent estimator of \( K(\theta) \).

We now establish the consistency of the estimate \( \hat{\theta}_T \). For this we need one additional condition on \( h(\lambda; \theta) \). We have

\[\text{Theorem 2.2.} \quad \text{Assume that Assumption 2.1 (for the unique identifiability of} \ \theta^0, \ \text{conditions of Theorem 2.1 (for the convergence of} \ \hat{K}_T(\theta) \ \text{to} \ K(\theta) \) \text{hold and} \]
(a) For any $\varepsilon > 0$, $\exists \eta = \eta(\varepsilon) > 0$ such that $|\log g(\lambda; \theta') - \log g(\lambda; \theta'')| < \varepsilon$ for all $\lambda$ whenever $\|\theta' - \theta''\| < \eta$.

Then whenever the cumulants of the processes $X$ and $N$ satisfy the integrability conditions of Assumption 2.2 we have $\hat{\theta}_T \to \theta^0$ in probability as $T \to \infty$.

We remark that condition (a) is satisfied when $X$ is an ARMA process and $\{\tau_k\}$ is a Poisson point process.

Proof. We follow the argument of Theorem 1 in Ibragimov [4] adapted to our setting. Let $\varepsilon > 0$ be an arbitrary fixed number. For each $\varepsilon_1 > 0$ choose a set of points $\theta_j$ in $\Theta_1 = \Theta - \{\theta : \|\theta - \theta^0\| < \varepsilon\}$ such that the sphere $S_j$ with centers $\theta_j$ and radius $\varepsilon_1$ cover the set $\Theta_1$. Since $\Theta$ is compact by (a) of Assumption 2.1, the number of $\theta_j$'s can be finite. Now

$$P[\|\hat{\theta}_T - \theta^0\| \geq \varepsilon] \leq \sum_j P[\|\hat{\theta}_T - \theta_j\| < \varepsilon_1].$$

As in the proof of Theorem 1 in [4] we have, by using Assumption 2.1(b) and condition (a) in Theorem 2.1, for every $\varepsilon_2 > 0$,

$$P[\|\hat{\theta}_T - \theta^0\| \geq \varepsilon] \leq \sum_j P[|\hat{K}_T(\hat{\theta}_T) - \hat{K}_T(\theta_j)| < \varepsilon_2]$$

$$+ P[\max_{j, \theta \in S_j} |\hat{K}_T(\theta) - \hat{K}_T(\theta_j)| \geq \varepsilon_2].$$

(2.23)

With the choice $\varepsilon_2 = \frac{1}{2} \min_{\theta_1} |K(\theta) - K(\theta^0)|$, we have

$$P[|\hat{K}_T(\hat{\theta}_T) - \hat{K}_T(\theta_j)| < \varepsilon_2] \leq P[|\hat{K}_T(\theta^0) - K(\theta^0)| \geq 2\varepsilon_2]$$

$$+ P[|\hat{K}_T(\theta_j) - K(\theta_j)| \geq \varepsilon_2]$$

$$\to 0 \quad \text{as} \quad T \to \infty$$

(2.24)

by Theorem 2.1. For the second term on the right side of (2.23) we proceed as follows:

$$J \triangleq \max_{\lambda, \theta \in S_j} |\hat{K}_T(\theta) - \hat{K}_T(\theta_j)|$$

$$= \max_{\lambda, \theta \in S_j} \left| \int_{-\infty}^{\infty} \log \left( \frac{g(\lambda; \theta)}{g(\lambda; \theta)} \right) \frac{I_{Z,T}(\lambda)}{1 + \lambda^2} d\lambda \right|$$

$$\leq \sup_{\lambda, \theta \in S_j} \left| \log \frac{g(\lambda; \theta)}{g(\lambda; \theta_j)} \right| \int_{-\infty}^{\infty} \frac{I_{Z,T}(\lambda)}{1 + \lambda^2} d\lambda$$

$$= \rho(\varepsilon_1) \int_{-\infty}^{\infty} \frac{I_{Z,T}(\lambda)}{1 + \lambda^2} d\lambda.$$
Note that by condition (a) on \( g(\lambda; \theta) \), \( \rho(\epsilon_1) \to 0 \) as \( \epsilon_1 \to 0 \). Also, since 
\[
E[I_{Z,T}(\lambda)] = \phi_Z(\lambda; \theta^0) + O(1/T),
\]
we have
\[
E[J] \leq \rho(\epsilon_1) \left\{ \int_{-\infty}^{\infty} \frac{\phi_Z(\lambda; \theta^0) + O(1)}{1 + \lambda^2} d\lambda \right\} \leq M \rho(\epsilon_1)
\]
for some constant \( M \), since \( \phi_Z(\lambda; \theta^0) \) is bounded. Hence, by the Markov inequality,
\[
P[J \geq \epsilon_2] \leq \frac{E[J]}{\epsilon_2} \leq \frac{M \rho(\epsilon_1)}{\epsilon_2}.
\]
Now choose \( \epsilon_1 \) so that \( \rho(\epsilon_1) \leq \delta \epsilon_2 / M \), where \( \delta > 0 \) is a fixed arbitrarily small number. Then
\[
P[\max_{\lambda, \theta \in S} |\hat{K}_T(\theta) - K_T(\theta)| \geq \epsilon_2] \leq \delta
\]
and, by (2.23) and (2.24),
\[
P[\|\theta_T - \theta^0\| \geq \epsilon] \leq o(1) + \delta.
\]
We now consider the estimation of the multiplicative parameter \( \eta \) in \( \phi_X(\lambda) \). Suppose \( \phi_X(\lambda) = \eta \phi_X(\lambda; \theta) \). That is, we assume that the parametric form of the spectral density \( \phi_X(\lambda) \) is known and given by \( \phi(\lambda; \theta) \) through parameter \( \theta \) up to an unknown multiplicative constant \( \eta \). In the previous discussion, \( \phi_Z(\lambda; \theta) \), which corresponds to \( \phi_X(\lambda; \theta) \), was used in (2.8) where \( \eta \) is not involved. Then, similarly, \( \phi_Z(\lambda) = \eta \phi_Z(\lambda; \theta) \) so that
\[
\sigma^2(\theta; \eta) = \int_{-\infty}^{\infty} \frac{\phi_Z(\lambda; \theta)}{1 + \lambda^2} d\lambda = \eta \int_{-\infty}^{\infty} \frac{\phi_Z(\lambda; \theta^0)}{1 + \lambda^2} d\lambda \equiv \eta \bar{\sigma}^2(\theta),
\]
where \( \bar{\sigma}^2(\theta) \) is obtained through the definition of \( \bar{\sigma}^2(\theta) \) in (2.25). Since \( \bar{\sigma}^2(\theta) \) is a continuous and positive function, \( \bar{\sigma}^2(\theta_T) \to \bar{\sigma}^2(\theta^0) \) in probability. Hence \( \hat{\eta}_T \to \eta \) in probability as \( T \to \infty \).

An alternative approach is to base \( \hat{\eta}_T \) on an estimate of \( R_X(0) \). Set
\[
R_X(0; \theta, \eta) = \int_{-\infty}^{\infty} \phi_X(\lambda) d\lambda = \eta \int_{-\infty}^{\infty} \phi_X(\lambda; \theta) d\lambda \equiv \eta R_X(0; \theta).
\]
Note that \( \tilde{R}_X(0; \theta) \) is continuous in \( \theta \) whenever \( \phi_X(\lambda; \theta) \) is continuous in \( \theta \). We now estimate \( R_X(0; \theta^0, \eta) \) by

\[
\tilde{R}_X(0; \theta^0, \eta) = \frac{1}{\beta T} \left[ \int_0^T X^2(t) \, dN(t) \right] = \frac{1}{\beta T} \sum_{j=1}^{N(T)} X^2(\tau_j)
\]

and \( \eta \) by

\[
\tilde{\eta}_T = \frac{\tilde{R}_X(0; \theta^0, \eta)}{R_X(0; \bar{\theta}_T)},
\]

where \( \tilde{R}_X(0; \bar{\theta}_T) \) is obtained through \( \tilde{R}_X(0; \theta) \) in (2.26). It is easy to show that \( \tilde{R}_X(0; \theta^0, \eta) \) converges in quadratic-mean to \( R_X(0; \theta^0, \eta) \) and \( \tilde{\eta}_T \) is a consistent estimate of \( \eta \) as \( T \to \infty \).

3. JOINT ASYMPTOTIC NORMALITY

In this section we establish the asymptotic normality of the estimate \( \theta_T \); i.e., we wish to show that \( \sqrt{T}(\theta_T - \theta^0) \) is asymptotically normal with mean zero and covariance matrix \( \Sigma \),

\[
\Sigma = W^{-1}QW^{-1},
\]

where the entries of \( W \) are given by

\[
w_{i,j} = -\int_{-\infty}^{\infty} \frac{\partial \log g(\lambda; \theta^0)}{\partial \theta_i^0} \frac{\partial \log g(\lambda; \theta^0)}{\partial \theta_j^0} \phi_2(\lambda; \theta^0) \frac{1}{1 + \lambda^2} \, d\lambda, \quad i, j = 1, ..., s,
\]

and the entries of \( Q \) are given by

\[
q_{i,j} = 2\pi \left\{ 2 \int_{-\infty}^{\infty} \frac{\partial \log g(\lambda; \theta^0)}{\partial \theta_i^0} \frac{\partial \log g(\lambda; \theta^0)}{\partial \theta_j^0} \phi_2(\lambda; \theta^0) \frac{1}{1 + \lambda^2} \, d\lambda \\
+ \int_{-\infty}^{\infty} \frac{\partial \log g(\lambda_1; \theta^0)}{\partial \theta_i^0} \frac{\partial \log g(\lambda_2; \theta^0)}{\partial \theta_j^0} \phi_2(\lambda_1, \lambda_2; \theta^0) \frac{1}{(1 + \lambda_1^2)(1 + \lambda_2^2)} \, d\lambda_1 \, d\lambda_2 \right\}.
\]

In this section we assume that the process \( X \) has finite moments of all orders and that the counting process \( N(A) \) has finite moments of all orders over finite interval \( A \) with the following integrability conditions.
Assumption 3.1. For all \( k \geq 2 \) and \( j = 1, \ldots, k - 1 \), we have

\[
\int_{R^{k-1}} (1 + |u_j|) |c^{(k)}(u_1, \ldots, u_{k-1})| \, du_1 \cdots du_{k-1} < \infty
\]

\[
\int_{R^{k-1}} (1 + |u_j|) |c^{(k)}(u_1, \ldots, u_{k-1})| \, du_1 \cdots du_{k-1} < \infty.
\]

We next impose smoothness conditions on \( g(\lambda; \theta) \) which allows differentiation of \( K(\theta) \) and \( \hat{K}_T(\theta) \) with respect to \( \theta \) under the integral sign.

Assumption 3.2. Let \( B_{\delta}(\theta^0) \) be a \( \delta \)-neighborhood of \( \theta^0 \). The partial derivatives \( \partial g(\lambda; \theta)/\partial \theta_i, \partial^2 g(\lambda; \theta)/\partial \theta_i \partial \theta_j, \partial^3 g(\lambda; \theta)/\partial \theta_i \partial \theta_j \partial \theta_k \) exist for \( \theta \in B_{\delta}(\theta^0) \). Set

\[
h^{(i)}(\lambda; \theta) = \frac{\partial \log g(\lambda; \theta)}{\partial \theta_i}, \quad i = 1, \ldots, s
\]

\[
h^{(i,j)}(\lambda; \theta) = \frac{\partial^2 \log g(\lambda; \theta)}{\partial \theta_i \partial \theta_j}, \quad i, j = 1, \ldots, s
\]

\[
h^{(i,j,k)}(\lambda; \theta) = \frac{\partial^3 \log g(\lambda; \theta)}{\partial \theta_i \partial \theta_j \partial \theta_k}, \quad i, j, k = 1, \ldots, s.
\]

Assume that for \( \theta \in B_{\delta}(\theta^0) \), \( h^{(i)}(\lambda; \theta) \) and \( h^{(i,j)}(\lambda; \theta) \) are bounded and continuous in \( \lambda \) and

\[
|h^{(i)}(\lambda; \theta)| < H^{(i)}(\lambda) \in L_1 \left( \frac{d\lambda}{1 + \lambda^2} \right)
\]

(3.4a)

\[
|h^{(i,j)}(\lambda; \theta)| < H^{(i,j)}(\lambda) \in L_1 \left( \frac{d\lambda}{1 + \lambda^2} \right)
\]

(3.4b)

\[
\sum_{k=1}^{s} \sup_{\lambda, \theta \in B_{\delta}(\theta^0)} |h^{(i,j,k)}(\lambda; \theta)| = M_{i,j} < \infty.
\]

(3.4c)

Under Assumption 3.2, (3.4a) and (3.4b), we have, by dominated convergence,

\[
\hat{K}_T^{(i)}(\theta) \triangleq \frac{\partial \hat{K}_T(\theta)}{\partial \theta_i} = \int_{-\infty}^{\infty} h^{(i)}(\lambda; \theta) \frac{I_{Z,T}(\lambda)}{1 + \lambda^2} \, d\lambda
\]

(3.5)

and

\[
\hat{K}_T^{(i,j)}(\theta) \triangleq \frac{\partial^2 \hat{K}_T(\theta)}{\partial \theta_i \partial \theta_j} = \int_{-\infty}^{\infty} h^{(i,j)}(\lambda; \theta) \frac{I_{Z,T}(\lambda)}{1 + \lambda^2} \, d\lambda,
\]

(3.6)
using the fact that $I_{Z,T}(\lambda)$ is bounded by a random variable (with a finite second moment) uniformly in $\lambda$. We have

**Theorem 3.1.** Assume that conditions of Theorem 2.2 are satisfied. Then under Assumptions 3.1 and 3.2 with the matrix $W = [w_{i,j}]$ of (3.2) being nonsingular we have that $\sqrt{T}(\hat{\theta}_T - \theta^0)$ is asymptotically normal with mean zero and covariance matrix given in (3.1).

The following lemma, whose proof is similar to that of Theorem 2.1, will be used repeatedly.

**Lemma 3.1.** Let $G(\lambda) \in L_1(d\lambda)$ be a bounded and continuous function on the real line. Set

$$J_T = \int_{-\infty}^{\infty} G(\lambda) I_{Z,T}(\lambda) \, d\lambda, \quad J = \int_{-\infty}^{\infty} G(\lambda) \phi_Z(\lambda; \theta^0) \, d\lambda.$$  \hspace{1cm} (3.7)

Then under Assumption 3.1 (with $k = 2, 3, 4$) we have

$$E[J_T] = J + O(1/T)$$

$$T \text{var}[J_T] \rightarrow 2\pi \int_{\mathbb{R}^2} G(\lambda_1) G(\lambda_2) \phi_Z^{(4)}(\lambda_1, -\lambda_1, \lambda_2, \lambda_3; \theta^0) \, d\lambda_1 \, d\lambda_2$$

$$+ 4\pi \int_{-\infty}^{\infty} G^2(\lambda) \phi_Z^2(\lambda; \theta^0) \, d\lambda$$  \hspace{1cm} (3.8)

as $T \rightarrow \infty$, where the $O(1/T)$ term is uniform in $\lambda$ and $\phi_Z(\lambda; \theta)$ and $\phi_Z^{(4)}(\lambda_1, \lambda_2, \lambda_3; \theta)$ are given in (2.5) and (2.17), respectively, with $\theta$ implicit.

The following lemma is needed.

**Lemma 3.2.** Under Assumption 3.1 and the assumption on $G(\lambda)$ in Lemma 3.1, we have $\sqrt{T}(J_T - J)$ is asymptotically normal with mean zero and variance given by the right side of (3.8).

In proving Lemma 3.2 we utilize the following proposition whose proof is omitted.

**Proposition 3.1.** Let $G \in L_1(d\lambda)$, $|G(\lambda)| \leq K < \infty$ for all $\lambda$, and

$$D_T(\lambda) = \int_0^T e^{-i\lambda t} \, dt = e^{-i\lambda T/2} T \left( \frac{\sin \lambda T/2}{\lambda T/2} \right),$$

then

$$\int_{-\infty}^{\infty} G(\lambda) D_T(\lambda - u) \, d\lambda = O(\log T) \text{ uniformly in } u.$$
Proof of Lemma 3.2. We first note that under Assumption 3.1 the integrability of (2.16) holds for all \(k \geq 2\) and consequently (2.18) also holds for all \(k \geq 2\) (cf. [1]).

In order to prove the lemma, it suffices to show that all cumulants of \(\sqrt{T} J_T\) of order \(k \geq 3\) tend to zero as \(T \to \infty\). Fix \(k \geq 3\); using the notation in the proof of Theorem 2.1 and (2.12), we have

\[
T^{k/2} \text{cum}_k(J_T, \ldots, J_T) = \int_{R^k} \left[ \prod_{j=1}^k G(\lambda_j) \right] h_k(\lambda_1, \ldots, \lambda_k) \, d\lambda_1 \cdots d\lambda_k, \tag{3.9}
\]

where

\[
h_k(\lambda_1, \ldots, \lambda_k) \triangleq T^{k/2} \text{cum}\{J_{Z,T}(\lambda_1), \ldots, J_{Z,T}(\lambda_k)\} \\
= \frac{1}{(2\pi \sqrt{T})^k} \sum_{p=1}^k \sum_v \text{cum}(\{d_{Z,T}(\lambda_i); l \in v_1\}) \cdots \text{cum}(\{d_{Z,T}(\lambda_i); l \in v_p\}) \tag{3.10}
\]

and the inner sum is over all indecomposable partitions \(v = v_1 \cup \cdots \cup v_p\) of the transformed table:

\[
\begin{array}{cccc}
\lambda_1 & -\lambda_1 & 1 & -1 \\
\lambda_2 & -\lambda_2 & 2 & -2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_k & -\lambda_k & k & -k
\end{array}
\]

(see [2, Theorem 2.3.2]). Note that in using the transformed table, \(\text{cum}(\{d_{Z,T}(\lambda_i); l \in v_j\})\) denotes the cumulant of all random variables \(\{d_{Z,T}(\lambda_i), l \in v_j\}\) with the convention \(\lambda_i = -\lambda_{|i|}\) for \(l < 0\). Any partition \(v\) which has a single element in a subset \(v_j\) contributes zero, since \(\text{cum}(d_{Z,T}(\lambda_i)) = E[d_{Z,T}(\lambda_i)] = 0\). Hence in any partition \(v\), all subsets \(v_j\) must have at least two elements from the transformed table and thus the sum in (3.10) over \(p\) has an upper limit \(k\). Denote by \(#(v_j)\) the number of elements in the subset \(v_j\).

Consider first the case \(p = 1\) in (3.10). Then by (2.18) we have

\[
h_k(\lambda_1, \ldots, \lambda_k) = \frac{(2\pi)^{2k-1}}{(2\pi \sqrt{T})^k} D_T(0) \phi_z^{(2k)} \\
\times \left\{ \{\lambda_i, l = \pm 1, \ldots, \pm (k-1), l = k\} \right\} + O(1) \\
= O(T^{-k/2}),
\]

since \(\phi_z^{(2k)}\) is bounded and \(D_T(0) = T\). Hence the contribution to (3.9) of
the case \( p = 1 \) is of the order \( O(T^{1-k/2}) \to 0 \) as \( T \to \infty \) for \( k \geq 3 \), using the
integrability of \( G \).

Assume now that \( 1 < p < k \). Given an indecomposable partition \( v = v_1 \cup \cdots \cup v_p \) of the transformed table, there exists at least one \( j \) such that
\( j \in v_m \) and \( -j \in v_n \) with \( m \neq n \); otherwise the partition is not indecomposable. Without loss of generality let \( j = 1, m = 1, \) and \( n = 2 \) for notational convenience. By (2.18) and (3.10) the contribution of this partition to (3.9) is

\[
B = \frac{1}{(2\pi \sqrt{T})^k} \int_{\mathbb{R}^k} \left[ \prod_{j=1}^{k} G(\lambda_j) \right] \\
\times \left[ (2\pi)^{\#(v_1)-1} DT \left( \lambda_1 + \sum_{l \in v_1, l \neq 1} \lambda_l \right) \phi_Z(\lambda_1, \{\lambda_l, l \in v_1'\}) + O(1) \right] \\
\times \left[ (2\pi)^{\#(v_2)-1} DT \left( -\lambda_1 + \sum_{l \in v_2, l \neq -1} \lambda_l \right) \right. \\
\times \left. \phi_Z(-\lambda_1, \{\lambda_l, l \in v_2'\}) + O(1) \right] \\
\times \prod_{i=3}^{p} \left[ (2\pi)^{\#(v_i)-1} DT \left( \sum_{l \in v_i} \lambda_l \right) \right. \\
\times \left. \phi_Z(\{\lambda_l, l \in v_i\}) + O(1) \right] \, d\lambda_1 \cdots d\lambda_k 
\tag{3.11}
\]

with the expanded notation that \( \phi_Z(\{\lambda_l, l \in v_i\}) \) being the cumulant spectrum of order \( \#(v_i) \) and thus has \( \#(v_i)-1 \) arguments. Also
\( \phi_Z(\lambda_j, \{\lambda_l, l \in v'_i\}) = \phi_Z(\lambda_j, \{\lambda_l, l \in v_i \text{ but } l \neq j\}) \) is a cumulant spectrum of order \( \#(v_i) \) and thus has \( \#(v_i)-1 \) arguments. The expansion of the product of the form \( \prod_{i=1}^{p} [a_i + O(1)] \) in the integrand of (3.11) has many terms. The most significant term is \( \prod_{i=1}^{p} a_i \), which involves the product of all the Dirichlet kernels \( D_T \)'s. Let \( B_1 \) be its contribution to (3.11) and let \( B_2 \) be the contribution of all other terms. We have

\[
B_1 = \frac{(2\pi)^{2k-p}}{(2\pi \sqrt{T})^k} \int_{\mathbb{R}^{k-1}} \left\{ \int_{-\infty}^{\infty} DT \left( \lambda_1 + \sum_{l \in v_1, l \neq 1} \lambda_l \right) \right. \\
\times \phi_Z(\lambda_1, \{\lambda_l, l \in v_1'\}) \left. \right] \\
\times \left. DT \left( -\lambda_1 + \sum_{l \in v_2, l \neq -1} \lambda_l \right) \phi_Z(-\lambda_1, \{\lambda_l, l \in v_2'\}) \right. \\
\times \left. G(\lambda_1) \, d\lambda_1 \right\} \\
\times \prod_{i=3}^{p} DT \left( \sum_{l \in v_i} \lambda_l \right) \phi_Z(\{\lambda_l, l \in v_i\}) \prod_{i=2}^{k} G(\lambda_i) \, d\lambda_2 \cdots d\lambda_k. \tag{3.12}
\]
The inner integral with respect to $\lambda_1$ can be bounded by the Cauchy–Schwarz inequality, using the boundedness of $\phi_Z$ and $G$, by

$$\text{Const} \left\{ \int_{-\infty}^{\infty} |D_T(\lambda_1 + u)|^2 d\lambda_1 \int_{-\infty}^{\infty} |D_T(-\lambda_1 + v)| d\lambda_1 \right\}^{1/2}$$

$$= \text{Const} \int_{-\infty}^{\infty} \frac{1}{T} |D_T(\lambda)|^2 d\lambda$$

$$= \text{Const} T$$  \hspace{1cm} (3.13)

where $\text{Const}$ means a constant factor. Hence

$$|B_1| \leq \frac{\text{Const}}{T^{k/2 - 1}} \int_{\mathbb{R}^k} \prod_{j=3}^{p} \left| D_T \left( \sum_{l \in v_j} \lambda_l \right) \phi_Z(\{\lambda_l, l \in v_j\}) \right|$$

$$\times \prod_{i=2}^{k} |G(\lambda_i)| \, d\lambda_2 \cdots d\lambda_k$$

and, by the boundedness of $\phi_Z$,

$$|B_1| \leq \frac{\text{Const}}{T^{k/2 - 1}} \int_{\mathbb{R}^k} \prod_{j=3}^{p} \left| D_T \left( \sum_{l \in v_j} \lambda_l \right) \right| \prod_{i=2}^{k} |G(\lambda_i)| \, d\lambda_2 \cdots d\lambda_k. \quad (3.14)$$

If $p = 2$ there will be no Dirichlet kernels $D_T$'s left in the integrand and, by the integrability of the $G$'s, we would have $B_1 = O(T^{1 - k/2}) \to 0$ as $T \to \infty$ for $k \geq 3$. If $p > 2$ then there exists an $l \neq 1$ such that $l \in v_j$ for $j \geq 3$ and $-l$ does not appear in any other remaining subset of the partition $v_i$; otherwise the partition would be decomposable. Without loss of generality, let $j = 3$ and $l = 2$. Now using Proposition 3.1 in (3.14) with respect to the integration of $d\lambda_2$ we obtain

$$|B_1| \leq \frac{\text{Const}}{T^{k/2 - 1}} (\log T) \int_{\mathbb{R}^{k-1}} \prod_{j=4}^{p} \left| D_T \left( \sum_{l \in v_j} \lambda_l \right) \right| \prod_{i=3}^{k} |G(\lambda_i)| \, d\lambda_3 \cdots d\lambda_k.$$

If $p = 3$, a final integration of $|G(\lambda_3)|$ gives

$$B_1 = O \left( \frac{\log T}{T^{k/2 - 1}} \right) \to 0 \quad \text{as} \quad T \to \infty \quad \text{for} \quad k \geq 3.$$

Continuing in this manner, using Proposition 3.1 repeatedly, we will have $p - 4$ additional integrations involving $D_T G$ and $k - 3 - (p - 4) = k - p + 1$
additional integrations involving \( G \)'s only. Hence the dominant term in (3.9) for a fixed \( 1 < p \leq k \) is

\[
|B_1| \leq \frac{\text{Const}}{T^{k/2-1}} (\log T)^{p-2} = \frac{\text{Const}}{(\sqrt{T})^{k-p}} \left( \frac{\log T}{\sqrt{T}} \right)^{p-2} \to 0
\]

as \( T \to \infty \) whenever \( 1 < p \leq k, k \geq 3 \). (3.15)

The same argument can be applied when at least one \( O(1) \) term is present in the expansion of the product of the form \( \prod_{i=1}^{p} \left[ a_i + O(1) \right] \) in the integrand of (3.11). Since at least one \( D_T \) term will be replaced by \( O(1) \), this will result in at least one factor \( \log T \) less than that in the bound for the dominant term \( B_1 \) of (3.15). Thus all other terms in the product expansion in the integrand of (3.11) are bounded by

\[
B_2 \leq \frac{\text{Const}}{(\sqrt{T})^{k-p}} (\log T)^{p-2} \to 0
\]

as \( T \to \infty \) whenever \( 1 < p \leq k \).  

**Proof of Theorem 3.1.** By the consistency of \( \hat{\theta}_T \) established in Theorem 2.2 we have for \( \delta > 0 \), \( P[\|\hat{\theta}_T - \theta^0\| < \delta] \to 1 \) as \( T \to \infty \). Suppose \( \hat{\theta}_T \in B_{\delta}(\theta^0) \). It is clear that \( \hat{K}_T^{(i)}(\hat{\theta}_T) = 0 \) for \( i = 1, \ldots, s \), since \( \hat{\theta}_T \) maximizes \( K_T(\theta) \) so that, by the mean value theorem,

\[
0 = \left. \hat{K}_T^{(i)}(\theta^0) + \sum_{j=1}^{s} (\hat{\theta}_{T,j} - \theta^0_j) \hat{K}_T^{(i,j)}(\theta^0) \right|_{\theta^0} = \left. \sum_{j=1}^{s} \left[ -\hat{K}_T^{(i,j)}(\theta^0) \right] \sqrt{T} (\hat{\theta}_{T,j} - \theta^0_j) \right|_{\theta^0} = \sqrt{T} \hat{K}_T^{(i)}(\theta^0).
\]

with

\[
\theta^* = \alpha \hat{\theta}_T + (1 - \alpha) \theta^0 \in B_{\delta}(\theta^0)
\]

(\( \theta^* \) actually depends on \( i \)). Hence

\[
\sum_{j=1}^{s} \left[ -\hat{K}_T^{(i,j)}(\theta^*) \right] \sqrt{T} (\hat{\theta}_{T,j} - \theta^0_j) = \sqrt{T} \hat{K}_T^{(i)}(\theta^0).
\]

We now wish to show that

(a) \( \hat{K}_T^{(i)}(\theta^*) \to \lim_{T \to \infty} E[\hat{K}_T^{(i)}(\theta^0)] = w_{i,j} \) in probability as \( T \to \infty \)

(b) \( \sqrt{T} \hat{K}_T^{(i)}(\theta^0), i = 1, \ldots, s \) is asymptotically normal with mean zero and covariance matrix \( Q \) given by (3.3).

The conclusion of the theorem would then follow from (3.18).
For (3.19a) it suffices to show

\[ |\hat{K}_T^{(i,j)}(\theta^*) - \hat{K}_T^{(i,j)}(\theta^0)| \to 0 \quad \text{in probability}, \]

(3.20a)

\[ \text{var}[\hat{K}_T^{(i,j)}(\theta^0)] \to 0, \]

(3.20b)

and

\[ \lim_{T \to \infty} E[\hat{K}_T^{(i,j)}(\theta^0)] = \omega_{i,j}. \]

(3.20c)

For (3.20a) we have by (3.6)

\[ |\hat{K}_T^{(i,j)}(\theta^*) - \hat{K}_T^{(i,j)}(\theta^0)| \]

\[ \leq \int_{-\infty}^{\infty} |h^{(i,j)}(\lambda; \theta^*) - h^{(i,j)}(\lambda; \theta^0)| \frac{I_{Z,T}(\lambda)}{1 + \lambda^2} d\lambda. \]

(3.21)

By the mean value theorem we have

\[ h^{(i,j)}(\lambda, \theta^*) - h^{(i,j)}(\lambda; \theta^0) = \sum_{k=1}^{s} (\theta^*_{i,k} - \theta^0_{i,k}) h^{(i,j,k)}(\lambda, \eta \theta^0 + (1 - \eta) \theta^*). \]

for some 0 < \eta < 1. Thus,

\[ |h^{(i,j)}(\lambda; \theta^*) - h^{(i,j)}(\lambda; \theta^0)| \]

\[ \leq \|\theta^* - \theta^0\| \left\{ \sum_{k=1}^{s} \left[ h^{(i,j,k)}(\lambda, \eta \theta^0 + (1 - \eta) \theta^*) \right]^2 \right\}^{1/2} \]

and, by condition (3.4c) (Assumption 3.2), this is bounded by

\[ M_{i,j} \|\theta^* - \theta^0\|. \]

It then follows from (3.21) that

\[ |\hat{K}_T^{(i,j)}(\theta^*) - \hat{K}_T^{(i,j)}(\theta^0)| \leq M_{i,j} \|\theta^* - \theta^0\| \int_{-\infty}^{\infty} \frac{I_{Z,T}(\lambda)}{1 + \lambda^2} d\lambda. \]

(3.22)

By Lemma 3.1 with \( G(\lambda) = 1/(1 + \lambda^2) \) we have

\[ \int_{-\infty}^{\infty} \frac{I_{Z,T}(\lambda)}{1 + \lambda^2} d\lambda \to \int_{-\infty}^{\infty} \frac{\phi_Z(\lambda; \theta^0)}{1 + \lambda^2} d\lambda \]

in quadratic-mean (and thus in probability) as \( T \to \infty \). Thus the right side of (3.22) tends to zero in probability as \( T \to \infty \) by the consistency of \( \hat{\theta}_T \) and thus (3.20a) is established. To prove (3.20b) we have by (3.6)

\[ \hat{K}_T^{(i,j)}(\theta^0) = \int_{-\infty}^{\infty} \frac{h^{(i,j)}(\lambda; \theta^0)}{1 + \lambda^2} I_{Z,T}(\lambda) d\lambda. \]
With $G(\lambda) = h^{(i,j)}(\lambda; \theta^0)/(1 + \lambda^2)$, we have by Assumption 3.2 that $G(\lambda)$ satisfies the required conditions of Lemma 3.1 and, thus, $\text{var}[\hat{K}_T^{(i,j)}(\theta^0)] \to 0$ as $T \to \infty$, establishing (3.20b). Next we proceed to prove (3.20c). We have, by (2.19) and (3.4b),

$$E[\hat{K}_T^{(i,j)}(\theta^0)] = \int_{-\infty}^{\infty} \frac{h^{(i,j)}(\lambda; \theta^0)}{1 + \lambda^2} E[I_{Z,r}(\lambda)] \, d\lambda = \int_{-\infty}^{\infty} \frac{h^{(i,j)}(\lambda; \theta^0)}{1 + \lambda^2} \phi_Z(\lambda; \theta^0) \, d\lambda + O\left(\frac{1}{T}\right).$$

(3.23)

We next show that the integral on the right side of (3.23) is equal to $w_{i,j}$ from which (3.20c) would follow. Now

$$h^{(i,j)}(\lambda; \theta^0) = \frac{g^{(i,j)}(\lambda; \theta^0)}{g(\lambda; \theta^0)} \frac{g^{(j)}(\lambda; \theta^0) g^{(j)}(\lambda; \theta^0)}{g^2(\lambda; \theta^0)}$$

and

$$\int_{-\infty}^{\infty} \frac{g^{(i)}(\lambda; \theta^0) \phi_Z(\lambda; \theta^0)}{g(\lambda; \theta^0)} \, d\lambda = \sigma^2(\theta^0) \int_{-\infty}^{\infty} \frac{g^{(i)}(\lambda; \theta^0)}{1 + \lambda^2} \, d\lambda$$

and the right side is equal to zero, since $\int_{-\infty}^{\infty} (g(\lambda; \theta^0)/(1 + \lambda^2)) \, d\lambda = 1$ (the differentiation under the integral sign is justified in view of Assumption 3.2 on the log of $g(\lambda; \theta^0)$). Hence by (3.23)

$$\lim_{T \to \infty} E[\hat{K}_T^{(i,j)}(\theta^0)] = -\int_{-\infty}^{\infty} \frac{\partial \log g(\lambda; \theta^0)}{\partial \theta_i^0} \frac{\partial \log g(\lambda; \theta^0) \phi_Z(\lambda; \theta^0)}{1 + \lambda^2} \, d\lambda = w_{i,j}.$$

Having established (3.20a)–(3.20c), property (3.19a) follows.

We now turn to the proof of (3.19b). We have

$$\hat{K}_T^{(i,j)}(\theta^0) = \int_{-\infty}^{\infty} \frac{h^{(i,j)}(\lambda; \theta^0)}{1 + \lambda^2} I_{Z,r}(\lambda) \, d\lambda$$

(3.24)

and set

$$G(\lambda) \triangleq \left[ \sum_{i=1}^{s} x_i h^{(i)}(\lambda; \theta^0) \right]/(1 + \lambda^2)$$

(3.25)
in (3.7), where \( \alpha_i \)'s are arbitrary constants. Clearly it suffices to show that \( J_T \) in (3.7) is asymptotically normal with mean zero and variance \( \sum_{i=1}^{s} \sum_{j=1}^{s} \alpha_i \alpha_j \). By Assumption 3.2 it is clear that \( G(\lambda) \) of (3.25) is bounded, continuous, and in \( L_1(d\lambda) \). Hence by Lemma 3.2, \( \sqrt{T}(J_T - J) \) is asymptotically normal with mean zero and variance

\[
4\pi \int_{-\infty}^{\infty} G^2(\lambda) \phi_Z^2(\lambda; \theta^0) d\lambda + 2\pi \int_{R^2} G(\lambda_1) G(\lambda_2) \phi_Z^4(\lambda_1, -\lambda_1, \lambda_2; \theta^0) d\lambda_1 d\lambda_2
\]

\[
= \sum_{i=1}^{s} \sum_{j=1}^{s} \alpha_i \alpha_j \left\{ 4\pi \int_{-\infty}^{\infty} h^{(i)}(\lambda; \theta^0) h^{(j)}(\lambda; \theta^0) \phi_Z^2(\lambda; \theta^0) \frac{d\lambda}{1 + \lambda^2} \right\}
\]

\[
= \sum_{i=1}^{s} \sum_{j=1}^{s} \alpha_i \alpha_j q_{i,j}
\]

by (3.25) and (3.3). To complete the proof of (3.19b) we need to show that \( J \) of (3.7) with \( G \) of (3.25) is equal to zero. Now

\[
\int_{-\infty}^{\infty} \frac{h^{(i)}(\lambda; \theta^0)}{1 + \lambda^2} \phi_Z(\lambda; \theta^0) d\lambda = \int_{-\infty}^{\infty} \frac{\partial g(\lambda; \theta^0) \phi_Z(\lambda; \theta^0)}{\partial \theta_i} \frac{d\lambda}{g(\lambda; \theta^0) (1 + \lambda^2)}
\]

\[
= \sigma^2(\theta^0) \int_{-\infty}^{\infty} \frac{\partial g(\lambda; \theta^0) / \partial \theta_i}{1 + \lambda^2} d\lambda = 0,
\]

since \( \int_{-\infty}^{\infty} \frac{g(\lambda; \theta^0)}{(1 + \lambda^2)} d\lambda = 1 \). This concludes the proof of the theorem. ■

REFERENCES


