ITERATED PERFECT-SET FORCING

James E. BAUMGARTNER* and Richard LAVER†

Department of Mathematics, Dartmouth College, Hanover, NH03755, U.S.A.

Received 4 December 1978; revised version received 5 September 1979

0. Introduction

In [7] Sacks introduced the notion of forcing with perfect closed sets of real numbers as conditions. This notion has since found wide application, particularly in descriptive set theory. Forcing with perfect sets has become known as Sacks forcing, and the associated generic real numbers are called Sacks reals.

Many people discovered independently that it is possible to enlarge the continuum by adding an arbitrary number $\kappa$ of Sacks reals. Conditions are functions with countable support belonging to the product of $\kappa$ copies of $P$, the partial ordering of perfect sets of reals. This notion has also proved useful, but it is still unknown whether it shares many of the combinatorial properties of forcing with $P$ alone. This notion of forcing is said to add $\kappa$ Sacks reals simultaneously, or side-by-side.

Here we will discuss another way of enlarging the continuum with Sacks reals, namely adding Sacks reals iteratively, so that each real is generic over the model obtained by adjoining the ones before it. Details are in Sections 1 to 3.

Our principal results are as follows:

First we show (Theorem 4.5) that if one starts with a model of ZFC+$2^{\aleph_0}=\aleph_1$ and adjoins $\aleph_2$ Sacks reals iteratively, then in the resulting model $2^{\aleph_0}=\aleph_2$, every selective ultrafilter is $\aleph_1$-generated, and every selective ultrafilter in the ground model generates a selective ultrafilter in the extension. Hence in the extension there are only $2^{\aleph_0}$ selective ultrafilters. It is also true (Theorem 4.3) that if $f: \omega \to 2$ and $f$ is in the extension, then some infinite subset of $f$ lies in the ground model.

Analogous results are unknown for simultaneous Sacks forcing (unless $\kappa < \omega!$).

We also show (Theorem 5.3) that if one forces with $P$ as defined in the model mentioned above, then cardinals are collapsed. Hence it is relatively consistent that perfect-set forcing collapses cardinals. It is not known whether it is relatively consistent that $2^{\aleph_0} > \aleph_1$ and perfect-set forcing preserves cardinals.

* Research partially supported by NSF grant MCS76-08231.
† Research partially supported by NSF grant MCS76-06942.
Finally we show (Theorem 6.4) that if $\kappa$ is weakly compact in the ground model and $\kappa$ Sacks reals are added iteratively, then in the resulting model $\kappa$ becomes $\aleph_2$ and there are no $\omega_2$-Aronszajn trees. The relative consistency of the latter assertion was first proved by Silver in a model of Mitchell [6]. This model is quite different.

We assume that the reader is familiar with the theory of forcing and generic sets (see [4], for example) but not the theory of iterated forcing. In a partial ordering $(P, \leq)$ we write $p$ extends $q$ as $p \geq q$. Thus, for example, a set $D \subseteq P$ is dense in $P$ iff $\forall p \in P \exists q \in D \ q \geq p$.

We consider forcing as taking place over $V$, the universe of all sets. If $P$ is a partial ordering and $G$ is $P$-generic, then $V[G]$ denotes the generic extension. Forcing with respect to $P$ is denoted by $\mathrel{\Vdash}_P$. We abbreviate $(\forall p) \ p \mathrel{\Vdash}_P \varphi$ to $\mathrel{\Vdash}_P \varphi$. For typographical convenience we generally use $\mathrel{\Vdash}_\alpha$ instead of $\mathrel{\Vdash}_{P_{\alpha}}$.

If $Q$ is a term such that $\mathrel{\Vdash}_P Q$ is a partial ordering, then the iterated forcing first by $P$, then by $Q$, can be accomplished by a single partial ordering $P \otimes Q$. Elements of $P \otimes Q$ are pairs $(p, q)$ such that $p \in P$ and $\mathrel{\Vdash}_P q \in Q$. We let $(p_1, q_1) \geq (p_2, q_2)$ iff $p_1 \geq p_2$ and $p_1 \mathrel{\Vdash}_P q_1 \geq q_2$. We identify pairs $(p_1, q_1)$ and $((p_2, q_2))$ such that $(p_1, q_1) \geq (p_2, q_2) \geq (p_1, q_1)$. More generally, if $\leq$ is a pre-ordering being used for forcing, then we consider $\leq$ as a partial ordering by identifying $p$ and $q$ when $p \geq q$.

If $a \in V$, then we use $a$ itself as the canonical term of the language of forcing which denotes $a$. Terms not necessarily denoting specific elements of $V$ usually have dots over the top. For instance, it is quite possible to have $\mathrel{\Vdash}_P \dot{a} \in V$ without having $\mathrel{\Vdash}_P a = a$ for any specific $a \in V$ (although necessarily $(\forall p \in P)(\exists a \in V) \ q \mathrel{\Vdash}_P \dot{a} = a$). Following this convention, the $Q$ of the preceding paragraph should have been $\dot{Q}$.

We shall assume that the set of terms of the language of forcing is full in the sense that if $p \mathrel{\Vdash}_P \forall x \varphi(x)$, then there is a term $\dot{x}$ such that $p \mathrel{\Vdash}_P \varphi(\dot{x})$. This requires the axiom of choice, which is assumed throughout the paper (see [4]).

Finally, if $G$ is $P$-generic and $\dot{x}$ is a term of the language of forcing with $P \otimes Q$ over $V$, then $\dot{x}$ canonically determines a term of the language of forcing with $Q$ over $V[G]$. The latter term will also be denoted by $\dot{x}$.

1. **The partial ordering**

Let $\text{Sq} = \bigcup \{\{2^n \colon n \in \omega\}$, the set of finite sequences of zeroes and ones. A set $p \subseteq \text{Sq}$ is perfect iff

1. $\forall s \in p \ \forall n \ s \upharpoonright n \in p$, and
2. $\forall s \in p \ \exists t, u \in p \ s \subseteq t, u$ and $t \notin u$ and $u \notin t$.

We express (2) by saying that $p$ forks below each $s \in p$. Note that $p \subseteq \text{Sq}$ is perfect iff $\{f \in \omega^\omega \colon \forall n \ f \upharpoonright n \in p\}$ is a perfect subset of $\omega^\omega$ with the product topology.

Let $P = \{p \subseteq \text{Sq} \colon p$ is perfect\}. We order $P$ by letting $p \geq q$ (read "$p$ extends $q$") iff $p \subseteq q$. Forcing with this partial ordering was introduced by Sacks in [7] and has found many applications.
We will be interested in iterated forcing with $P$. For ordinals $\alpha \geq 1$, we define $P_\alpha$ by induction as follows. Let $P_1 = P$. Given $P_\alpha$, let $P_{\alpha+1} = P_\alpha \otimes P$, the canonical partial ordering associated with the extension obtained by forcing first with $P_\alpha$ and then with $P$ as defined in the extension via $P_\alpha$. (Note that the definition of $P$ is not absolute.) If $\alpha$ is a limit ordinal, let $P_\alpha$ be the inverse limit of $\langle P_\beta : \beta < \alpha \rangle$ if $\text{cf} \alpha = \omega$, and the direct limit otherwise.

As is well-known, $P_\alpha$ may be realized as the set of all functions $p$ such that domain $(p)$ is a countable subset of $\alpha$ and $\forall \beta \in \text{domain} (p) \, \upharpoonright \beta p(\beta) \in P$, where $\upharpoonright \beta$ denotes forcing with respect to $P_\beta$. We let $p \succeq q$ iff domain $(q) \subseteq \text{domain} (p)$ and $\forall \beta \in \text{domain} (q) \, p \upharpoonright \beta p(\beta) \supseteq q(\beta)$. We identify $p$ and $q$ if $p \preceq q$ and $q \preceq p$. We also identify $P_1$ and $P$.

The reader should be warned that we will frequently declare that $p \in P_\alpha$ when we have only checked that $\forall \beta \in \text{domain} (p) \, p \upharpoonright \beta p(\beta) \in P$. The reason for doing this is that if $p \upharpoonright \beta p(\beta) \in P$, then there is $q(\beta)$ such that $\upharpoonright \beta q(\beta) \in P$ and $p \upharpoonright \beta p(\beta) \supseteq q(\beta)$, and therefore $p$ could be replaced by $q$.

If $p, q \in P$ and $m, n \in \omega$, let $(p, m) > (q, n)$ iff $p \succeq q$, $m > n$, and $(\forall s \in q \cap \omega^2) (\exists t, u \in p \cap \omega^2) s \subseteq t, u$ and $t \neq u$. The following is a variant of Lemma 1.4 of [7], frequently called the Fusion Lemma.

**Lemma 1.1.** Suppose $\langle (p_i, n_i) : i \in \omega \rangle$ is a sequence such that $p_i \in P$, $n_i \in \omega$ and for each $i$, $(p_{i+1}, n_{i+1}) \succ (p_i, n_i)$. Then $\bigcap \{p_i : i \in \omega \} \in P$ (and clearly $\bigcap \{p_i : i \in \omega \} \succ p_i$ for all $i$).

The proof is left to the reader.

We will need a version of Lemma 1.1 for the partial orderings $P_\alpha$. If $p, q \in P_\alpha$, $m, n \in \omega$, and $F$ is a finite subset of domain $(q)$, let $(p, m) >_F (q, n)$ iff $p \succeq q$, $m > n$, and $(\forall \beta \in F) \, p \upharpoonright \beta (p(\beta), m) \succ (q(\beta), n)$. Note that if $k \geq m$ and $(p, m) \succ_F (q, n)$, then $(p, k) \succ_F (q, n)$.

**Lemma 1.2 (Fusion Lemma).** Suppose $\langle (p_i, n_i, F_i) : i \in \omega \rangle$ is a sequence such that $p_i \in P_\alpha$, $n_i \in \omega$, $F_i \subseteq F_{i+1}$, $\bigcup \{F_i : i \in \omega \} = \bigcup \{\text{domain} (p_i) : i \in \omega \}$, and for each $i$, $(p_{i+1}, n_{i+1}) \succ_F (p_i, n_i)$. Define $p$ so that domain $(p) = \bigcup \{\text{domain} (p_i) : i \in \omega \}$ and $\forall \beta \in \text{domain} (p) \, p(\beta) = \bigcap \{p_i(\beta) : i \in \omega, \beta \in \text{domain} (p_i)\}$. Then $p \in P_\alpha$ and $p \succ p_i$ for each $i$.

**Proof.** To show $p \in P_\alpha$ we will show that $\forall \beta \in \text{domain} (p) \, p \upharpoonright \beta p(\beta) \in P$. This is an example of the abuse of terminology referred to earlier. But the proof is a trivial induction on $\beta$, using Lemma 1.1 at each stage.

A sequence $\langle (p_i, n_i, F_i) : i \in \omega \rangle$ satisfying the hypothesis of Lemma 1.2 will be called a *fusion sequence*, and $p$ will be referred to as the *fusion* of the sequence.

Now if $\beta < \alpha$, let $P_{\beta \alpha} = \{p \in P_\alpha : \text{domain} (p) \subseteq \{\gamma : \beta \leq \gamma < \alpha\}\}$. If $p \in P_\alpha$, then $p^\beta = p - (p \upharpoonright \beta) \in P_{\beta \alpha}$, and the mapping which carries $p$ into $(p \upharpoonright \beta, p^\beta)$ is an isomorphism of $P_\alpha$ with a dense subset of $P_\beta \otimes P_{\beta \alpha}$, where the ordering on $P_{\beta \alpha}$ in $V[G_\beta]$ is
given by letting \( f \geq g \) iff \((\exists p \in G_a) \ p \cup f \supseteq p \cup g\). To see that the image is dense, suppose \((p, \dot{f}) \in P_\beta \otimes P_\beta\). Then there must be \(p' \supseteq p\) and \(f \in P_\beta\) such that \(p' \forces_\beta \dot{f} = f\). Now let \(q \in P_\alpha\) be such that \(q \mid \beta = p'\) and \(q^\beta = f\). Then clearly \(q \mid \beta, q^\beta \supseteq (p, \dot{f})\).

Thus forcing with \(P_\alpha\) is the same as forcing with \(P_\beta \otimes P_\beta\).

2. Preservation of \(\omega_1\)

In this section we will show that forcing with \(P_\alpha\) does not collapse \(\omega_1\).

Some explanation of our forcing terminology may be in order first. We always consider forcing as taking place over \(V\), the universe of set theory, and we denote by \(V[G_\alpha]\) the extension of \(V\) via forcing with \(P_\alpha\) (of course \(G_\alpha\) denotes the canonical generic subset of \(P_\alpha\)). The reader uncomfortable with this approach may substitute for \(V\) any countable transitive model \(M\) of a sufficiently large fragment of ZFC.

If \(p \in P\) and \(s \in \omega\) for some \(n \in \omega\), then let \(p_s = \{t \in p : s \subseteq t\ \text{or} \ t \subseteq s\}\). Of course \(p_s \in P\) iff \(s \in p\).

**Lemma 2.1.** Suppose \(p \in P\), \(n \in \omega\), and \(p \forces \dot{a} \in V\). Then there are \(q \in P\), \(m \neq \omega\), and finite \(x \in V\) such that \((q, m) > (p, n)\) and \(q \forces \dot{a} = x\). In fact, \(q\) can be chosen so that for every \(s \in \omega \cap p\) there is \(a_s \in V\) such that \(q_s \forces \dot{a} = a_s\) (so \(x = \{a_s : s \in \omega \cap p\}\)).

**Proof.** For each \(s \in \omega \cap p\), find \(q^s \supseteq p_s\) and \(a_s \in V\) such that \(q^s \forces \dot{a} = a_s\). Let \(q = \bigcup\{q^s : s \in \omega \cap p\}\). Then \(q_s = q^s\) for each \(s \in \omega \cap p\). Now choose \(m\) large enough so that \((q, m) > (p, n)\).

In order to generalize Lemma 2.1 to arbitrary \(P_\alpha\) (see Lemma 2.3(ii) below) we need an analogue of \(p_s\).

Suppose \(p \in P_\alpha\), \(F\) is a finite subset of domain \((p)\), and \(\sigma : F \rightarrow \omega\). Then \(p \mid \sigma\) is the function with the same domain as \(p\) such that

\[
p \mid \sigma(\beta) = \begin{cases} p(\beta) & \text{if } \beta \notin F, \\ p(\beta)_{\sigma(\beta)} & \text{if } \beta \in F. \end{cases}
\]

As with \(p_s\) it need not be the case that \(p \mid \sigma \in P_\alpha\). Let us say that \(\sigma\) is **consistent with** \(p\) iff \(p \mid \sigma \in P_\alpha\), i.e., iff \(\forall \beta \in F \ (p \mid \sigma) \mid \beta \forces_\beta \sigma(\beta) \in p(\beta)\).

Let us say that \(p\) is \((F, n)\)-**determined** iff for any \(\sigma : F \rightarrow \omega\), either \(\sigma\) is consistent with \(p\) or else there is \(\beta \in F\) such that \(\sigma \mid (F \cap \beta)\) is consistent with \(p\) and \((p \mid \sigma) \mid \beta \forces_\beta \sigma(\beta) \notin p(\beta)\).

The following lemma collects some easy properties of these notions.

**Lemma 2.2.** Suppose \(p \in P_\alpha\), \(F\) is a finite subset of domain \((p)\), \(n \in \omega\), and \(\sigma : F \rightarrow \omega\). Then:

(i) If \(\max F < \beta < \alpha\), then \((p \mid \sigma) \mid \beta = (p \mid \beta) \mid \sigma\).

(ii) If \(\alpha = 1\), then \(p\) is \((\{0\}, n)\)-determined for every \(n \in \omega\).
(iii) If \( k \geq n \), \( G \supseteq F \), \((q, m) \succ_{G} (p, k)\) and \( p \) is \((F, n)\)-determined, then so is \( q \).
(iv) If \( \max F < \beta < \alpha \), then \( p \) is \((F, n)\)-determined iff \( p \upharpoonright \beta \) is \((F, n)\)-determined.
(v) There is \( q \in P_{\alpha} \) such that \( q \succeq p \) and for some \( \sigma : F \rightarrow {}^{< \omega}2 \), \( q = q \upharpoonright \sigma \).
(vi) If \( p \) is \((F, n)\)-determined and \( q \succeq p \), then there is \( \sigma : F \rightarrow {}^{< \omega}2 \) such that \( \sigma \) is consistent with \( p \), and \( q \) and \( p \upharpoonright \sigma \) are compatible.

**Proof.** The only non-trivial parts are (v) and (vi), and (v) easily implies (vi) so we only prove (v). Suppose the elements of \( F \) are \( \beta_1, \ldots, \beta_k \) in increasing order. By induction on \( i \leq k \) we find \( q_i \) and \( \sigma_i : \{ \beta_1, \ldots, \beta_i \} \rightarrow {}^{< \omega}2 \) so that \( p \leq q_0 \leq \cdots \leq q_k \) and \( q_i \upharpoonright \sigma_i = q_i \). Let \( q_0 = p \) and \( \sigma_0 = 0 \). Given \( q_i \) and \( \sigma_i \), find \( q \succeq q_i \upharpoonright \beta_{i+1} \) and \( s \in {}^{< \omega}2 \) so that \( q \upharpoonright \beta_{i+1}, s \in q_i(\beta_{i+1}) \). Then let \( \sigma_{i+1} = \sigma_i \upharpoonright \{ (\beta_{i+1}, s) \} \), and let \( q_{i+1} \) be defined by \( q_{i+1} \upharpoonright \beta_{i+1} = q_i \), \( q_{i+1}(\beta_{i+1}) = q_i(\beta_{i+1})_s \) and \( q_{i+1}(\gamma) = q_i(\gamma) \) for \( \gamma > \beta_{i+1} \).

**Lemma 2.3.** For each \( \alpha \geq 1 \), the following hold:

(i) If \( p \in P_{\alpha} \), \( n \in \omega \), \( p \upharpoonright \alpha \models \dot{a} \in V \), and \( F \) is a finite subset of domain \( p \), then there exists \((q, m)\) such that \((q, m) \succ_{F} (p, n)\), \( q \) is \((F, n)\)-determined, and \( \forall \sigma : F \rightarrow {}^{< \omega}2 \), if \( \sigma \) is consistent with \( q \), then there exists \( a_{\sigma} \) such that \( q \upharpoonright \sigma \models \dot{a} = a_{\sigma} \). Hence if \( x = \{ a_{\sigma} : \sigma \ \text{is consistent with} \ q \} \), then \( q \upharpoonright \alpha \models \dot{a} \in x \).

(ii) If \( p \in P_{\alpha} \), \( n \in \omega \), \( p \upharpoonright \alpha \models \dot{f} : \omega \rightarrow V \), and \( F \) is a finite subset of domain \( p \), then there exists \((q, m)\) and a sequence \((x_i : i \in \omega)\) of finite sets such that \((q, m) \succ_{F} (p, n)\) and \( q \upharpoonright \alpha \models \forall i f(i) \in x_i \).

(iii) If \( p \in P_{\alpha} \), \( n \in \omega \), \( p \upharpoonright \alpha \models \ \text{"\( \dot{a} \) is a countable subset of \( V \)"} \), and \( F \) is a finite subset of domain \( p \), then there exists \((q, m)\) and a countable set \( A \) such that \((q, m) \succ_{F} (p, n)\) and \( q \upharpoonright \alpha \models \dot{a} \subseteq A \).

(iv) If \( p \in P_{\alpha} \), \( n \in \omega \), \( \alpha < \gamma \), \( p \upharpoonright \alpha \models \dot{f} \in P_{\alpha \gamma} \), and \( F \) is a finite subset of domain \( p \), then there exist \((q, m)\) and \( f \in P_{\alpha \gamma} \) such that \((q, m) \succ_{F} (p, n)\) and \( q \upharpoonright \alpha \models \dot{f} = f \).

**Proof.** The proof is by induction on \( \alpha \).

(i) For \( \alpha = 1 \) this is Lemma 2.1 (and Lemma 2.2(ii)). Assume \( \alpha > 1 \).

Suppose \( \alpha = \beta + 1 \). Without loss of generality we may assume \( \beta \in F \). By Lemma 2.1 applied in \( V[G_{\beta}] \), we see that there are terms \( \dot{q}, \dot{m}, \) and \( \dot{a}_{\sigma} \) for each \( s \in {}^{< \omega}2 \) such that \( p \upharpoonright \beta \models \varphi \), where \( \varphi \) is the following assertion:

\[ (((\dot{q}, \dot{m}) \succ (p(\beta), n), \dot{a}_{\sigma} \in V), \) and \( \forall s \in p(\beta) \cap {}^{< \omega}2 \) \( \dot{q} \upharpoonright \dot{a}_{\sigma} = \dot{a}_{\sigma} \).\]

Now, by inductive hypothesis applied to \( P_{\beta} \), we can find \((q', m') \succ_{F-\{\beta\}} (p \upharpoonright \beta, n)\) such that \( q' \) is \((F-\{\beta\}, n)\)-determined and for each \( \sigma : F-\{\beta\} \rightarrow {}^{< \omega}2 \), if \( \sigma \) is consistent with \( q' \), then there are \( x_{\sigma}, m_{\sigma}, \) and \( \{ a_{\sigma-s} : s \in x_{\sigma} \} \subseteq V \) such that \( q' \upharpoonright \sigma \models \dot{q} \cap {}^{< \omega}2 = x_{\sigma}, \) \( m = m_{\sigma} \), and \( \forall s \in {}^{< \omega}2 \) \( \dot{a}_{\sigma} = a_{\sigma-s} \).

Here \( \sigma \rightarrow s \) denotes the function \( \tau : F \rightarrow {}^{< \omega}2 \) such that \( \tau \upharpoonright \beta = \sigma \) and \( \tau(\beta) = s \).

Now define \( q \in P_{\beta} \) by \( q \upharpoonright \beta = q' \) and \( q(\beta) = \dot{q} \), and let \( m \in \omega \) be large enough so that \( m \geq m' \) and \( m \geq m_{\sigma} \) for all \( \sigma \) consistent with \( q' \). But then \((q, m) \succ_{F} (p, n)\), \( q \) is \((F, n)\)-determined, and for any \( \sigma : F \rightarrow {}^{< \omega}2 \), if \( \sigma \) is consistent with \( q \), then \( q \upharpoonright \sigma \models \dot{a} = a_{\sigma} \). This completes the proof if \( \alpha \) is a successor ordinal.
Assume $\alpha$ is a limit ordinal. Choose $\beta$ so that $\max(F) < \beta < \alpha$. We use the symbol $\Vdash^*$ to denote forcing over $V[G_\alpha]$ with respect to $P_{\beta\alpha}$. Using the isomorphism between $P_\alpha$ and the dense subset of $P_\beta \otimes P_{\beta\alpha}$ constructed earlier, we see that

$$p \Vdash_\beta \langle \exists f \in P_{\beta\alpha} \rangle (\exists b \in V) f \geq p^\beta \text{ and } f \Vdash^* \dot{a} = b'. $$

It follows that there are terms $\dot{f}$ and $\dot{b}$ of the language of forcing with respect to $P_\beta$ such that

$$p \Vdash_\beta \dot{f} \geq p^\beta, \dot{b} \in V, \text{ and } \dot{f} \Vdash^* \dot{a} = \dot{b}' $$

Applying the inductive hypothesis to $P_\beta$ we find $(q_1, m) \succ_F (p \mid \beta, n)$ such that $q_1$ is $(F, n)$-determined and for every $\sigma : F \rightarrow 2$, if $\sigma$ is consistent with $q_1$, then there is $b_\sigma \in V$ such that $q_1 \mid \sigma \Vdash \dot{b} = b_\sigma$.

Now, applying part (iv) of the lemma to $P_\beta$, we find $f \in P_{\beta\alpha}$ and $(q_2, k) \succ_F (q_1, m)$ such that $q_2 \Vdash_\beta \dot{f} = f$. But then $(q_2, k) \succ_F (p \mid \beta, n)$, so if we let $q = q_2 \cup f$, then $(q, k) \succ_F (p, n)$, $q$ is $(F, n)$-determined (by Lemma 2.2(iii)) and for any $\sigma : F \rightarrow 2$, if $\sigma$ is consistent with $q$, then $q \mid \sigma \Vdash \dot{a} = b_\sigma$.

The final sentence of (i) follows from Lemma 2.2(vi).

(ii) Using part (i) repeatedly, it is easy to construct a fusion sequence $((p_i, n_i, F_i) : i \in \omega)$, and a sequence $(x_n : n \in \omega)$ so that $p_0 = p$, $n_0 = n$, $F_0 = F$ and $\forall i p_{i+1} \Vdash f(i) \in x_i$. If $q$ is the fusion of the sequence, then by Lemma 1.2, $q \Vdash_\alpha \forall i f(i) \in x_i$.

(iii) is a trivial consequence of (ii).

(iv) By part (iii) there is $(q, m) \succ_F (p, n)$ and a countable set $A$ such that $q \Vdash_\alpha \text{domain}(\dot{f}) \subseteq A$. Now define $f \in P_{\alpha\gamma}$ so that $\forall \beta \in A$, $f(\beta)$ is a term denoting the same object denoted by $\dot{f}(\beta)$ if $\beta \in \text{domain}(\dot{f})$ (in $V[G_\alpha]$), and otherwise $\dot{f}(\beta) = 0$. Then clearly $q \Vdash_\alpha \dot{f} \leq f \geq \dot{f}$ so $q \Vdash_\alpha \dot{f} = f$ by our convention for identifying equivalent objects.

Theorem 2.4. Forcing with $P_\alpha$ does not collapse $\omega_1$.

Proof. By Lemma 2.3(ii) or (iii).

Remark. Lemma 2.3 is all we will need for the rest of the paper, but it should be remarked that for many applications of perfect-set forcing a somewhat stronger version of part (ii) is required, namely: if $p \Vdash_\alpha \dot{f} : \omega \rightarrow V$, and if $g \in {}^\omega \omega$ is eventually arbitrarily large (i.e., $\forall m \exists n \forall k \geq n \ g(k) > m$) and $\forall n g(n) > 0$, then $\exists q \geq p \exists (z_n : n \in \omega) q \Vdash_\alpha \forall n \dot{f} \upharpoonright n \in z_n$ and $\forall n |z_n| \leq g(n)$. The proof is rather involved, since one must ensure that each condition "forks very slowly", but the reader familiar with perfect-set forcing should have little difficulty reconstructing it.

Lemma 2.3 also allows us to conclude that forcing with the $P_\alpha$ is iterated forcing in the full sense of the word.

Theorem 2.5. Let $P_\beta$ denote the result of defining $P_\beta$ in $V[G_\alpha]$. Then for any $\alpha, \beta \geq 1$, $\Vdash_\alpha P_{\alpha\alpha + \beta}$ is isomorphic to $P_\beta$. 

Proof. The proof is by induction on $\beta$. It is easy to see that $\Vdash_\alpha P_{\alpha, \alpha+\beta} \subseteq \{\bar{f} : \exists f \in \dot{P}_\beta \forall \gamma \in \text{domain}(f) \; \bar{f}(\alpha + \gamma) = f(\gamma)\}$. The reverse inclusion follows from Lemma 2.3(iv).

3. The $\kappa_2$-chain condition

In this section we show that $2^{\kappa_0} = \kappa_1$ implies that for each $\alpha \leq \omega_2$, $P_\alpha$ has the $\kappa_2$-chain condition, and hence preserves cardinals. This result is sharp, for it is shown in Section 5 that $P_{\omega_2+1}$ collapses $\omega_2$ onto $\omega_1$ (if $2^{\kappa_0} = \kappa_1$). We also remark that if $2^{\kappa_0} = \kappa_1$, then $\Vdash_{\omega_2} 2^{\kappa_0} = \kappa_2$.

Lemma 3.1. For each $\alpha < \omega_2$, there is a dense set $W_\alpha \subseteq P_\alpha$ such that $|W_\alpha| = 2^{\kappa_0}$.

Proof. Let $W_\alpha = \{p \in P_\alpha : \forall n \in \omega \forall \beta \in \text{domain}(p) \; \exists m \geq n \exists \text{ finite } F \subseteq \text{domain}(p) \; p \in (F, m)-\text{determined and } \beta \in F\}.$

First we show $W_\alpha$ is dense in $P_\alpha$. Let $p \in P_\alpha$ be arbitrary. Using Lemma 2.3(i) repeatedly, it is easy to find a fusion sequence $\langle (p_i, n_i, F_i) : i \in \omega \rangle$ such that $p_0 = P$, $F_0$ and $n_0$ are arbitrary, and for each $i$, $p_{i+1}$ is $(F_i, n_i)$-determined. Let $q$ be the fusion of the sequence. We claim $q \in W_\alpha$. Fix $n \in \omega$ and $\beta \in \text{domain}(q)$. Choose $i$ large enough so that $n_i \geq n$ and $\beta \in F_i$. Then since $p_{i+1}$ is $(F_i, n_i)$-determined it follows that $q$ is $(F_i, n_i)$-determined (by Lemma 2.2(iii)).

Now we show by induction on $\alpha < \omega_2$ that $|W_\alpha| = 2^{\kappa_0}$. Clearly $W_1 = P_1$ (by Lemma 2.2(ii)) so $|W_1| = 2^{\kappa_0}$. Note that if $p \in W_\alpha$ and $\beta < \alpha$, then $p \not\Vdash \beta \in W_\beta$. Hence $|W_\alpha| = 2^{\kappa_0}$ if $\alpha$ is a limit ordinal. Suppose $\alpha = \beta + 1$. If $p \in W_\alpha$, then $\forall n \exists m \geq n \exists \text{ finite } F \subseteq \text{domain}(p) \; p \in (\beta, m)-\text{determined and } \beta \in F$. Then clearly $p$ is completely determined by $p \not\Vdash \beta$, the sequence $\langle (m_n, F_n) : n \in \omega \rangle$, and the correspondence that sends $\sigma$ to $x_\sigma$. Hence $|W_\alpha| = 2^{\kappa_0}$.

Theorem 3.2. Assume $2^{\kappa_0} = \kappa_1$. Then for every $\alpha \leq \omega_2$, $P_\alpha$ has the $\kappa_2$-chain condition, and therefore preserves cardinals.

Proof. For $\alpha < \omega_2$ this follows from Lemma 3.1, since $P_\alpha$ has a dense subset $W_\alpha$ of cardinality $\kappa_1$. For $\alpha = \omega_2$ this follows from an easy $\Delta$-system argument.

Theorem 3.3. (a) If $f \in {}^\omega 2 \cap V[G_\alpha]$ and $\text{cf } \alpha \geq \omega_1$, then $\exists \beta < \alpha \; f \in V[G_\beta]$.

(b) Assume $2^{\kappa_0} = \kappa_1$. Then $\Vdash_{\omega_2} 2^{\kappa_0} = \kappa_2$.

Proof. Suppose $\Vdash_{\omega_2} f : \omega \rightarrow 2$. As in the proof of Lemma 3.1, there is $q \geq p$ and a sequence $\langle (F_n, m_n) : n \in \omega \rangle$ such that $\forall n \forall \sigma : F_n \rightarrow \langle m_n \rangle \cdot 2$ if $\sigma$ is consistent with $q$. Then for some $x_\sigma q \Vdash_\alpha \bar{f} \upharpoonright n = x_\sigma$. Then $\bar{f}$ is completely determined by $q$, $\langle (F_n, m_n) : n \in \omega \rangle$, and the mapping carrying each $\sigma$ to $x_\sigma$, provided that $q \in G_\alpha$.

If $q \in G_\alpha$ and $\text{cf } \alpha \geq \omega_1$, then $\exists \beta < \alpha \; q \in G_\beta$. This proves (a).

To prove (b) simply note that $q$ may be chosen to be in $W_\alpha$ for some $\alpha < \omega_2$.\]
4. Selective ideals and ultrafilters

If \( I \) is an ideal of subsets of \( \omega \), then let \( I^+ = \{ A \subseteq \omega : A \notin I \} \). An ideal \( I \) is \textit{selective} iff \( (\forall A \in I^+) (\forall f : A \rightarrow \omega) f \) is one-to-one or constant on a set in \( I^+ \). An ultrafilter on \( \omega \) is \textit{selective} (or Ramsey) iff its dual ideal is selective. (Note: Some authors use the term "weakly selective" instead of "selective"; our terminology is that of Grigorieff [3]. Throughout this section we consider only non-trivial ideals, i.e., ideals containing all finite subsets of \( \omega \).)

The object of this section is to show that under forcing with respect to the partial orderings \( P_\alpha \), selective ideals remain, in some sense, selective ideals. A precise statement is in Theorem 4.2.

Examples of selective ideals are:
(a) the dual ideal to a selective ultrafilter,
(b) the ideal of finite subsets of \( \omega \), and
(c) the ideal generated by a maximal (infinite) collection of almost-disjoint subsets of \( \omega \).

It should be remarked that selective ultrafilters can be proved to exist only under special set-theoretic assumptions, such as the continuum hypothesis or Martin's Axiom (see [1] or [4]). For a model with no selective ultrafilters, see Kunen [5].

It will be convenient to observe that \( I \) is a selective ideal iff

1. \( \forall A \in I, \forall B \subseteq A, B \in I \),
2. \( \forall A \in I, \forall \text{ finite } X \subseteq \omega, A \cup X \in I \), and
3. \( \forall A \in I^+, \forall f : A \rightarrow \omega, \text{ if } \forall n \{ a \in A : f(a) \leq n \} \in I \), then \( \exists B \in I^+ \ f \text{ is one-to-one on } B \).

The proof is trivial except for the fact that any set \( I \) satisfying (1)-(3) is closed under finite unions. If \( A, B \in I \) but \( A \cup B \notin I \), then choose \( f : A \cup B \rightarrow \omega \) such that \( f^{-1}(0) = A \) and \( f \) is one-to-one on \( B - A \), and apply (3) for a contradiction.

Another characterization of selective ideals, to be found in Grigorieff [3], is the following:

A set \( T \subseteq \bigcup \{ n \omega : n \in \omega \} \) is a \textit{tree} iff \( \forall s \in T \ \forall n \in \omega \ s | n \in T \). For each \( s \in n \omega \) and each \( m \in \omega \), let \( s^-\langle m \rangle \) denote the concatenation of \( s \) and \( m \). If \( T \) is a tree and \( s \in T \), let \( T_s = \{ m : s^-\langle m \rangle \in T \} \). If \( I \) is an ideal and \( T \) is a tree, then \( T \) is a \textit{strong} \( I \)-\textit{tree} iff every finite intersection of sets in \( \{ T_s : s \in T \} \) lies in \( I^+ \). A function \( f \in {}^\omega \omega \) is a \textit{branch} through a tree \( T \) iff \( \forall n f | n \in T \).

Then \( I \) is a selective ideal iff every strong \( I \)-tree has a branch \( f \) with range \( (f) \in I^+ \).

If \( X \) is a collection of subsets of \( \omega \), then the \textit{upward closure} of \( X \) is \( \{ A \subseteq \omega : \exists B \in X B \subseteq A \} \).

**Lemma 4.1.** Suppose \( I \) is a selective ideal. Then

\[ \models \text{"the upward closure of } I^+ \text{ is the complement of a selective ideal"}, \]

where \( \models " \) refers to forcing with \( P \).
Proof. Let $\dot{J}$ be a name for the complement of the upward closure of $I^+$. Then clearly $\models \dot{J}$ satisfies (1), (2) above. We need only check $\models \dot{J}$ satisfies (3).

Suppose $A \in I^+$, $p \in P$, and

$$p \Vdash \dot{f}: A \rightarrow \omega \text{ and } \forall n \{m : \dot{f}(m) \leq n\} \in \dot{J}.$$ 

It will suffice to find $q \geq p$ and $B \in I^+$ such that

$$q \Vdash \dot{f} \text{ is one-to-one on } B.$$ 

**Sublemma.** Let $p' \geq p$ and $n, k \in \omega$. Let

$$Z(p', n, k) = \{i \in A : (\exists q, m, l)(q, m) > (p', n) \text{ and } q \Vdash k \leq \dot{f}(i) < l\}.$$ 

Then $A - Z(p', n, k) \in I$.

**Proof.** For each $s \in p' \cap \omega^2$, let

$$Z_s = \{i \in A : \exists q \geq p'_s \text{ and } q \Vdash \dot{f}(i) \geq k\}.$$ 

Then clearly $A - Z_s \in I$. We claim $\cap \{Z_s : s \in p' \cap \omega^2\} \subseteq Z(p', n, k)$, which will complete the proof. Let $i \in \cap \{Z_s : s \in p' \cap \omega^2\}$. For each $s \in p' \cap \omega^2$ choose $q^s \geq p'_s$ and $k_s \geq k$ such that $q^s \Vdash \dot{f}(i) = k_s$. Let $q = \bigcup \{q^s : s \in p' \cap \omega^2\}$. If $m$ and $l$ are chosen large enough so that $(q, m) > (p', n)$ and $\max \{k_s : s \in p' \cap \omega^2\} < l$, then clearly $q \Vdash k \leq \dot{f}(i) < l$. Hence $i \in Z(p', n, k)$.

We return to the proof of Lemma 4.1.

Let us define a strong $I$-tree $T$ as follows. By induction on $n$, we will determine $T \cap \omega^n$ and, for each $s \in T \cap \omega^n$, we will find $p_s \in P$ and $n_s$ and $k_s \in \omega$ such that if $t = s \triangleleft (i)$ and $i \in T$, then $(p_t, n_t) > (p_s, n_s)$ and $p_t \Vdash k_s \leq \dot{f}(i) < k_t$. The only element of $T \cap \omega^2$ is the empty sequence 0. Let $p^0 = p$ and $n_0 = k_0 = 0$. Given $s \in T \cap \omega^2$, put $t = s \triangleleft (i) \in T$ iff $i > \max \text{ range } (s)$ and $i \in Z(p^s, n_s, k_s)$. Then $p^t, n_t, k_t$ may be found easily. By the Sublemma, $T$ is a strong $I$-tree. Since $I$ is selective, $T$ has a branch $g$ with $B = \text{ range } g \in I^+$. Note that $g$ must enumerate $B$ in increasing order. But now if $q = \bigcap \{p^i : i \in \omega\}$, it is clear by Lemma 1.1 (since $(p^i, n_i, k_i) > (p^g, n_g, k_g)$ for all $i$) that $q \in P$, and $q \Vdash \dot{f}$ is one-to-one on $B$.

**Theorem 4.2.** Suppose $I$ is a selective ideal. Then for any $\alpha \geq 1$,

$$\models \alpha \text{ "the upward closure of } I^+ \text{ is the complement of a selective ideal"}.$$ 

**Proof.** The proof is by induction on $\alpha$. For $\alpha = 1$ this is Lemma 4.1, and if $\alpha = \beta + 1$, $\beta \geq 1$, then since $P_\alpha$ is isomorphic to a dense subset of $P_\beta \otimes P$ we are done by inductive hypothesis and Lemma 4.1.

Thus we need only treat the case where $\alpha$ is a limit ordinal. The proof in this case is essentially the same as in Lemma 4.1, provided we can prove an appropriate version of the Sublemma. Therefore we prove the following, and leave everything else to the reader.
Sublemma. Let $A \in I^+$, $p \in P_\alpha$, and assume

$$p \Vdash_A \hat{f} : A \to \omega \quad \text{and} \quad \forall n \{m : \hat{f}(m) \leq n\} \in \hat{J}$$

as before. Suppose $n, k \in \omega$ and $F$ is a finite subset of domain $(p)$. Let

$$Z(p, n, F, k) = \{i \in A : (\exists q, m, l)(q, m) > F (p, n) \text{ and } q \Vdash_A k \leq \hat{f}(i) < l\}.$$ 

Then $A - Z(p, n, F, k) \in I$.

Proof. Choose $\beta$ so that $\sup F < \beta < \alpha$. Let $\hat{J}_\beta$ be a name for the complement of the upward closure of $I^+$ in $\mathcal{V}[G_\beta]$. By inductive hypothesis, $\Vdash_B \hat{J}_\beta$ is selective.

Let $\hat{Z}$ be such that

$$p \mid \beta \Vdash_B \hat{Z} = \{i \in A : \exists g \in P_{\beta^*} \forall \beta \geq \beta \text{ and } g \Vdash_A \hat{f}(i) \geq k\},$$

where $\Vdash_A$ refers to forcing with $P_{\beta^*}$. Since

$$p \Vdash_A \forall n \{m : \hat{f}(m) \leq n\} \in \hat{J},$$

we must have $p \mid \beta \Vdash_B A - \hat{Z} \in \hat{J}_\beta$.

To show that $A - Z(p, n, F, k) \in I$ we need only show that if $B \in I^+ \cap P(A)$, then $B \cap Z(p, n, F, k) \neq 0$. We begin by showing that there is $i \in B$ and $(q, m) > F (p \mid \beta, n)$ such that $q \Vdash_B i \in \hat{Z}$.

By Lemma 2.3(i) we may assume $p$ is $(F, n)$-determined. Let $\sigma_1, \ldots, \sigma_r$ enumerate all functions $\sigma : F \to \omega$ such that $\sigma$ is consistent with $p$. We construct a sequence $\langle (p_0, n_0, B_0) : r \rangle$ by induction so that $(p_{i+1}, n_{i+1}) > F (p_i, n_i)$, $B_i \in I^+$, $B_{i+1} \subseteq B_i$, and $p_{i+1} \mid \sigma_i \Vdash_B B_{i+1} \subseteq \hat{Z}$. Let $p_0 = p \mid \beta$, $n_0 = n$, $B_0 = B$. Since $p_0 \Vdash_B A - \hat{Z} \in \hat{J}_\beta$, we must have

$$p_1 \Vdash_B (\exists B' \in I^+) \ V \subseteq B_1 \cap \hat{Z}.$$

By Lemma 2.3(iii) there is $(p_{i+1}, n_{i+1}) > F (p_i, n_i)$ and $B_{i+1} \in I^+$ such that $p_{i+1} \mid \sigma_i \Vdash_B B_{i+1} \subseteq B_i \cap \hat{Z}$. But now if $q = p_r$ and $m = m_r$ we have $(q, m) > F (p \mid \beta, n)$ and $q \Vdash_B B_r \subseteq \hat{Z}$.

We assert that if $i \in B_r$, then $i \in Z(p, n, F, k)$, and this will complete the proof. By Lemma 2.3(iv) there is $g \in P_{\beta^*}$ and $(q', m') > F (q, m)$ such that $q' \Vdash_B "g \geq p^B"$ and $g \Vdash_A \hat{f}(i) \geq k"$. Hence $q' \cup g \Vdash_A \hat{f}(i) \geq k$. Now by Lemma 2.3(i) there is $(q'', m'') > F (q' \cup g, m')$ such that for some $l$, $q'' \Vdash_A \hat{f}(i) < l$. Hence $i \in Z(p, n, F, k)$.

Theorem 4.3. For any $\alpha \geq 1$,

$$\Vdash_A \forall A \subseteq \omega \exists B \in V B \text{ is infinite and } B \subseteq A \text{ or } B \subseteq \omega - A.$$ 

Proof. Let $I$ be the ideal of finite subsets of $\omega$ and apply Theorem 4.2, using the characteristic function in $\omega$ for $A$.

Theorem 4.4. Let $U$ be a selective ultrafilter on $\omega$. Then for any $\alpha \geq 1$,

$$\Vdash_A U \text{ generates a selective ultrafilter.}$$
Proof. Let $I$ be the dual ideal of $U$, and apply Theorem 4.2.

Theorem 4.5. Assume that $2^\kappa_0 = \kappa_1$ and $2^\kappa_1 = \kappa_2$. Then in $V[G_{\omega_2}]$ it is true that $2^{\kappa_0} = 2^\kappa_1 = \kappa_2$ and every selective ultrafilter is $\kappa_1$-generated. Hence in $V[G_{\omega_2}]$ there are exactly $2^{\kappa_0}$ selective ultrafilters.

Proof. By Theorem 3.3(b), $I_{\omega_2} 2^{\kappa_0} = \kappa_2$. Since $2^{\kappa_1} = \kappa_2$, $P_{\omega_2}$ has the $\kappa_2$-chain condition, and $P_{\omega_2}$ has a dense subset of cardinality $\kappa_2$ (by Lemma 3.1) it follows immediately that $I_{\omega_2} 2^{\kappa_1} = \kappa_2$. Thus we only need show that every selective ultrafilter in $V[G_{\omega_2}]$ is $\kappa_1$-generated.

Suppose $U$ is a selective ultrafilter in $V[G_{\omega_2}]$. Let $\langle A_\alpha : \alpha < \omega_2 \rangle$ enumerate $U$, and let $\dot{A}_{\alpha}$ be a name for $A_\alpha$ in the language of forcing with respect to $P_{\omega_2}$. For each $\alpha < \omega_2$ and each $n < \omega$, let $D_{\alpha n}$ be a maximal pairwise incompatible subset of $P_{\omega_2}$ such that $\forall p \in D_{\alpha n}$ either $p \Vdash_{\omega_2} n \in \dot{A}_{\alpha}$ or $p \Vdash_{\omega_2} n \notin \dot{A}_{\alpha}$. Since $P_{\omega_2}$ has the $\omega_2$-chain condition, $\exists \beta < \omega_2 \forall n D_{\alpha n} \subseteq P_{\beta}$.

By Theorem 3.3(a), if $\exists \beta < \omega_2$ and $f \in {}^{\omega_2} \cap V[G_{\alpha}]$, then $\exists \beta < \omega_2 f \in V[G_{\beta}]$.

It follows that in $V[G_{\omega_2}]$, for each $\alpha < \omega_2$ there is $\pi(\alpha) < \omega_2$ such that $\forall f \in {}^{\omega_2} \cap V[G_{\alpha}] \exists \beta < \pi(\alpha) f$ is one-to-one or constant on $A_\beta$. Since $P_{\omega_2}$ has the $\kappa_2$-chain condition, there is $\bar{\pi} : \omega_2 \to \omega_2$ such that $\bar{\pi} \in V$ and $\forall \alpha \pi(\alpha) = \bar{\pi}(\alpha)$.

But now it is easy to see that there must be some $\beta < \omega_2$ such that $\exists \beta = \omega_1$ and $\forall \alpha < \beta \bar{\pi}(\alpha) < \beta$ and $\forall n D_{\alpha n} \subseteq P_{\beta}$. But then clearly $U \cap V[G_{\beta}] \in V[G_{\beta}]$ and $U \cap V[G_{\beta}]$ is a selective ultrafilter in $V[G_{\beta}]$.

By Theorem 2.5, $I_{\beta} P_{\beta, \omega_2}$ is isomorphic to $P_{\omega_2}$, so by Theorem 4.4 applied in $V[G_{\beta}]$, $U \cap V[G_{\beta}]$ generates $U$. Since $|U \cap V[G_{\beta}]| = \kappa_1$, we are done.

Putting Theorems 4.4 and 4.5 together, we have, for example,

Theorem 4.6. If ZF is consistent, then so is ZFC + $2^{\kappa_0} = \kappa_2$ + every selective ultrafilter in $L$ generates a selective ultrafilter + every selective ultrafilter is $\kappa_1$-generated.

Remarks. (1) Kunen (unpublished) was the first to prove the consistency with $2^{\kappa_0} > \kappa_1$ of the existence of an $\kappa_1$-generated selective ultrafilter, but his model is quite different. He begins with a model of $2^{\kappa_0} > \kappa_1$ and adds a generating set iteratively in $\omega_1$ steps, each time preserving the countable chain condition.

(2) The existence of an $\kappa_1$-generated ultrafilter while $2^{\kappa_0} = \kappa_2$ also settles problems 26 and 40 of Erdös and Hajnal [8]. The notation $(\kappa) \to (\lambda)^1_{\alpha, \beta}$ mean that for any $f : \kappa \times \lambda \to 2$ there are $A \subseteq \kappa$ and $B \subseteq \lambda$ such that $|A| = \kappa$, $|B| = \lambda$ and $f$ is constant on $A \times B$. A family $F$ of sets has property $B$ if there is a set $A$ such that for all $B \in F$, both $A \cap B$ and $A - B$ are nonempty. Problem 26 asks whether

$$
\left(\begin{array}{c}
2^{\kappa_0} \\
\kappa_0
\end{array}\right) \to \left(\begin{array}{c}
2^{\kappa_0} \\
\kappa_0
\end{array}\right)^{1,1}_{2,2}
$$
always holds and problem 40 asks whether every collection $F$ of sets such that $|F| < 2^{\aleph_0}$ has property $B$. Both problems clearly have negative answers if $2^{\aleph_0} = \aleph_2$, and there is an $\aleph_1$-generated ultrafilter. In fact, as in Theorem 4.6 we can even obtain a slight improvement, namely

$$\left(\frac{2^{\aleph_0}}{\aleph_0}\right) \rightarrow \left(\frac{2^{\aleph_0}}{\aleph_0} \text{ in } L\right)_{1,1},$$

which is defined as above except that the set $B$ is required to be in $L$.

5. Collapsing cardinals

In this section we prove, assuming the continuum hypothesis, that forcing with $P_{\omega_1}$ collapses $\omega_2$ onto $\omega_1$, and hence it is relatively consistent with ZFC that forcing with $P$ collapses cardinals. This also shows that Theorem 3.2 cannot be improved.

Let us say that $p \in P_{\omega_1}$ splits if there is a sequence $\langle (F_n, m_n, i_n) : n \in \omega \rangle$ such that

1. each $F_n$ is finite and $\bigcup \{F_n : n \in \omega\} = \text{domain} (p \upharpoonright \omega_2)$.
2. $\forall n \in \omega$, $m_n, i_n \in \omega$, $F_n \subseteq F_{n+1}$, $i_n < i_{n+1}$, and $p \upharpoonright \omega_2$ is $(F_n, m_n)$-determined.
3. $\forall n \in \omega$, $\forall \sigma : F_n \rightarrow \mathord{\langle \omega \rangle} \omega_2$ if $\sigma$ is consistent with $p \upharpoonright \omega_2$, then there is $x_\sigma \in V$ such that $p \upharpoonright (\omega_2) \upharpoonright \sigma \upharpoonright \omega_2 \upharpoonright (\omega_2)$ $\downarrow \sigma$.
4. if $\sigma, \tau, x_\sigma$, and $x_\tau$ are as in (3) and $\sigma \neq \tau$, then $x_\sigma \cap x_\tau = 0$.

Suppose $p$ splits. Let $\alpha_p$ be the unique countable ordinal such that there is an order-preserving mapping $f_p$ carrying $\alpha_p$ onto domain $p$. Let $E_\omega = f_p^{-1}(F_n)$ and for each $n$ and each $\sigma : E_n \rightarrow \mathord{\langle \omega \rangle} \omega_2$ let $g_p(\sigma) = x_{\sigma \circ f_p^{-1}}$, where the $x_\sigma$ are as in (3) and (4) above. We refer to $g_p$ as the splitting type of $p$. Note that $\alpha_p$ and $\langle (E_n, m_n, i_n) : n \in \omega \rangle$ are recoverable from $g_p$.

Lemma 5.1. If $p$ and $q$ both split and have the same splitting type, and if $p$ and $q$ are compatible, then domain $(p)$ = domain $(q)$.

Proof. Suppose domain $(p) \neq$ domain $(q)$. Let $r \supseteq p \upharpoonright \omega_2$, $q \upharpoonright \omega_2$. It will suffice to prove that there is $r' \supseteq r$ such that

$r' \upharpoonright \omega_2 \upharpoonright p(\omega_2)$ and $q(\omega_2)$ are incompatible.

Since $g_p = g_q$ we must have $\alpha_p = \alpha_q$ and the sequence $\langle (E_n, m_n, i_n) : n \in \omega \rangle$ must be the same for both $p$ and $q$.

Let $\beta < \alpha_p$ be the least ordinal such that $f_p(\beta) \neq f_q(\beta)$. Without loss of generality we may assume $f_p(\beta) < f_q(\beta)$. Let $\gamma = f_p(\beta)$, $\delta = f_q(\beta)$. Clearly there is $r_\gamma \supseteq r \upharpoonright \delta$ and $s \in \mathord{\langle \omega \rangle} \omega_2$ such that $s \neq t$, $\beta \in E_n$, and

$r_\delta \upharpoonright s \upharpoonright t \in r(\delta)$. 

Let \( F = (f_p^\omega E_n) \cap \delta \). By Lemma 2.2(v) we may assume that there is \( \sigma : F \rightarrow \langle \omega_1 \rangle^2 \) such that \( r_1 \models \sigma \in P_\delta \). Define \( \tau : F \cup \{ \delta \} \rightarrow \langle \omega_1 \rangle^2 \) by \( \tau | F = \sigma \) and \( \tau(\delta) = s \) if either \( \gamma \notin F \) or \( \sigma(\gamma) \neq \delta \), and \( \tau(\delta) = t \) otherwise.

Finally, take \( r' \supseteq r | \tau \) such that \( r' | \delta \supseteq r_1 | \sigma \) and such that for some \( \sigma_1, \sigma_2 : E_n \rightarrow \langle \omega_1 \rangle^2 \) we have \( r' | \sigma_1 \circ f_p^{-1} = r' = r' | \sigma_2 \circ f_p^{-1} \) (by Lemma 2.2(v)). By the way \( \tau \) was defined we must have \( \sigma_1 \neq \sigma_2 \) (since \( \sigma_1(\beta) \neq \sigma_2(\beta) \)). Hence \( g_\sigma(\sigma_1) \cap g_\tau(\sigma_2) = 0 \) and we are done, since

\[
r \models \omega_2 p(\omega_2) \cap \langle \omega_1 \rangle^2 = g_\sigma(\sigma_1)
\]

while

\[
r \models \omega_2 q(\omega_2) \cap \langle \omega_1 \rangle^2 = g_\tau(\sigma_2).
\]

Note that since \( p | \omega_2 \) is \( (f_p^\omega E_n, m_n) \)-determined, \( \sigma_1 \circ f_p^{-1} \) must be consistent with \( p | \omega_2 \); similarly for \( \sigma_2 \).

Lemma 5.2. \( \{ p \in P_{\omega_2} : p \text{ splits} \} \) is dense in \( P_{\omega_2} \).

**Proof.** Let \( q \in P_{\omega_2^+} \). We will find \( p \models q \) such that \( p \) splits.

Let \( k \in \omega^\omega \) be given. Then, using Lemma 2.3(i) repeatedly, it is easy to find a sequence \( \langle (p_n, m_n, F_n, i_n) : n \in \omega \rangle \) such that \( \langle (p_n, m_n, F_n) : n \in \omega \rangle \) is a fusion sequence, \( p_0 = q | \omega_2 \), \( m_0 = F_0 = i_0 = 0 \), \( p_{n+1} \) is \( (F_n, m_n) \)-determined, and for all \( \sigma : F_n \rightarrow \langle \omega_1 \rangle^2 \) there is \( y_\sigma \in V \) such that if \( \sigma \) is consistent with \( p_{n+1} \), then

\[
p_{n+1} | \sigma \models \omega_2 q(\omega_2) \cap \langle \omega_1 \rangle^2 = y_\sigma,
\]

and

\[
p_{n+1} \models \omega_2 \text{ every element of } q(\omega_2) \cap \langle \omega_1 \rangle^2 \text{ has at least } k(n) \text{ extensions in } q(\omega_2) \cap \langle \omega_1 \rangle^2.
\]

Moreover, such a sequence can still be found even if \( k(n) \) depends on \( F_n, m_n, \) and \( i_n \). The function \( k \) we shall need is determined as follows. Let \( l \) be the number of sequences \( \sigma : F_n \rightarrow \langle \omega_1 \rangle^2 \), and let \( k(n) = l \cdot 2^{l+1} \).

Then, given a sequence as above that works for this \( k \), it is easy to find \( x_\sigma \subseteq y_\sigma \) for each \( \sigma \) such that each \( x_\sigma \neq 0 \) and (a) if \( \sigma : F_n \rightarrow \langle \omega_1 \rangle^2 \) and \( \tau : F_{n+1} \rightarrow \langle \omega_1 \rangle^2 \) and \( \forall \beta \in F_n \sigma(\beta) \subseteq \tau(\beta) \), then every element of \( x_\sigma \) has at least two extensions in \( x_\tau \), and (b) if \( \sigma \neq \sigma' \), then \( x_\sigma \cap x_{\sigma'} = 0 \).

Now define \( p \models q \) by letting \( p | \omega_2 \) be the fusion of the sequence \( \langle (p_n, m_n, F_n) : n \in \omega \rangle \) and defining \( p(\omega_2) \) by stipulating

\[
(p | \omega_2) | \sigma \models \omega_2 p(\omega_2) \cap \langle \omega_1 \rangle^2 = x_\sigma
\]

for every \( \sigma \). Then \( p \) splits and \( p \models q \).

**Theorem 5.3.** Assume the continuum hypothesis. Then forcing with \( P_{\omega_2^+} \) collapses \( \omega_2 \) onto \( \omega_1 \).
Proof. The continuum hypothesis trivially implies that if $T$ is the set of all splitting types of elements of $P_{\omega_1+1}$, then $|T| = \aleph_1$. Now, working in $V[G_{\omega_1+1}]$, define $f: T \to \dot{\omega}_2$ (where $\dot{\omega}_2$ denotes the $\omega_2$ of $V$) by $f(g) = \xi$ iff $\exists p \in G_{\omega_1+1}$ $p$ splits and has splitting type $g$ and supdomain $(p) = \xi$. By Lemma 5.1, $f$ is a function with domain of cardinality $\aleph_1$. By Lemma 5.2, the range of $f$ is cofinal in $\dot{\omega}_2$. This completes the proof.

Corollary 5.4. It is relatively consistent with ZFC that forcing with $P$ fails to preserve cardinals.

Proof. $P_{\omega_1+1}$ is isomorphic to a dense subset of $P_{\omega_2} \otimes P$.

6. Aronszajn trees

The cardinal-collapsing results of the previous section may seem unfortunate, but they can be put to use to show that if $\kappa$ is a weakly compact cardinal then forcing with $P_{\kappa}$ yields a model in which there are no $\omega_2$-Aronszajn trees.

For a more complete discussion of the problem of the existence of Aronszajn trees, see Mitchell [6]. The first model in which there are no $\omega_2$-Aronszajn trees is also to be found in [6]. The forcing conditions here are quite different; nevertheless in Lemma 6.2 we show that they have the same key property as Mitchell's.

Mitchell's model can easily be modified to yield a model in which there are no $\omega_3$-Aronszajn trees, and the same can be done for the methods here. In recent unpublished work, A. Kanamori has shown how to generalize iterated perfect-set forcing to larger cardinals.

Lemma 6.1. Suppose $\alpha > \omega$, $p \in P$, and

$$ p \forces \check{f} \text{ is a function on } \alpha \text{ and } (\forall \beta < \alpha) \check{f}|\beta \in V. $$

Then $p \forces \check{f} \in V$.

Proof. Suppose the conclusion is false. Then there is $q \supseteq p$ such that $q \forces \check{f} \notin V$.

Recall that for any $r \in P$ and $s \in {}^\kappa \{2\} \cap r$, $r_s = \{t \in r : t \subseteq s \text{ or } s \subseteq t\}$.

We will construct a fusion sequence $\langle (p_i, n_i) : i \in \omega \rangle$ such that

1. $(\forall i)(\forall s \in p_i \cap {}^{(n_i)}2)(\exists \beta \in \check{g}_s)(p_i \forces \check{f}|\beta = \check{g}_s)$,
2. $(\forall i)(\forall s, t \in p_i \cap {}^{(n_i)}2)$ if $s \neq t$, then $g_s$ and $g_t$ disagree at some point.

Let $p_0 = q$, $n_0 = 0$. Now suppose $(p_i, n_i)$ is given.

Let $k$ be the cardinality of $p_i \cap {}^{(n_i)}2$, and fix $s \in p_i \cap {}^{(n_i)}2$. Since $(p_i) \forces \check{f} \notin V$, it is easy to find an ordinal $\beta(s) < \alpha$ and a sequence $\langle (p(s, j), g(s, j)) : 0 \leq j \leq k \rangle$ such that for each $j$, $p(s, j) \supseteq (p_i)_s$, domain $g(s, j) = \beta(s)$, $p(s, j) \forces \check{g}(s, j) \subseteq \check{f}$, and if $j_1 \neq j_2$, then $g(s, j_1)$ and $g(s, j_2)$ disagree at some point. Moreover, we may assume that if
Iterated perfect-set forcing

s, t \in p_i \cap (n)^2$, then $\beta(s) = \beta(t)$. Hence there is a function $z$ such that if $s, t \in p_i \cap (n)^2$ and $s \neq t$, then $g(s, z(s))$ and $g(t, z(t))$ disagree somewhere. Let $p_{i+1} = \bigcup \{p(s, z(s)) : s \in p_i \cap (n)^2\}$, and choose $n_{i+1}$ so large that $(p_{i+1}, n_{i+1}) > (p_i, n_i)$. Let $g_s = g(s, z(s))$. This completes the construction.

Let $r$ be the fusion of $(p_i, n_i) : i \in \omega)$. Since $\text{cf } \alpha > \omega$ there is $\beta < \alpha$ such that for every $s$ occurring in the construction of the $p_i$, $\beta(s) < \beta$. Let $r' \equiv r$ be such that for some $g$, $r' \Vdash \beta = g$. But since $r' \in P$ there is $i \in \omega$ and $s, t \in r' \cap (n)^2$ such that $s \neq t$. But then we must have $g_s, g_t \subseteq g$, contradicting the construction of $z$.

Now we step up Lemma 6.1 to arbitrary $P_e$.

**Lemma 6.2.** Let $\xi \geq 1$ be arbitrary. Suppose $\text{cf } \alpha > \omega$, $p \in P_e$, and $p \Vdash_{\xi} \dot{f}$ is a function on $\alpha$ and $(\forall \beta < \alpha) \dot{f} \mid \beta \in V$. Then $p \Vdash_{\xi} \dot{f} \in V$.

**Proof.** The proof is by induction on $\xi$. By Lemma 6.1, we need treat only the case where $\xi$ is a limit ordinal. The proof in this case is quite similar to the proof of Lemma 6.1; we assume $p \Vdash_{\xi} \dot{f} \in V$ and build a fusion sequence in which the crucial step is given by the following Sublemma. We will prove the Sublemma and leave the rest to the reader.

**Sublemma.** Suppose $q \Vdash_{\xi} \dot{f} \notin V$, $n \in \omega$, and $F$ is a finite subset of domain $(q)$. Then there is $(r, m)$ such that $(r, m) >_F (q, n)$, $r$ is $(F, n)$-determined, and for every $\sigma : F \rightarrow n^2$, if $\sigma$ is consistent with $r$, then there is a function $g, \sigma \in V$ such that

(i) $r \mid \sigma \Vdash_{\xi} (\exists \beta < \alpha) g, \sigma = \dot{f} \mid \beta$.

(ii) if $\sigma \neq \tau$, then $g, \sigma$ and $g, \tau$ disagree at some point.

**Proof.** Choose $\eta$ so that $\max(F) < \eta < \xi$. Then by Lemma 6.2 for $\eta$, which we may assume as inductive hypothesis, we must have

$$q \mid \eta \Vdash \eta \Vdash (q \Vdash \dot{f} \notin V)[G_n]$$

where $\Vdash$ refers to forcing over $V[G_n]$ with respect to $P_{\eta e}$.

By Lemma 2.3(i), we may assume without loss of generality that $q$ is $(F, n)$-determined. Let $\sigma_1, \ldots, \sigma_k$ enumerate all $\sigma : F \rightarrow n^2$ such that $\sigma$ is consistent with $q$.

Thus we have

$$q \mid \eta \Vdash (\exists h_1, \ldots, h_k \geq q^n)(\exists g_1, \ldots, g_k \in V) \text{ if } i \neq j, \text{ then } h_i \text{ and } g_i \text{ disagree at some point and if } 1 \leq i \leq k, \text{ then } h_i \Vdash (\exists \beta < \alpha) \dot{f} \mid \beta = g_i$$. 

Fix terms $h_1, \ldots, h_k, \dot{g}_1, \ldots, \dot{g}_k$ so that

$$q \mid \eta \Vdash _\eta (\exists h_1, \ldots, h_k \geq q^n) \text{ and } \dot{g}_1, \ldots, \dot{g}_k \in V \text{ and if } i \neq j, \text{ then } \dot{g}_i \text{ and } \dot{g}_j \text{ disagree at some point and if } 1 \leq i \leq k, \text{ then } h_i \Vdash (\exists \beta < \alpha) \dot{f} \mid \beta = \dot{g}_i$$. 

By Lemma 2.3(i) and (iv) there is \((q', n') >_F (q, n)\) such that for all \(j\), if \(1 \leq j \leq k\), then there are \(h_{ij} \in P_\eta\) and \(g_{ij}\) for \(i = 1, \ldots, k\) such that

\[
q' \upharpoonright \sigma_j \models \forall i \, h_i = h_{ij} \quad \text{and} \quad g_i = g_{ij}.
\]

Now it is easy to find a mapping \(\pi : \{1, \ldots, k\} \to \{1, \ldots, k\}\) such that for all \(j, j'\), if \(j \neq j'\), then \(g_{\pi(j)}\) and \(g_{\pi(j')}\) disagree at some point. Determine \(r \in P_\xi\) as follows. Let \(r \models \eta = q'\), and let \(r^n\) be such that for all \(j, q' \upharpoonright \sigma_j \models \eta = h_{\pi(j)}\). Then clearly \((r, n') >_F (q, n)\), and if \(1 \leq j \leq k\), then

\[
r \upharpoonright \sigma_j \models \exists \beta <_\alpha g_{\pi(j)} = f \models \beta.
\]

**Lemma 6.3.** Suppose \(\kappa\) is strongly inaccessible. Then \(\kappa = \omega^V_2\).

**Proof.** An easy \(\Delta\)-system argument shows that if \(\kappa\) is strongly inaccessible, then \(P_\kappa\) has the \(\kappa\)-chain condition. Hence \(\kappa \geq \omega_2^V\). To complete the proof we must show that every cardinal \(\lambda\) such that \(\aleph_1 < \lambda < \kappa\) is collapsed onto \(\aleph_1\) in \(V[G_\lambda]\). This can be proved along the lines of Section 5 (assuming \(2^{< \aleph_1} = \aleph_1\) in \(V\)), but a much simpler proof is available, as pointed out by J. Roitman. All we need is the following (combined with Theorem 2.5).

**Sublemma.** If \(2^{\aleph_1} > \aleph_1\) in \(V\), then \(P_{\aleph_1}\) collapses \(2^{\aleph_1}\) onto \(\aleph_1\).

**Proof.** For \(\alpha < \omega_1\) and \(i \in \{0, 1\}\), let \(p_{\alpha i}\) be a term such that \(\models \alpha p_{\alpha i} = \{t \in \bigcup \{2^n : n \in \omega\} : t(0) = i\}\). For each \(f \in \omega\) and each \(\alpha < \omega_1\), let \(q_{\alpha f} \in P_{\omega_1}\) be such that \(\text{dom}(q_{\alpha f}) = \{\omega \cdot \alpha + n : n \in \omega\}\) and if \(\beta = \omega \cdot \alpha + n\), then \(q_{\alpha f}(\beta) = p_{\alpha f(n)}\). If \(f \neq g\), then \(q_{\alpha f}\) and \(q_{\alpha g}\) are incompatible. Now in \(V[G_{\omega_1}]\) define a function \(h : \omega_1 \to (\omega^V)^V\) by

\[
h(\alpha) = \begin{cases} f & \text{if } q_{\alpha f} \in G_{\omega_1}, \\ f_0 & \text{otherwise,} \end{cases}
\]

where \(f_0 \in (\omega^V)^V\) is fixed. Clearly \(h\) is well-defined and onto by the genericity of \(G_{\omega_1}\).

**Remark.** This lemma is applicable to many other notions of forcing besides perfect-set forcing.

A partial ordering \((T, \leq_T)\) is a tree iff \(\forall t \in T \{s \in T : s \leq_T t\}\) is well-ordered. If \((T, \leq_T)\) is a tree, let \(T^\alpha = \{t \in T : \{s \in T : s \leq_T t\}\}\) has order type \(\alpha\). A branch through \((T, \leq_T)\) is a linearly ordered subset of \(T\). A tree \((T, \leq_T)\) is an \(\omega_2\)-Aronszajn tree iff \(|T| = \aleph_2\), \(|T^\alpha| \leq \aleph_1\) for all \(\alpha < \omega_2\), and \(T\) has no branches of cardinality \(\aleph_2\). An \(\omega_2\)-Aronszajn tree \(T\) is special if there is a mapping \(\pi : T \to \{f \in \bigcup \{\omega_1 : \alpha < \omega_2\} : f\) is one-to-one\} such that \(s \leq_T t\) iff \(\pi(s) \leq \pi(t)\). We assume the reader is familiar with the theory of weakly compact cardinals. See [2].
**Theorem 6.4.** (i) Suppose $\kappa$ is weakly compact. Then $\models_{\kappa}$ there are no $\omega_2$-Aronszajn trees.

(ii) Suppose $\kappa$ is Mahlo. Then $\models_{\kappa}$ there are no special $\omega_2$-Aronszajn trees.

**Proof.** We proceed as in Mitchell [6]. It is easy to see that if $(T, \preceq_T)$ is an $\omega_2$-Aronszajn tree, then we may assume that $T = \omega_2, \forall \alpha \in \omega_2 \ T^\alpha = \omega_1 \cdot \alpha$ (the ordinal product), and $\forall \alpha, \beta \in T$ if $\alpha \preceq_T \beta$, then $\alpha \preceq \beta$. Accordingly, suppose by way of contradiction that $p \models_{\kappa} \dot{T} = (\kappa, \preceq_T)$ is an $\omega_2$-Aronszajn tree, $\forall \alpha \in \kappa \ \dot{T}^\alpha = \omega_1 \cdot \alpha$, and $\forall \alpha, \beta \in \kappa$ if

$$\alpha \preceq_T \beta, \text{ then } \alpha \preceq \beta.$$

Since $P_\kappa$ has the $\kappa$-chain condition there is a function $\pi : \kappa \rightarrow \kappa$ such that for each $\alpha$, if $q \supseteq p$, $\beta < \alpha$, and $q \models_{\kappa} \beta \preceq_T \alpha$, then $q \restriction \pi(\alpha) \models_{\kappa} \beta \preceq_T \alpha$. Let $C = \{\alpha < \kappa : \omega_1 \cdot \alpha = \alpha$ and $\forall \beta < \alpha \pi(\beta) < \alpha\}$. Then $C$ is closed and unbounded, and if $\alpha \in C$, then clearly $p \models_{\kappa} \dot{T}^\alpha \in V[\dot{G}_\alpha]$ where $\dot{G}_\alpha$ is a name for $G_\alpha$.

Let $R = \{(q, \alpha, \beta) : q \supseteq p$ and $q \models_{\kappa} \alpha \preceq_T \beta\}$. Let $V_\kappa$ denote the set of all sets of rank $< \kappa$, and let $\varphi$ denote the assertion $p \models_{\kappa}$ every cofinal subset of $\kappa$ fails to be a branch through $(\kappa, \preceq_T)$.

Then $\varphi$ may be written as a $\Pi^1_1$-assertion over the structure

$$(V_\kappa, \in, P_\kappa, \geq, R, p)$$

as follows: $(\forall q \supseteq p)(\forall f \in V_\kappa) \text{ if } f \text{ is a function on } \kappa, (\forall \alpha \in \kappa) (f(\alpha) \text{ is an antichain in } P_\kappa \text{ and } (\forall q' \supseteq q)(\exists \beta > \alpha)(\exists r \in f(\beta))$ $q'$ and $r$ are compatible, then $(\forall q' \supseteq q)(\exists \alpha, \beta \in \kappa)(\exists s \in f(\alpha))(\exists t \in f(\beta)) \alpha < \beta, r \supseteq s, t \text{ and } \forall q''$ (if $(q'', \alpha, \beta) \in R$ then $q''$ and $r$ are incompatible).

Note that the function $f$ above determines a set $X \subseteq \kappa$ in $V[G_\kappa]$ by letting $X = \{\alpha : G_\kappa \cap f(\alpha) \neq \emptyset\}$. With this in mind, it should be easy to see that the assertion above says that $p \models_{\kappa}$ if $q$ forces $X$ to be cofinal in $\kappa$, then $q$ forces $X$ not to be a branch through $(\kappa, \preceq_T)$. Furthermore, any $X \subseteq \kappa$ in $V[G_\kappa]$ may be represented by such a function $f \in V$. If $X$ is a term denoting $X$ then simply let $f(\alpha)$ be a maximal incompatible subset of $\{q \in P_\kappa : q \models_{\kappa} \alpha \in X\}$. Since $P_\kappa$ has the $\kappa$-chain condition, $f(\alpha) \in V_\kappa$. (Note also that we assume $P_\kappa \subseteq V_\kappa$; whether or not this is literally true depends on the definition of the notion of a term of the language of forcing. Certainly this can be arranged. If the reader is working with a definition of forcing which makes this literally false then simply replace $P_\kappa$ by an appropriate isomorph.)

Since $\kappa$ is weakly compact, there is a strongly inaccessible cardinal $\lambda \in C$ such that

$$(V_\lambda, \in, P_\kappa \cap V_\lambda, \geq \cap V_\lambda, R \cap V_\lambda, p) \models \varphi.$$
But that means

\[ p \Vdash_\lambda \text{ every cofinal subset of } \lambda \text{ fails to be a branch through } (\lambda, \leq_T | \lambda). \]

Hence

\[ p \Vdash \dot{T}^\lambda \in V[\dot{\mathcal{G}}] \text{ and in } V[\dot{\mathcal{G}}], \dot{T}^\lambda \text{ has no branches cofinal in } \lambda(= \omega_2). \]

By Lemma 6.2,

\[ p \Vdash_\kappa \text{ every maximal branch through } \dot{T}^\lambda \text{ lies in } V[\dot{\mathcal{G}}]. \]

But this is a contradiction, since

\[ p \Vdash_\kappa \forall \alpha \in \dot{T}^{\lambda+1} - \dot{T}^\lambda, \text{ then } \{\beta : \beta \leq_T \alpha\} \text{ is a maximal branch through } \dot{T}^\lambda \text{ cofinal in } \lambda. \]

The second part of the theorem has a similar proof, which is left to the reader.

Added in proof

Shelah has recently shown that it is relatively consistent that \(2^{\aleph_0} > \aleph_1\) and perfect-set forcing preserves cardinals.

References