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Pairs of matrices that preserve the value of a generalized matrix function on the set of the upper triangular matrices☆

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ABSTRACT

Let H be a subgroup of the symmetric group of degree m and let χ be an irreducible character of H. In this paper we give conditions that characterize the pairs of matrices that leave invariant the value of a generalized matrix function associated with H and χ, on the set of the upper triangular matrices.

1. Introduction and auxiliary results

Let S_n be the symmetric group of degree n. Let $\mathbb{F}$ be an arbitrary field of characteristic zero and c be a function, not identically zero, from S_n into $\mathbb{F}$. If $X = [x_{ij}]$ is an $n \times n$ matrix over $\mathbb{F}$, the generalized Schur function $d_c(X)$ is defined by [4]

$$d_c(X) = \sum_{\sigma \in S_n} c(\sigma) \prod_{i=1}^{n} x_{i\sigma(i)}.$$

When c coincides with an irreducible character χ of a subgroup H of S_n, we denote $d_c(X)$ by $d_\chi^H(X)$ and we say that $d_\chi^H(X)$ is the generalized matrix function associated with H and χ. Throughout this paper we see χ as a map from S_n over $\mathbb{F}$ taking $\chi(\sigma) = 0$ if $\sigma \in S_n - H$.

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Let $M_n(\mathbb{F})$ be the linear space of n-square matrices with elements in $\mathbb{F}$. In [2] the first author studied the set, $\mathcal{T}(H, \chi)$, of the matrices $A \in M_n(\mathbb{F})$ satisfying

$$d_\chi^H(AX) = d_\chi^H(X)$$

for all $X \in T_n^U(\mathbb{F})$, where $T_n^U(\mathbb{F})$ is the set of n-square upper triangular matrices (the set of n-square lower triangular matrices is denoted by $T_n^L(\mathbb{F})$). It is motivated by de Oliveira and Dias da Silva [5] which studies the matrices $A \in M_n(\mathbb{F})$ satisfying

$$d_\chi^H(AX) = d_\chi^H(X),$$

for all $X \in M_n(\mathbb{F})$. Similar problems were proposed in [6]. One of those problems is the characterization of the pairs of matrices $(A, B) \in M_n(\mathbb{F}) \times M_n(\mathbb{F})$ that satisfy

$$d_\chi^H(AXB) = d_\chi^H(X),$$

for all $X \in M_n(\mathbb{F})$.

These pairs was first studied by Duffner and de Oliveira in [1]. They denote by $\overline{\mathcal{C}}(H, \chi)$ the set of all these pairs of matrices, that is,

$$\overline{\mathcal{C}}(H, \chi) = \{(A, B) : d_\chi^H(AXB) = d_\chi^H(X), \text{ for all } X \in M_n(\mathbb{F})\}.$$

It is easy to see that for all H and χ, $\overline{\mathcal{C}}(H, \chi) \neq \emptyset$ because $(I_n, I_n) \in \overline{\mathcal{C}}(H, \chi)$, where I_n denote the identity matrix of $M_n(\mathbb{F})$. In [1] it was proved that if $(A, B) \in \overline{\mathcal{C}}(H, \chi)$ then A and B are nonsingular and $\overline{\mathcal{C}}(H, \chi)$ can be made a group by defining the product $(A, B)(C, D) = (AC, DB)$.

In this paper, we use the following notations: By $(H)_{id}^T$ we denote the subgroup of S_n generated by all transpositions τ such that $\chi(\tau) = -\chi(id)$, and by $\overline{\mathcal{U}}(H, \chi)$ we denote the set

$$\overline{\mathcal{U}}(H, \chi) = \{(\gamma_1, \gamma_2) \in S_n \times S_n : \chi(id)\chi(\gamma_1\sigma\gamma_2) = \chi(\sigma)\chi(\gamma_1\gamma_2), \text{ for all } \sigma \in S_n\}.$$

Note that $\overline{\mathcal{U}}(H, \chi)$ is never an empty set because $(id, id) \in \overline{\mathcal{U}}(H, \chi)$, where id is the identity of S_n. We also have that if $(\gamma_1, \gamma_2) \in \overline{\mathcal{U}}(H, \chi)$ then $\chi(\gamma_1\gamma_2) \neq 0$ and $(\gamma_1^{-1}, \gamma_2^{-1}) \in \overline{\mathcal{U}}(H, \chi)$. The set $\overline{\mathcal{U}}(H, \chi)$ equipped with the product $(\gamma_1, \gamma_2)(\pi_1, \pi_2) = (\gamma_1\pi_1, \pi_2\gamma_2)$ is a group.

If $\theta \in S_n$, we denote by $P(\theta)$ the $n \times n$ permutation matrix whose (i, j) entry is

$$P(\theta)_{ij} = \delta_{i\theta(j)}, \quad i, j \in \{1, \ldots, n\}.$$

The next theorem is the main result of [1] and it gives a characterization of $\overline{\mathcal{C}}(H, \chi)$:

Theorem 1.1 [1]. *Let H be a subgroup of S_n and let χ be an irreducible character of S_n. Then, $(A, B) \in \overline{\mathcal{C}}(H, \chi)$ if and only if A and B can be written as*

$$A = M_1 P(\gamma_1), \quad B = P(\gamma_2) M_2$$

and the following conditions are satisfied:

1. *The (i, j) elements of M_1 and M_2 are zero whenever i and j are in different orbits of $(H)_{id}^T$;*
2. $(\gamma_2, \gamma_1) \in \overline{\mathcal{U}}(H, \chi)$;
3. $\det(M_1 M_2) = \frac{\chi(\gamma_2\gamma_1)}{\chi(id)}$.

The main purpose of this paper is to solve a problem based on this one. We are going to characterize the pais of matrices $(A, B) \in M_n(\mathbb{F}) \times M_n(\mathbb{F})$ that satisfy

$$d_\chi^H(AXB) = d_\chi^H(X),$$

for all $X \in T_n^U(\mathbb{F})$.

We denote by $\overline{\mathcal{T}}(H, \chi)$ the set of all this pairs of matrices, that is

$$\overline{\mathcal{T}}(H, \chi) = \{(A, B) \in M_n(\mathbb{F}) \times M_n(\mathbb{F}) : d_\chi^H(AXB) = d_\chi^H(X), \text{ for all } X \in T_n^U(\mathbb{F})\}.$$

This set is not a group because $\mathcal{T}(H,\chi)$ is not a group (see [2]). However if $(A,B) \in \overline{\mathcal{T}}(H,\chi)$ then A and B are nonsingular matrices:

Proposition 1.2. *If $(A,B) \in \overline{\mathcal{T}}(H,\chi)$ then A and B are nonsingular matrices.*

Proof. Assume that A is a singular matrix. Then, there exists a nonzero column $u = [u_i]$ such that $Au = 0$. Let x be an arbitrary element of $\mathbb{F}$ and let p be the great integer such that $u_p \neq 0$. Let X be the $n \times n$ matrix whose pth column is xu, and the kth column, $k \neq p$, has a 1 in the kth entry and the remaining entries are null. Then $X \in T_n^U(\mathbb{F})$, and so,

$$d_\chi^H(AXB) = d_\chi^H(X).$$

The matrix AX does not depend of x because the pth column of AX is null. Then AXB does not depend of x and the same happens for $d_\chi^H(AXB)$. However,

$$d_\chi^H(X) = \chi(id)xu_p,$$

which is a contradiction. In a similar way we prove that B is nonsingular. Assume that B is singular. Then there exists a nonzero row $v = [v_i]$ such that $vB = 0$. Let x be an arbitrary element of $\mathbb{F}$ and let p be the small integer such that $v_p \neq 0$. Let Y be the $n \times n$ matrix whose pth row is xv, and the kth row, $k \neq p$, of Y has a 1 in the kth entry and the remaining entries are null. The rest of the proof goes in a similar way. □

2. The set $\overline{\mathcal{T}}(H,\chi)$

In this section we are going to present a characterization of the set $\overline{\mathcal{T}}(H,\chi)$. So the Theorem 2.3 in the main result of this section.

Remark that if $\sigma_1, \sigma_2 \in S_n$, χ is an irreducible character of H (subgroup of S_n) such that $\chi((\sigma_1\sigma_2)^{-1}) \neq 0$ then $\sigma_1\sigma_2 \in H$.

Proposition 2.1. *Let H be a subgroup of S_n and χ be an irreducible character of H. The pair (A,B) is in $\overline{\mathcal{T}}(H,\chi)$ if and only if there exists $\sigma_1, \sigma_2 \in S_n$ such that $\chi((\sigma_1\sigma_2)^{-1}) \neq 0$, and lower triangular matrices L_1 and L_2 with the entries of the main diagonal equal to 1 satisfying*

(1) $L_1^{-1}P(\sigma_1^{-1})A, L_2^{-1}P(\sigma_2^{-1})B \in T_n^U(\mathbb{F})$;
(2) $\det(AB) = \frac{\epsilon((\sigma_1\sigma_2)^{-1})\chi(id)}{\chi((\sigma_1\sigma_2)^{-1})}$;
(3) $d_\chi^H(P(\sigma_1)L_1ZL_2P(\sigma_2)) = \chi((\sigma_1\sigma_2)^{-1})\det(Z)$, *for all* $Z \in T_n^U(\mathbb{F})$.

Proof. Let $(A,B) \in \overline{\mathcal{T}}(H,\chi)$. Then there exists $\sigma_1, \sigma_2 \in S_n$ such that

$$A = P(\sigma_1)L_1R_1 \quad \text{and} \quad B = R_2L_2P(\sigma_2),$$

where L_1 and L_2 are lower triangular matrices with the entries of the main diagonal equal to 1, and R_1 and R_2 are upper triangular matrices. Then,

$$L_1^{-1}P(\sigma_1^{-1})A = R_1, \quad \text{and} \quad BP(\sigma_2^{-1})L_2^{-1} = R_2,$$

and so we have (1).

Let $X = [x_{ij}] \in T_n^U(\mathbb{F})$. Since $(A,B) \in \overline{\mathcal{T}}(H,\chi)$ we have,

$$d_\chi^H(AXB) = d_\chi^H(X).$$

Let $Z = R_1XR_2$. Since R_1 and R_2 are nonsingular, $Z \in T_n^U(\mathbb{F})$ is arbitrary and we have

$$d_\chi^H(P(\sigma_1)L_1ZL_2P(\sigma_2)) = d_\chi^H(R_1^{-1}ZR_2^{-1}).$$

Taking $Z = I_n$ we have

$$d_\chi^H(P(\sigma_1)L_1L_2P(\sigma_2)) = d_\chi^H(R_1^{-1}R_2^{-1}) = \chi(id)\prod_{i=1}^{n}(r_{ii}^{(1)})^{-1}(r_{ii}^{(2)})^{-1} \neq 0,$$

where $r_{ii}^{(1)}$ and $r_{ii}^{(2)}$ are the (i,i) entry of R_1 and R_2, respectively. But

$$d_\chi^H(P(\sigma_1)L_1L_2P(\sigma_2)) = \begin{cases}\chi((\sigma_1\sigma_2)^{-1}) & \text{if}\,(\sigma_1\sigma_2)^{-1} \in H\\ 0 & \text{otherwise}\end{cases},$$

so $(\sigma_1\sigma_2)^{-1} \in H$ and $\chi((\sigma_1\sigma_2)^{-1}) \neq 0$. Hence,

$$\det(R_1R_2) = \frac{\chi(id)}{\chi((\sigma_1\sigma_2)^{-1})}.$$

Now it is easy to get (2).

Since

$$d_\chi^H(R_1^{-1}ZR_2^{-1}) = \chi(id)\prod_{i=1}^{n}(R_1^{-1}ZR_2^{-1})_{ii} = \chi(id)\det(R_1^{-1})\det(R_2^{-1})\det(Z),$$

we have

$$\begin{aligned} d_\chi^H(P(\sigma_1)L_1ZL_2P(\sigma_2)) &= d_\chi^H(P(\sigma_1)L_1R_1R_1^{-1}ZR_2^{-1}R_2L_2P(\sigma_2))\\ &= d_\chi^H(R_1^{-1}ZR_2^{-1})\\ &= \chi(id)\det(R_1^{-1})\det(R_2^{-1})\det(Z), \end{aligned}$$

and so

$$\begin{aligned} d_\chi^H(P(\sigma_1)L_1ZL_2P(\sigma_2)) &= \chi(id)\frac{\chi((\sigma_1\sigma_2)^{-1})}{\chi(id)}\det(Z)\\ &= \chi((\sigma_1\sigma_2)^{-1})\det(Z) \end{aligned}$$

and the proof of the necessity of the conditions is complete.

Assume now that the matrices $A, B, L_1, L_2, P(\sigma_1)$ and $P(\sigma_2)$ satisfy the three conditions. Let $X \in T_n^U(\mathbb{F})$. Then

$$d_\chi^H(AXB) = d_\chi^H(P(\sigma_1)L_1L_1^{-1}P(\sigma_1^{-1})AXBP(\sigma_2^{-1})L_2^{-1}L_2P(\sigma_2)).$$

By (1), $L_1^{-1}P(\sigma_1^{-1})A,\ L_2^{-1}P(\sigma_2^{-1})B \in T_n^U(\mathbb{F})$, and using (2) and (3) we have

$$\begin{aligned} d_\chi^H(P(\sigma_1)L_1L_1^{-1}P(\sigma_1)^{-1}AXBP(\sigma_2^{-1})L_2^{-1}L_2P(\sigma_2)) &= \chi((\sigma_1\sigma_2)^{-1})\det(L_1^{-1}P(\sigma_1)^{-1}AXBP(\sigma_2^{-1})L_2^{-1})\\ &= \chi((\sigma_1\sigma_2)^{-1})\epsilon(\sigma_1\sigma_2)\frac{\epsilon(\sigma_1\sigma_2)\chi(id)}{\chi((\sigma_1\sigma_2)^{-1})}\det(X)\\ &= \chi(id)\det(X)\\ &= d_\chi^H(X) \end{aligned}$$

and the proof is complete. □

Notation 2.2. Let $\sigma \in H$ such that $\chi(\sigma)^{-1} \neq 0$. In [2] it was defined the set $V_\sigma(H,\chi)$ of the lower triangular matrices L with diagonal elements equal to 1, satisfying

$$d_\chi^H(P(\sigma)LX) = \chi(\sigma^{-1})\det(X),$$

for all $X \in T_n^U(\mathbb{F})$. So, if $\sigma_1,\ \sigma_2 \in S_n$ are such that $\chi((\sigma_1\sigma_2)^{-1}) \neq 0$, we denote by $V_{(\sigma_1,\sigma_2)}(H,\chi)$ the set of the pairs of lower triangular matrices (L_1, L_2), with 1 in the entries of the main diagonal, that satisfy

$$d_\chi^H(P(\sigma_1)L_1XL_2P(\sigma_2)) = \chi((\sigma_1\sigma_2)^{-1})\det(X),$$

for all $X \in T_n^U(\mathbb{F})$.

Using Proposition 2.1 and the previous notation we conclude that

Theorem 2.3

$$\overline{\mathcal{T}}(H,\chi) = \bigcup_{\substack{\sigma_1,\sigma_2\in H,\\ \chi((\sigma_1\sigma_2)^{-1})\neq 0}} \left\{ (P(\sigma_1)L_1R_1,\ R_2L_2P(\sigma_2)) : (L_1,L_2) \in V_{(\sigma_1,\sigma_2)}(H,\chi), \right. \\ \left. R_1, R_2 \in T_n^U(\mathbb{F}) \textit{ and } \det(R_1R_2) = \frac{\chi(id)}{\chi((\sigma_1\sigma_2)^{-1})} \right\}$$

By this result we conclude that if we want to characterize the set $\overline{\mathcal{T}}(H,\chi)$ we have somehow to obtain a characterization of the set $V_{(\sigma_1,\sigma_2)}(H,\chi)$.

Proposition 2.4. *Let $\sigma_1,\ \sigma_2 \in H$ such that $\chi((\sigma_2\sigma_1)^{-1}) \neq 0$. Then*

$$V_{(\sigma_1,\sigma_2)}(H,\chi) = V_{(\sigma_2\sigma_1,id)}(H,\chi) = V_{(id,\sigma_2\sigma_1)}(H,\chi).$$

Proof. Let $X \in T_n^U(\mathbb{F})$. By definition

$$d_\chi^H(P(\sigma_1)L_1XL_2P(\sigma_2)) = \sum_{\rho\in S_n} \chi(\sigma_2^{-1}\rho\sigma_1^{-1}) \prod_{i=1}^n (L_1XL_2)_{i\rho(i)}.$$

Since χ is a class function of H, we have

$$\chi(\sigma_2^{-1}\rho\sigma_1^{-1}) = \chi((\sigma_2\sigma_1)^{-1}\rho) = \chi(\rho(\sigma_2\sigma_1)^{-1}).$$

Therefore,

$$d_\chi^H(P(\sigma_1)L_1XL_2P(\sigma_2)) = \sum_{\rho\in S_n} \chi((\sigma_2\sigma_1)^{-1}\rho) \prod_{i=1}^n (L_1XL_2)_{i\rho(i)} = d_\chi^H(L_1XL_2P(\sigma_2\sigma_1))$$

and

$$d_\chi^H(P(\sigma_1)L_1XL_2P(\sigma_2)) = \sum_{\rho\in S_n} \chi(\rho(\sigma_2\sigma_1)^{-1}) \prod_{i=1}^n (L_1XL_2)_{i\rho(i)} = d_\chi^H(P(\sigma_2\sigma_1)L_1XL_2).$$

Consequently, we have the result. □

Proposition 2.5. *Let $\sigma \in H$ such that $\chi(\sigma^{-1}) \neq 0$. Then*

$$V_\sigma(H,\chi) \times \{I_n\} \subseteq V_{(\sigma,id)}(H,\chi).$$

Proof. Let $L \in V_\sigma(H,\chi)$ and $X \in T_n^U(\mathbb{F})$. Since

$$\chi(\sigma^{-1})\det(X) = d_\chi^H(P(\sigma)LX) = d_\chi^H(P(\sigma)LXI_n),$$

then $(L, I_n) \in V_{(\sigma,id)}(H,\chi)$. □

Notation 2.6. Let $\sigma \in H$ such $\chi(\sigma^{-1}) \neq 0$. We denote by $(H)^T_\sigma$ the subgroup of H spanned by all transpositions τ such that $\chi(\sigma^{-1}\tau) = -\chi(\sigma^{-1})$.

Let $\sigma_1,\ \sigma_2 \in S_n$ such $\chi((\sigma_1\sigma_2)^{-1}) \neq 0$. We denote by $(H)^T_{(\sigma_1,\sigma_2)}$ the subgroup of S_n spanned by all transpositions τ such that $\chi(\sigma_2^{-1}\tau\sigma_1^{-1}) = -\chi(\sigma_2^{-1}\sigma_1^{-1})$.

Since χ is a class function in H, it is easy to prove the following proposition.

Proposition 2.7. *Let $\sigma_1,\ \sigma_2 \in H$ such $\chi((\sigma_1\sigma_2)^{-1}) \neq 0$. Then*

$$(H)^T_{(\sigma_1,\sigma_2)} = (H)^T_{\sigma_2\sigma_1}.$$

Notation 2.8. Let x be an indeterminate over $\mathbb{F}$. Then $E^{(i)+x(j)}$ is the matrix obtained from the identity matrix by adding x times column j to column i.

Theorem 2.9. *Let $L_1 = [l^{(1)}_{ij}]$ and $L_2 = [l^{(2)}_{ij}]$ be two lower triangular matrices with 1 in the entries of the main diagonal and let $\sigma_1, \sigma_2 \in S_n$ such that $\chi((\sigma_1\sigma_2)^{-1}) \neq 0$. If $(L_1, L_2) \in V_{(\sigma_1,\sigma_2)}(H, \chi)$ then $l^{(1)}_{ij} = l^{(2)}_{ij} = 0$ whenever i and j are in different orbits of $(H)^T_{(\sigma_1,\sigma_2)}$.*

Proof. Let $k \in \{1, \dots, n\}$ such that for all $s > k$, $s \in \{1, \dots, n\}$, the transposition $(k, s) \notin (H)^T_{(\sigma_1,\sigma_2)}$. Then s and k are in different orbits of $(H)^T_{(\sigma_1,\sigma_2)}$. We are going to show that

$$l^{(1)}_{k+1,k} = \cdots = l^{(1)}_{n,k} = l^{(2)}_{k+1,k} = \cdots = l^{(2)}_{n,k} = 0.$$

Let x be an arbitrary element of $\mathbb{F}$ and let

$$X = E^{(k+1)+x(k)}.$$

Since $(L_1, L_2) \in V_{(\sigma_1,\sigma_2)}(H, \chi)$ and $X \in T^U_n(\mathbb{F})$ we have

$$d^H_\chi(P(\sigma_1)L_1XL_2P(\sigma_2)) = d^H_\chi(P(\sigma_1)XP(\sigma_2)) = \chi((\sigma_1\sigma_2)^{-1}).$$

The $(k, k+1)$ entry of L_1XL_2 is x, the $(k+1, k)$ entry of L_1XL_2 is $l^{(1)}_{k+1,k} + l^{(1)}_{k+1,k}l^{(2)}_{k+1,k}x + l^{(2)}_{k+1,k}$, and the entries $(k+1, k+1)$ and (k, k) are $l^{(1)}_{k+1,k}x + 1$ and $l^{(2)}_{k+1,k}x + 1$, respectively. Assume that $(k+1, k) \notin H$. Then,

$$d^H_\chi(P(\sigma_1)L_1XL_2P(\sigma_2)) = \chi((\sigma_1\sigma_2)^{-1})(l^{(1)}_{k+1,k}l^{(2)}_{k+1,k}x^2 + l^{(1)}_{k+1,k}x + l^{(2)}_{k+1,k}x + 1),$$

and since $(L_1, L_2) \in V_{(\sigma_1,\sigma_2)}(H, \chi)$ we obtain

$$l^{(1)}_{k+1,k}l^{(2)}_{k+1,k}x^2 + l^{(1)}_{k+1,k}x + l^{(2)}_{k+1,k}x = 0.$$

Then

$$l^{(1)}_{k+1,k} = l^{(2)}_{k+1,k} = 0.$$

If $(k+1, k) \in H$ then

$$\begin{aligned} d^H_\chi(P(\sigma_1)L_1XL_2P(\sigma_2)) &= \chi((\sigma_1\sigma_2)^{-1})(l^{(1)}_{k+1,k}l^{(2)}_{k+1,k}x^2 + l^{(1)}_{k+1,k}x + l^{(2)}_{k+1,k}x + 1)\\ &\quad + \chi(\sigma_2^{-1}(k, k+1)\sigma_1^{-1})(l^{(1)}_{k+1,k}l^{(2)}_{k+1,k}x^2 + l^{(1)}_{k+1,k}x + l^{(2)}_{k+1,k}x)\\ &= \chi((\sigma_1\sigma_2)^{-1})(l^{(1)}_{k+1,k}l^{(2)}_{k+1,k}x^2 + l^{(1)}_{k+1,k}x + l^{(2)}_{k+1,k}x + 1)\\ &\quad + \chi((\sigma_2\sigma_1)^{-1}(k, k+1))(l^{(1)}_{k+1,k}l^{(2)}_{k+1,k}x^2 + l^{(1)}_{k+1,k}x + l^{(2)}_{k+1,k}x). \end{aligned}$$

Since $(k+1,k) \notin (H)^T_{(\sigma_1,\sigma_2)}$ and $(L_1,L_2) \in V_{(\sigma_1,\sigma_2)}(H,\chi)$ we have

$$l^{(1)}_{k+1,k} l^{(2)}_{k+1,k} x^2 + l^{(1)}_{k+1,k} x + l^{(2)}_{k+1,k} x = 0,$$

and then

$$l^{(1)}_{k+1,k} = l^{(2)}_{k+1,k} = 0.$$

Next, using $E^{(k+2)+x(k)}$, because $(k+2,k) \notin (H)^T_{(\sigma_1,\sigma_2)}$ we conclude that

$$l^{(1)}_{k+2,k} = l^{(2)}_{k+2,k} = 0.$$

Now it is easy to complete the proof. □

The converse of this result is not true (see [2]). However in certain situations the converse holds:

Proposition 2.10. *If* $(\sigma_2,\sigma_1) \in \overline{\mathcal{U}}(H,\chi)$ *then*

$$V_{(\sigma_1,\sigma_2)}(H,\chi) = V_{(id,id)}(H,\chi).$$

Proof. Let $X \in T^U_n(\mathbb{F})$. By definition

$$d^H_\chi(P(\sigma_1)L_1XL_2P(\sigma_2)) = \sum_{\rho\in S_n} \chi(\sigma_2^{-1}\rho\sigma_1^{-1}) \prod_{i=1}^n (L_1XL_2)_{i\rho(i)}.$$

Since $(\sigma_2,\sigma_1) \in \overline{\mathcal{U}}(H,\chi)$ we have

$$\chi(\sigma_2^{-1}\rho\sigma_1^{-1}) = \frac{\chi((\sigma_1\sigma_2)^{-1})}{\chi(id)}\chi(\rho),$$

and so

$$d^H_\chi(P(\sigma_1)L_1XL_2P(\sigma_2)) = \frac{\chi((\sigma_1\sigma_2)^{-1})}{\chi(id)} \sum_{\rho\in S_n} \chi(\rho) \prod_{i=1}^n (L_1XL_2)_{i,\rho(i)}.$$

Hence, $(L_1,L_2) \in V_{(\sigma_1,\sigma_2)}(H,\chi)$ if and only if

$$d^H_\chi(P(\sigma_1)L_1XL_2P(\sigma_2)) = \frac{\chi((\sigma_1\sigma_2)^{-1})}{\chi(id)} d^H_\chi(X),$$

that is, if and only if

$$\frac{\chi((\sigma_1\sigma_2)^{-1})}{\chi(id)} d^H_\chi(L_1XL_2) = \frac{\chi((\sigma_1\sigma_2)^{-1})}{\chi(id)} d^H_\chi(X),$$

if and only if

$$(L_1,L_2) \in V_{(id,id)}(H,\chi). \quad \square$$

Theorem 2.11. *Let* $L_1 = [l^{(1)}_{ij}]$ *and* $L_2 = [l^{(2)}_{ij}]$ *be two lower triangular matrices with* 1 *in the main diagonal and let* $\sigma_1,\sigma_2 \in H$ *such that* $(\sigma_2,\sigma_1) \in \overline{\mathcal{U}}(H,\chi)$. *Then,* $(L_1,L_2) \in V_{(\sigma_1,\sigma_2)}(H,\chi)$ *if and only if* $l^{(1)}_{ij} = l^{(2)}_{ij} = 0$ *whenever i and j are in different orbits of* $(H)^T_{\sigma_2\sigma_1}$.

Proof. If $(\sigma_2,\sigma_1) \in \overline{\mathcal{U}}(H,\chi)$ is easy to prove that

$$(H)^T_{\sigma_2\sigma_1} = (H)^T_{id}.$$

Let $L_1 = [l^{(1)}_{ij}]$ and $L_2 = [l^{(2)}_{ij}]$ be two lower triangular matrices with 1 in the main diagonal and such that $l^{(1)}_{ij} = l^{(2)}_{ij} = 0$ whenever i and j are in different orbits of $(H)^T_{\sigma_2\sigma_1}$. If $i,j \in \{1,\ldots,n\}$ are in different orbits of $(H)^T_{\sigma_2\sigma_1}$ then i and j are in different orbits of $(H)^T_{id}$ and so, by Theorem 1.1, $(L_1, L_2) \in \overline{\mathcal{C}}(H, \chi)$. Let $X \in T_n^U(\mathbb{F})$. Then

$$\begin{aligned} d_\chi^H(P(\sigma_1)L_1XL_2P(\sigma_2)) &= \frac{\chi((\sigma_1\sigma_2)^{-1})}{\chi(id)} d_\chi^H(L_1XL_2) \\ &= \frac{\chi((\sigma_1\sigma_2)^{-1})}{\chi(id)} d_\chi^H(X) \\ &= \chi((\sigma_1\sigma_2)^{-1}) \det(X), \end{aligned}$$

and so $(L_1, L_2) \in V_{(\sigma_1,\sigma_2)}(H, \chi)$.

The converse is Theorem 2.9. □

3. The set $V_{(\sigma_1,\sigma_2)}(S_n, \chi)$

Let χ be an irreducible character of S_n and $\sigma_1,\ \sigma_2 \in S_n$ such that $\chi(\sigma_1\sigma_2) \neq 0$. In this section we are going to present a characterization of some sets $V_{(\sigma_1,\sigma_2)}(S_n, \chi)$.

In [2] it was proved the following Theorems and Propositions.

Proposition 3.1 [2]. *Let $\pi \in S_n$ such that $\chi(\pi) \neq 0$. If χ is self-associated or χ is the principal character of S_n, then there is no transposition τ such that*

$$\chi(\pi\tau) = -\chi(\pi),$$

i.e., $(S_n)^T_\pi = \{id\}$.

Theorem 3.2 [2]. *Let χ be an irreducible character of S_n. Then*

$$\bigcup_{\sigma\in S_n,\ \chi(\sigma)\neq 0} V_\sigma(S_n, \chi) = \{I_n\}$$

if and only if

$$\chi = 1 \text{ or } \chi \text{ is self-associated}.$$

Proposition 3.3 [2]. *Let $\chi = (s, 1^{n-s})$ be the irreducible character of S_n where*

(1) $s - 1 > n - s \geqslant 1$
(2) if $s \geqslant 5$ and s is odd then $2(n - s) \neq s - 1$
(3) if $s = 6$ then $n \notin \{9, 10\}$.

Let $(a,\ b)$ be a transposition of S_n and $\sigma \in S_n$ be a cycle with length $s - 1$ such that $\chi(\sigma) \neq 0$. Then

$$\chi(\sigma(a,\ b)) = -\chi(\sigma) \text{ if and only if } \sigma(a) = a,\ \sigma(b) = b.$$

Theorem 3.4 [2]. *Let $\chi = (n - 1, 1)$ be the irreducible character of S_n with $n > 3$. Let $\sigma \in S_n$ be a cycle with length $n - 2$ and $L = [l_{ij}] \in T_n^L(F)$ with diagonal elements equal to 1. Then*

$$L \in V_\sigma(S_n, \chi)$$

if and only if L satisfies the condition:

"For $r > p$, if there exists an integer k such that $p \leqslant k \leqslant r$ and $\sigma(k) \neq k$ then $l_{rp} = 0$."

Theorem 3.5 [2]. *Let $\chi = (s, 1^{n-s})$ be the irreducible character of S_n satisfying*

(i) $s - 1 > n - s \geqslant 1$

(ii) if $s = 6$ then $n \notin \{9, 10\}$
(iii) if s is odd and $s \geqslant 5$ then $2(n - s) \neq s - 1$.

Let $\sigma \in S_n$ be a cycle with length $s - 1$ such that

$$\{j : \sigma(j) \neq j\} = \{u, u + 1, \ldots, u + s - 2\}$$

for some integer $u < n - s + 2$. Let $L = [l_{ij}] \in T_n^L(F)$ with diagonal elements equal to 1. Then

$$L \in V_\sigma(S_n, \chi)$$

if and only if L satisfies the condition:
"For $r > p$, if there exists an integer k such that $p \leqslant k \leqslant r$ and $\sigma(k) \neq k$ then $l_{rp} = 0$."

Now, we are going to prove similar results in the set $V_{(\sigma_1,\sigma_2)}(S_n, \chi)$.
Using Theorem 2.11, Propositions 2.5 and 3.1 and Theorem 3.2 we have the following result.

Theorem 3.6. *Let χ be an irreducible character of S_n. Then*

$$\bigcup_{\sigma_1,\sigma_2 \in S_n,\ \chi(\sigma_1\sigma_2) \neq 0} V_{(\sigma_1,\sigma_2)}(S_n, \chi) = \bigcup_{\sigma \in S_n,\ \chi(\sigma) \neq 0} V_{(\sigma,id)}(S_n, \chi) = \{(I_n, I_n)\}$$

if and only if

$$\chi = 1 \text{ or } \chi \text{ is self-associated}.$$

Theorem 3.7. *Let $\chi = (n - 1, 1)$ be the irreducible character of S_n with $n > 3$. Let $\sigma_1, \sigma_2 \in S_n$ such that $\sigma_2\sigma_1$ is a cycle with length $n - 2$ and $L_1 = [l_{ij}^{(1)}],\ L_2 = [l_{ij}^{(2)}] \in T_n^L(F)$ with diagonal elements equal to 1. Then*

$$(L_1, L_2) \in V_{(\sigma_1,\sigma_2)}(S_n, \chi)$$

if and only if L_1 and L_2 satisfy the condition:
"For $r > p$, if there exists an integer k such that $p \leqslant k \leqslant r$ and $\sigma_2\sigma_1(k) \neq k$ then $l_{rp}^{(1)} = 0$ and $l_{rp}^{(2)} = 0$."

Proof. Since $n > 3$ then, if $n - 1 \geqslant 5$ and $n - 1$ is odd we have $2(n - (n - 1)) = 2 \neq n - 2$. If $n - 1 = 6$ then $n \notin \{9, 10\}$. Using Proposition 3.3, if $(a\ b)$ is a transposition of S_n, then $\chi(\sigma_2\sigma_1(a\ b)) = -\chi(\sigma_2\sigma_1)$ if and only if $\sigma_2\sigma_1(a) = a$ and $\sigma_2\sigma_1(b) = b$.

Since $\sigma_2\sigma_1$ is a cycle of length $n - 2$, there are only two integers $u, v \in \{1, \ldots, n\}$, $u > v$ such that $\sigma_2\sigma_1(u) = u$ and $\sigma_2\sigma_1(v) = v$. Consequently, $(S_n)^T_{\sigma_2\sigma_1} = \langle (u\ v) \rangle$.

Necessity. Suppose that $(L_1 = [l_{ij}^{(1)}], L_2 = [l_{ij}^{(2)}]) \in V_{(\sigma_1,\sigma_2)}(S_n, \chi)$. By Theorem 2.9, if $a > b$, $a, b \in \{1, \ldots, n\}$ and $\sigma_2\sigma_1(a) \neq a$ or $\sigma_2\sigma_1(b) \neq b$ then $l_{ab}^{(1)} = 0$ and $l_{ab}^{(2)} = 0$.

Suppose there exists an integer k such that $u > k > v$ and $\sigma_2\sigma_1(k) \neq k$. Let Z be the matrix whose $(v + 1)$th column is the vth column of I_n and the uth column of Z is the $(v + 1)$th column of I_n, the remaining columns of Z are the columns of I_n. Then

$$d_\chi^{S_n}(P(\sigma_1)L_1ZL_2P(\sigma_2)) = (\chi((\sigma_2\sigma_1)^{-1}(v + 1,\ u)) + \chi((\sigma_2\sigma_1)^{-1}(v + 1,\ u,\ v)))l_{uv}^{(1)}.$$

Since $(\sigma_2\sigma_1)^{-1}(v + 1) \neq v + 1$ and $(\sigma_2\sigma_1)^{-1}(u) = u$ then $(\sigma_2\sigma_1)^{-1}(v + 1,\ u)$ is a cycle with length $n - 1$. Using the Murnaghan–Nakayama rule,

$$\chi((\sigma_2\sigma_1)^{-1}(v + 1,\ u)) = 0.$$

But $\chi((\sigma_2\sigma_1)^{-1}(v + 1,\ u,\ v))$ is a cycle with length n, then $\chi((\sigma_2\sigma_1)^{-1}(v + 1,\ u,\ v)) = -1$. Therefore, $d_\chi^{S_n}(P(\sigma_1)L_1ZL_2P(\sigma_2)) = -l_{uv}^{(1)}$. Since $(L_1, L_2) \in V_{(\sigma_1,\sigma_2)}(S_n, \chi)$,

$$-l_{uv}^{(1)} = d_\chi^{S_n}(P(\sigma_1)L_1ZL_2P(\sigma_2)) = d_\chi^{S_n}(P(\sigma_1)ZP(\sigma_2)) = 0.$$

Consequently, $l_{uv}^{(1)} = 0$.

Let B be the matrix whose $(u-1)$th row is the uth row of I_n and the vth row of B is the $(u-1)$th row of I_n, the remaining rows of B are the rows of I_n. Then

$$d_\chi^{S_n}(P(\sigma_1)L_1BL_2P(\sigma_2)) = (\chi((\sigma_2\sigma_1)^{-1}(u-1,\ v)) + \chi((\sigma_2\sigma_1)^{-1}(u-1,\ u,\ v)))l_{uv}^{(2)}.$$

Since $(\sigma_2\sigma_1)^{-1}(u-1) \neq u-1$ and $(\sigma_2\sigma_1)^{-1}(v) = v$ then $(\sigma_2\sigma_1)^{-1}(u-1,\ v)$ is a cycle with length $n-1$. Using the Murnaghan–Nakayama rule,

$$\chi((\sigma_2\sigma_1)^{-1}(u-1,\ v)) = 0.$$

But $\chi((\sigma_2\sigma_1)^{-1}(u-1,\ u,\ v))$ is a cycle with length n, then $\chi((\sigma_2\sigma_1)^{-1}(u-1,\ u,\ v)) = -1$. Therefore, $d_\chi^{S_n}(P(\sigma_1)BL_2P(\sigma_2)) = -l_{uv}^{(2)}$. Since $(L_1, L_2) \in V_{(\sigma_1,\sigma_2)}(S_n, \chi)$,

$$-l_{uv}^{(2)} = d_\chi^{S_n}(P(\sigma_1)L_1BL_2P(\sigma_2)) = d_\chi^{S_n}(P(\sigma_1)BP(\sigma_2)) = 0.$$

Consequently, $l_{uv}^{(2)} = 0$ and we have the condition.

Sufficiency. Let $L_1 = [l_{ij}^{(1)}]$ and $L_2 = [l_{ij}^{(2)}]$ be matrices satisfying the condition of the theorem. Then

$$L_1 = \begin{cases} I_n & \text{if } u \neq v+1 \\ I_n + E^{v+l_{uv}^{(1)}(u)} & \text{if } u = v+1 \end{cases}$$

and

$$L_2 = \begin{cases} I_n & \text{if } u \neq v+1 \\ I_n + E^{v+l_{uv}^{(2)}(u)} & \text{if } u = v+1 \end{cases}$$

Let $X \in T_n^U$.

If $u \neq v+1$,

$$d_\chi^{S_n}(P(\sigma_1)L_1XL_2P(\sigma_2)) = d_\chi^{S_n}(P(\sigma_1)I_nXI_nP(\sigma_2)) = d_\chi^{S_n}(P(\sigma_1)XP(\sigma_2)).$$

If $u = v+1$,

$$d_\chi^{S_n}(P(\sigma_1)L_1XL_2P(\sigma_2)) = \chi((\sigma_2\sigma_1)^{-1})\prod_{s=1}^{n} x_{ss} = d_\chi^{S_n}(P(\sigma_1)XP(\sigma_2)).$$

Consequently, $(L_1, L_2) \in V_{(\sigma_1,\sigma_2)}(S_n, \chi)$. □

Theorem 3.8. *Let $\chi = (s, 1^{n-s})$ be the irreducible character of S_n satisfying*

(i) $s-1 > n-s \geqslant 1$
(ii) if $s = 6$ then $n \notin \{9,\ 10\}$
(iii) if s is odd and $s \geqslant 5$ then $2(n-s) \neq s-1$.

Let $\sigma_1, \sigma_2 \in S_n$ such that $\sigma_2\sigma_1$ is a cycle with length $s-1$ such that

$$\{j : \sigma_2\sigma_1(j) \neq j\} = \{u,\ u+1, \ldots, u+s-2\}$$

for some integer $u < n-s+2$. Let $L_1 = [l_{ij}^{(1)}],\ L_2 = [l_{ij}^{(2)}] \in T_n^L(\mathbb{F})$ with diagonal elements equal to 1. Then

$$(L_1,\ L_2) \in V_{(\sigma_1,\sigma_2)}(S_n, \chi)$$

if and only if L_1 and L_2 satisfy the condition:

"For $r > p$, if there exists an integer k such that $p \leqslant k \leqslant r$ and $\sigma_2\sigma_1(k) \neq k$ then $l_{rp}^{(1)} = 0$ and $l_{rp}^{(2)} = 0$."

Proof. Using Proposition 3.3, we see that $(S_n)_{\sigma_2\sigma_1}^T$ is generated by those transpositions, $(a\,b)$ such that $\sigma_2\sigma_1(a) = a$ and $\sigma_2\sigma_1(b) = b$. Consequently, if $\pi \in (S_n)_{\sigma_2\sigma_1}^T$, π, $(\sigma_2\sigma_1)^{-1}$ are disjoint permutations and by Murnaghan-Nakayama rule we have, $\chi((\sigma_2\sigma_1)^{-1}\pi) = \epsilon(\pi)\chi((\sigma_2\sigma_1)^{-1}) = \epsilon(\pi)$.

Necessity. Using Theorem 2.9, if $a > b$, $a, \ b \in \{1, \ldots, n\}$ and $\sigma_2\sigma_1(a) \neq a$ or $\sigma_2\sigma_1(b) \neq b$ then $l_{ab}^{(1)} = 0 = l_{ab}^{(2)}$.

Suppose that $i > j, i, j \in \{1, \ldots, n\}$ and there exists $i > k > j$ such that $\sigma_2\sigma_1(k) \neq k$. We are going to prove that $l_{ij}^{(1)} = 0 = l_{ij}^{(2)}$.

Using the hypothesis of the Theorem, then $j < u$ and $i > u + s - 2$. Let t, f two integers, $t, f \in \{u, \ldots, u + s - 2\}$ such that $t < f$ and $\sigma_2\sigma_1(t) = f$. We are seeing that $l_{u+s-1\ u-1}^{(1)} = 0 = l_{u+s-1\ u-1}^{(2)}$.

Let Z be the matrix those tth column is the $u - 1$th column of I_n, the fth column of Z is the tth column of I_n and the $(u + s - 1)$th column of Z is the fth column of I_n, the remaining columns of Z are the columns of I_n. Then

$$d_\chi^{S_n}(P(\sigma_1)L_1ZL_2P(\sigma_2)) = (\chi((\sigma_2\sigma_1)^{-1}(t,\ f,\ u+s-1)) \\ + \chi((\sigma_2\sigma_1)^{-1}(u-1,\ t,\ f,\ u+s-1)))l_{u+s-1\ u-1}^{(1)}.$$

Since $\sigma_2\sigma_1(t) = f$ and $(\sigma_2\sigma_1)^{-1}(t, f, \ u + s - 1)$ is a cycle with length $s - 1$, using the Murnaghan-Nakayama rule,

$$\chi((\sigma_2\sigma_1)^{-1}(t,\ f,\ u+s-1)) = 1.$$

But $\chi((\sigma_2\sigma_1)^{-1}(u-1,\ t,\ f,\ u+s-1))$ is a cycle with length s and $n - s \geqslant 1$, then $\chi((\sigma_2\sigma_1)^{-1}(u-1,\ t,\ f,\ u+s-1)) = 0$. Since $(L_1, L_2) \in V_{(\sigma_1,\sigma_2)}(S_n, \chi)$,

$$-l_{u+s-1\ u-1}^{(1)} = d_\chi^{S_n}(P(\sigma_1)L_1ZL_2P(\sigma_2)) = d_\chi^{S_n}(P(\sigma_1)ZP(\sigma_2)) = 0.$$

Consequently, $l_{u+s-1\ u-1}^{(1)} = 0$.

Let B be the matrix those $(u - 1)$th row is the tth row of I_n, the tth row of B is the fth row of I_n and the fth row of B is the $(u + s - 1)$th row of I_n, the remaining rows of B are the rows of I_n. Then

$$d_\chi^{S_n}(P(\sigma_1)L_1BL_2P(\sigma_2)) = (\chi((\sigma_2\sigma_1)^{-1}(t,\ f,\ u+s-1)) \\ + \chi((\sigma_2\sigma_1)^{-1}(u-1,\ t,\ f,\ u+s-1)))l_{u+s-1\ u-1}^{(2)}.$$

Since $\sigma_2\sigma_1(t) = f$ and $(\sigma_2\sigma_1)^{-1}(t, f, \ u + s - 1)$ is a cycle with length $s - 1$, using the Murnaghan-Nakayama rule,

$$\chi((\sigma_2\sigma_1)^{-1}(t,\ f,\ u+s-1)) = 1.$$

But $\chi((\sigma_2\sigma_1)^{-1}(u-1,\ t,\ f,\ u+s-1))$ is a cycle with length s and $n - s \geqslant 1$, then $\chi((\sigma_2\sigma_1)^{-1}(u-1,\ t,\ f,\ u+s-1)) = 0$. Since $(L_1, L_2) \in V_{(\sigma_1,\sigma_2)}(S_n, \chi)$,

$$-l_{u+s-1\ u-1}^{(2)} = d_\chi^{S_n}(P(\sigma_1)L_1ZL_2P(\sigma_2)) = d_\chi^{S_n}(P(\sigma_1)ZP(\sigma_2)) = 0.$$

Consequently, $l_{u+s-1\ u-1}^{(2)} = 0$.

Now, let Z be the matrix those tth column is the $(u - 2)$th column of I_n, the fth column of Z is the tth column of I_n and the $(u + s - 1)$th column of Z is the fth column of I_n, the remaining columns of Z are the columns of I_n. Then we can conclude that $l_{u+s-1\ u-2}^{(1)} = 0$. If B is the matrix those $(u - 2)$th row is the tth row of I_n, the tth row of B is the fth row of I_n and the fth row of B is the $(u + s - 1)$th row of I_n, the remaining rows of B are the rows of I_n. Then we can conclude that $l_{u+s-1\ u-2}^{(2)} = 0$. In this way we can show that $l_{u+s-1\ 1}^{(1)} = l_{u+s-1\ 1}^{(2)} = 0 \cdots l_{u+s-1\ u-1}^{(1)} = l_{u+s-1\ u-1}^{(2)} = 0$.

Next, in the same way we prove that $l_{u+s\ 1}^{(1)} = l_{u+s\ 1}^{(2)} = \cdots = l_{u+s\ u-1}^{(1)} = l_{u+s\ u-1}^{(2)} = 0$.

Therefore, we can conclude that $l_{ij}^{(1)} = 0 = l_{ij}^{(2)}$.

Sufficiency. Let L_1 and L_2 be matrices satisfying the condition of the theorem. Then

$$L_1 = \begin{bmatrix} L_{11} & 0 & 0 \\ 0 & L_{12} & 0 \\ 0 & 0 & L_{13} \end{bmatrix}$$

and

$$L_2 = \begin{bmatrix} L_{21} & 0 & 0 \\ 0 & L_{22} & 0 \\ 0 & 0 & L_{23} \end{bmatrix}$$

where L_{11}, $L_{21} \in T^L_{u-1}(\mathbb{F})$ with diagonal elements equal to 1, $L_{22} = L_{12} = I_{s-2}$ and $L_{13}, L_{23} \in T^L_{n-u-s+3}(\mathbb{F})$ with diagonal elements equal to 1. Let $Z \in T^U_n(\mathbb{F})$,

$$Z = \begin{bmatrix} Z_1 & * & * \\ 0 & Z_2 & * \\ 0 & 0 & Z_3 \end{bmatrix}$$

where $Z_1 \in T^U_{u-1}(\mathbb{F})$, $Z_2 \in T^U_{s-2}(\mathbb{F})$ and $Z_3 \in T^U_{n-u-s+3}(\mathbb{F})$. Since $\chi((\sigma_2\sigma_1)^{-1}) = 1$ and $\chi((\sigma_2\sigma_1)^{-1}\rho) = \epsilon(\rho)$ if $\rho \in (S_n)^T_{\sigma_2\sigma_1}$, then

$$\begin{aligned} d^{S_n}_\chi(P(\sigma_1)L_1ZL_2P(\sigma_2)) &= \chi((\sigma_2\sigma_1)^{-1})(det(L_{11}Z_1L_{12})det(Z_2)det(L_{13}Z_3L_{23})) \\ &= \chi((\sigma_2\sigma_1)^{-1})det(Z). \end{aligned}$$

Then $(L_1, L_2) \in V_{(\sigma_1,\sigma_2)}(S_n, \chi)$. □

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