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Self-Affine Tiles in $\mathbb{R}^n$

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A *self-affine tile* in $\mathbb{R}^n$ is a set T of positive measure with $\mathsf{A}(T) = \bigcup_{\mathbf{d} \in \mathscr{D}} (T + \mathbf{d})$, where A is an expanding $n \times n$ real matrix with $|\det(\mathsf{A})| = m$ an integer, and $\mathscr{D} = \{\mathbf{d}, \mathbf{d}_2, ..., \mathbf{d}_m\} \subseteq \mathbb{R}^n$ is a set of m digits. It is known that self-affine tiles always give tilings of $\mathbb{R}^n$ by translation. This paper extends known characterizations of digit sets $\mathscr{D}$ yielding self-affine tiles. It proves several results about the structure of tilings of $\mathbb{R}^n$ possible using such tiles, and gives examples showing the possible relations between self-replicating tilings and general tilings, which clarify results of Kenyon on self-replicating tilings.

1. INTRODUCTION

A *self-affine* tile is a compact set T in $\mathbb{R}^n$ of positive Lebesgue measure for which there is an expanding matrix A such that the affinely inflated copy $\mathsf{A}(T)$ of T can be perfectly tiled with essentially disjoint translates of T. Here an $n \times n$ real matrix A is *expanding* if all of its eigenvalues satisfy $|\lambda_i| > 1$. In other words T satisfies a set-valued functional equation

$$\mathsf{A}(T) = \bigcup_{i=1}^{m} (T + \mathbf{d}_i), \tag{1.1}$$

in which $\mathscr{D} = \{\mathbf{d}_1, \mathbf{d}_2, ..., \mathbf{d}_m\}$ is a set of vectors in $\mathbb{R}^n$, which we call a *digit set*, and "essentially disjoint" means that the measure of $(T + \mathbf{d}_i) \cap (T + \mathbf{d}_j)$ is zero when $i \neq j$. A necessary condition for the set T to have positive measure together with the "essentially disjoint" property is that $|\det(\mathsf{A})| = |\mathscr{D}| = m$.

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More generally, for any expanding matrix A, and any finite set $\mathscr{D}$ in $\mathbb{R}^n$ the functional equation (1.1) determines a unique compact set T, given by

$$T := \left\{ \sum_{j=1}^{\infty} \mathsf{A}^{-j} \mathbf{d}_{i_j} : \text{each } \mathbf{d}_{i_j} \in \mathscr{D} \right\}. \tag{1.2}$$

Uniqueness does not hold in the converse direction. In fact, any self-affine tile T arises from infinitely many different pairs $(\tilde{\mathsf{A}}, \tilde{\mathscr{D}})$.

Self-affine tiles arise in many contexts, including radix expansions ([12], [26], [27], [38]), the construction of multidimensional wavelet bases having compact support ([8], [14], [15], [35], [36]), and the construction of Markov partitions ([5], [28]). They also have been studied directly as objects giving interesting tilings of $\mathbb{R}^n$ ([1], [2], [9], [10], [11], [16], [19], [37]). Kenyon [19] gives general results on self-affine tiles, in a study of self-replicating tilings (defined in § 2). Many authors have considered self-affine properties for sets of tiles of several different shapes, for the construction of sets of tiles which only tile $\mathbb{R}^n$ aperiodically, see Grünbaum and Shepard [16] and Kenyon [19] for references. A *self-similar tile* is a special kind of self-affine tile for which the matrix A is a similarity, i.e., $\mathsf{A} = \lambda \mathsf{Q}$ where $\lambda > 1$ and Q is an orthogonal matrix. Self-similar tiles are somewhat easier to analyze than general self-affine tiles, and a number of results have been proved specific to them, see [33], [34], [36].

This paper presents three theorems about the existence, structure and tiling properties of general self-affine tiles. The first theorem gives conditions characterizing when a pair $(\mathsf{A}, \mathscr{D})$ gives a self-affine tile, generalizing known conditions, and also asserts that such tiles are set-theoretically rather nice objects. The second theorem reproves the well-known fact that every self-affine tile $T(\mathsf{A}, \mathscr{D})$ gives a tiling of $\mathbb{R}^n$ by translation, and adds the extra information that there is a tiling whose translations form a subset of a set $\Delta(\mathsf{A}, \mathscr{D})$ defined in § 2. It also shows that every self-affine tile T can be used as a prototile for a self-replicating tiling of $\mathbb{R}^n$ in the sense of Kenyon [19]. The third theorem adds a converse to Kenyon's rigidity theorem concerning quasiperiodic self-replicating tilings. The theorems are stated in § 2 and proved in § 3. We also present several examples indicating limits of the results. The examples are stated in § 2 and proofs of their properties are given in § 4.

Our results sharpen and complement some of the fundamental results of Kenyon [19] and Vince [38]. Our focus differs from Kenyon's in that we treat the self-affine tile $T = T(\mathsf{A}, \mathscr{D})$ as the fundamental object, and study all possible tilings by T, while Kenyon is concerned only with self-replicating tilings. Vince [38] studies the related question of when a tile $T(\mathsf{A}, \mathscr{D})$ has a self-replicating tiling that is a lattice tiling. It turns out that self-affine tiles give many different tilings of $\mathbb{R}^n$, including some that are not self-replicating tilings. In particular we show that there are self-affine tiles that have no self-replicating tiling that is a periodic tiling, but which have a non-self-replicating tiling that is a lattice tiling, see Example 2.3. This example shows that Theorem 12 of Kenyon [19] is not correct.

There remain many open questions about tilings with self-affine tiles. We state several of these in § 5. An interesting problem no addressed here concerns when a self-affine tile is connected or is a topological disk. Work on these questions appears in Bandt and Gelbrich [2] and Gröchenig and Haas [14].

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2. Statements of Results

The set-valued functional equation (1.1) for a self-affine tile has an equivalent form

$$T=\bigcup_{i=1}^{m} \mathsf{A}^{-1}(T+\mathbf{d}_i). \tag{2.1}$$

This equation makes sense more generally for any expanding matrix $\mathsf{A}\in M_n(\mathbb{R})$, not necessarily having an integer determinant, and for any finite set $\mathscr{D}=(\mathbf{d}_1, ..., \mathbf{d}_l\}$ in $\mathbb{R}^n$. The maps

$$\phi_i(\mathbf{x})=\mathsf{A}^{-1}(\mathbf{x}+\mathbf{d}_i), \qquad 1\leqslant i\leqslant l,$$

are all contractions in an appropriate metric on the space of all compact subsets of $\mathbb{R}^n$, as shown in § 3. Results of Hutchinson [17] then imply that there is a unique compact set $T(\mathsf{A}, \mathscr{D})$ satisfying (2.1). The collection $\{\phi_i\colon 1\leqslant i\leqslant l\}$ form a *hyperbolic affine iterated function system* in the terminology of Barnsley [3], and $T(\mathsf{A}, \mathscr{D})$ is its *attractor*. The attractor is explicitly given by

$$T(\mathsf{A}, \mathscr{D})=\left\{\sum_{j=1}^{\infty} \mathsf{A}^{-j}\mathbf{d}_{i_j} : \text{all } \mathbf{d}_{i_j}\in\mathscr{D}\right\}. \tag{2.2}$$

However only for special data $(\mathsf{A}, \mathscr{D})$ will $T(\mathbf{A}, \mathscr{D})$ have positive Lebesgue measure.

In what follows we restrict to the case that $\mathsf{A}\in M_n(\mathbb{R})$ with $|\det(\mathsf{A})|=m$, an integer greater than one, and to digit sets $\mathscr{D}=\{\mathbf{d}_1, \mathbf{d}_2, ..., \mathbf{d}_m\}$ containing exactly m digits. Furthermore in studying attractors $T(\mathsf{A}, \mathscr{D})$, we can always reduce to the case that $\mathbf{0} \in \mathscr{D}$, using the fact that

$$T(\mathsf{A}, \mathscr{D} + \mathbf{x}) = T(\mathsf{A}, \mathscr{D}) + \left(\sum_{j=1}^{\infty} \mathsf{A}^{-j} \right) \mathbf{x}. \tag{2.3}$$

That is, translating all the digits by $\mathbf{x}$ just translates the set $T(\mathsf{A}, \mathscr{D})$ by $(\mathsf{A}\text{-}\mathsf{I})^{-1}\mathbf{x}$.

The first theorem is a criterion for $(\mathsf{A}, \mathscr{D})$ to give a self-affine tile. Associate to $(\mathsf{A}, \mathscr{D})$ the sets

$$\mathscr{D}_{\mathsf{A}, k} = \left\{ \sum_{j=0}^{k-1} \mathsf{A}^j \mathbf{d}_{i_j} : \text{all } \mathbf{d}_{i_j} \in \mathscr{D} \right\} \tag{2.4}$$

$$\mathscr{D}_{\mathsf{A}, \infty} = \bigcup_{k=1}^{\infty} \mathscr{D}_{\mathsf{A}, k}. \tag{2.5}$$

Note that $\mathbf{0} \in \mathscr{D}$ implies $\mathscr{D}_{\mathsf{A}, k} \subseteq \mathscr{D}_{\mathsf{A}, k+1}$ for all $k \geqslant 1$. We say that a set $\chi \subseteq \mathbb{R}^n$ is *uniformly discrete* if there exists $\delta > 0$ such that $\mathbf{x}, \mathbf{x}' \in \chi$ implies $\|\mathbf{x}\text{-}\mathbf{x}'\| > \delta$.

THEOREM 1.1 (Interior Theorem). *Let $\mathsf{A} \in M_n(\mathbb{R})$ be an expanding matrix such that $|\det(\mathsf{A})| = m$ is an integer and let $\mathscr{D} \subseteq \mathbb{R}^n$ have cardinality m, and suppose that $\mathbf{0} \in \mathscr{D}$. The following four conditions are equivalent.*

(i) *$T(\mathsf{A}, \mathscr{D})$ has positive Lebesgue measure.*

(ii) *$T(\mathsf{A}, \mathscr{D})$ has nonempty interior.*

(iii) *$T(\mathsf{A}, \mathscr{D})$ is the closure of its interior T°, and its boundary $\partial T := T - T^\circ$ has Lebesgue measure zero.*

(iv) *For each $k \geqslant 1$, all m^k expansions in $\mathscr{D}_{\mathsf{A}, k}$ are distinct, and $\mathscr{D}_{\mathsf{A}, \infty}$ is a uniformly discrete set.*

The equivalence of (i) and (ii) is due to Kenyon [19], while the other two equivalences are apparently new in this generality. Proving the equivalence of (iv) to (i) uses an idea of Odlyzko [27, Lemma 9]. Conditions (i), (ii) and (iii) are still equivalent when $\mathbf{0} \notin \mathscr{D}$, but (iv) is not.

As a complement to this result, there are self-affine tiles consisting of an infinite number of disconnected pieces, even in one dimension. Figure 2.1 exhibits a two-dimensional example, similar to some pictured in Gröchenig and Madych [15].

The criterion (iv) of Theorem 1.1 yields the following corollary, recovering results of Kenyon [19] and Bandt [1].

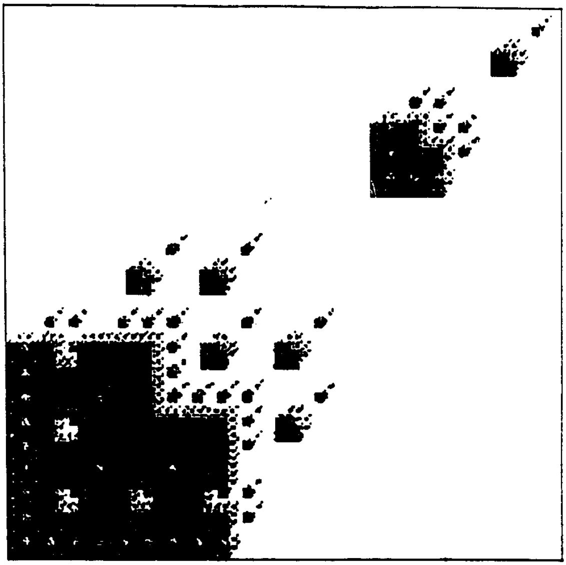

Fig. 2.1. $T(\mathsf{A}, \mathscr{D})$ for $\mathsf{A} = \left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right]$ and $\mathscr{D} = \left\{ \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 5 \\ 5 \end{smallmatrix}\right] \right\}$.

COROLLARY 1.1. *Suppose that* $\mathsf{A} \in M_n(\mathbb{Z})$ *and that* $\mathscr{D} \subseteq \mathbb{Z}^n$ *with* $|\mathscr{D}| = |\det(\mathsf{A})| = m$. *Then* $T(\mathsf{A}, \mathscr{D})$ *contains an open set if and only if* $\mathscr{D}_{\mathsf{A},k}$ *contains* m^k *distinct elements for all* $k \geqslant 1$. *This condition holds if* $\mathscr{D}$ *forms a complete residue system of* $\mathbb{Z}^n/\mathsf{A}(\mathbb{Z}^n)$.

The second theorem concerns tilings of $\mathbb{R}^n$ by self-affine tiles $T(\mathsf{A}, \mathscr{D})$. What is the nature of the tilings of $\mathbb{R}^n$ obtainable using self-affine tiles? Let $\mathscr{S}$ denote a set of translations such that $T + \mathscr{S}$ is a tiling of $\mathbb{R}^n$. We say that $\mathscr{S}$ is a *lattice tiling* if $\mathscr{S}$ is a lattice Λ in $\mathbb{R}^n$. We say that $\mathscr{S}$ is a *periodic tiling* if it is invariant under n linearly independent translations, i.e. $\mathscr{S}$ is a union of finitely many cosets of a lattice Λ, and it is *non-periodic* otherwise. More generally, $\mathscr{S}$ is a j/n-periodic tiling if the lattice of translations leaving $\mathscr{S}$ invariant is of rank j; in particular a 0-periodic tiling is called *aperiodic*. We say that a tiling $\mathscr{S}$ is a *quasiperiodic tiling*, if the following two conditions hold:

(i) *Local Finiteness Property*. For each integer $k \geqslant 1$ and each positive real r, there are only finitely many translation-inequivalent arrangements of k points in $\mathscr{S}$ which are contained in some ball of radius r.

(ii) *Local Isomorphism Property*.[1] For each "patch" Σ_k of k points in $\mathscr{S}$, there is a constant $R = R(\Sigma_k)$ such that inside every ball of radius R in $\mathbb{R}^n$ the tiling $\mathscr{S}$ contains a translate $\Sigma_k + t$ of Σ_k.

[1] Kenyon [19] calls this property "homogeneity," and a self-replicating tiling having this property he calls "pure." We follow the terminology of Radin and Wolff [30].

Note that lattice tilings are periodic tilings and periodic tilings are quasiperiodic tilings.

A tiling $T+\mathscr{S}$ of $\mathbb{R}^n$ by a tile T is a *self-replicating tiling* (abbreviation: SRT) *with matrix* B if $\mathsf{B} \in M_n(\mathbb{R})$ is expanding, and for each $\mathbf{s} \in \mathscr{S}$ there is a finite subset $J(\mathbf{s}) \subseteq \mathscr{S}$ with

$$\mathsf{B}(T+\mathbf{s}) = \bigcup_{\mathbf{s}' \in J(\mathbf{s})} (T+\mathbf{s}'). \tag{2.6}$$

It follows that a self-replicating tiling is completely determined by the finite set of tiles $\mathscr{C}_0$ in it that touch the origin $\mathbf{0}$, by repeated applications of (2.6). We call a self-replicating tiling *atomic* if $\mathscr{C}_0$ consists of a single tile, otherwise it is *non-atomic*. The concept of self-replicating tiling is due to Kenyon [19], who proves that all the tiles in any self-replicating tiling are necessarily self-affine tiles $T(\mathsf{B}, \mathscr{D})$ for some digit set $\mathscr{D}$. Theorem 1.2 below gives a converse, showing that every self-affine tile $T(\mathsf{A}, \mathscr{D})$ serves as a prototile for some atomic self-replicating tiling, taking $\mathsf{B} = \mathsf{A}^k$, for some sufficiently large k. There is no general inclusion relation between self-replicating tilings and any of the other tiling concepts above, as indicated by examples 2.1-2.3 below.

To state results on the existence of tilings, we need additional notation. To any set χ in $\mathbb{R}^n$ we associate the *difference set*

$$\Delta(\chi) := \chi - \chi = \{\mathbf{x} - \mathbf{x}' : \mathbf{x}, \mathbf{x}' \in \chi\}.$$

We now define the *differenced radix expansion set*

$$\begin{aligned} \Delta(\mathsf{A}, \mathscr{D}) &:= \bigcup_{k=1}^{\infty} \Delta(\mathscr{D}_{\mathsf{A}, k}) \\ &= \bigcup_{k=1}^{\infty} (\mathscr{D}_{\mathsf{A}, k} - \mathscr{D}_{\mathsf{A}, k}). \end{aligned} \tag{2.7}$$

It is clear that

$$\Delta(\mathsf{A}, \mathscr{D} + \mathbf{y}) = \Delta(\mathsf{A}, \mathscr{D}), \qquad \text{all} \quad \mathbf{y} \in \mathbb{R}^n,$$

and if $\mathbf{0} \in \mathscr{D}$ then $\Delta(\mathsf{A}, \mathscr{D}) = \Delta(\mathscr{D}_{\mathsf{A}, \infty})$.

THEOREM 1.2 (Tiling Theorem). *Let* $\mathsf{A} \in M_n(\mathbb{R})$ *be an expanding matrix and* $\mathscr{D}$ *a digit set with* $|\mathscr{D}| = |\det(\mathsf{A})|$, *and suppose that* $T(\mathsf{A}, \mathscr{D})$ *contains an open set. Then*:

(i) *There exists a set of translations* $\mathscr{S} \subseteq \Delta(\mathsf{A}, \mathscr{D})$ *such that* $T(\mathsf{A}, \mathscr{D}) + \mathscr{S}$ *tiles* $\mathbb{R}^n$. *Furthermore there is a translate* $\mathscr{S} + \mathbf{x}$ *of one such* *tiling that is an atomic self-replicating tiling of* $\mathbb{R}^n$ *with matrix* $\mathsf{B} = \mathsf{A}^k$ *for some sufficiently large* $k \geqslant 1$.

(ii) *If* $\Delta(\mathsf{A}, \mathscr{D})$ *is a lattice, then* $\Delta(\mathsf{A}, \mathscr{D})$ *is a tiling set for* $T(\mathsf{A}, \mathscr{D})$.

Concerning (i), it seems plausible that any self-affine tile $T(\mathsf{A}, \mathscr{D})$ actually admits a self-replicating tiling with matrix A. There are however examples of such tiles having no atomic SRT with matrix A, cf. Example 2.2.

Concerning (ii), we note that the set $\Delta := \Delta(\mathsf{A}, \mathscr{D})$ is always A-*invariant* in the sense that $\mathsf{A}(\Delta) \subseteq \Delta$. Consequently a necessary condition for $\Delta(\mathsf{A}, \mathscr{D})$ to be a lattice is that the *differenced digit set*

$$\Delta(\mathscr{D}) := \mathscr{D} - \mathscr{D} = \mathscr{D}_{\mathsf{A},1} - \mathscr{D}_{\mathsf{A},1} \subseteq \Delta(\mathsf{A}, \mathscr{D}), \tag{2.8}$$

be contained in an A-invariant lattice. We call $T(\mathsf{A}, \mathscr{D})$ a *lattice self-affine tile* if $\Delta(\mathscr{D})$ is contained in some A-invariant lattice. If so, let $\mathbb{Z}(\mathsf{A}, \mathscr{D})$ denote the smallest A-invariant lattice containing $\Delta(\mathscr{D})$, which is

$$\mathbb{Z}(\mathsf{A}, \mathscr{D}) := \mathbb{Z}[\Delta(\mathscr{D}), \mathsf{A}(\Delta(\mathscr{D})), \mathsf{A}^2(\Delta(\mathscr{D})), ...]. \tag{2.9}$$

It is clear that

$$\Delta(\mathsf{A}, \mathscr{D}) \subseteq \mathbb{Z}(\mathsf{A}, \mathscr{D}). \tag{2.10}$$

It follows that $\Delta(\mathsf{A}, \mathscr{D})$ can be a lattice if and only if $\Delta(\mathsf{A}, \mathscr{D}) = \mathbb{Z}(\mathsf{A}, \mathscr{D})$, because $\Delta(\mathsf{A}, \mathscr{D})$ is A-invariant and contains $\Delta(\mathscr{D})$. We have:

COROLLARY 1.2. *A lattice self-affine tile* $T(\mathsf{A}, \mathscr{D})$ *has a lattice tiling of* $\mathbb{R}^n$ *with the lattice* $\mathbb{Z}(\mathsf{A}, \mathscr{D})$ *if and only if* $\Delta(\mathsf{A}, \mathscr{D}) = \mathbb{Z}(\mathsf{A}, \mathscr{D})$.

This corollary follows immediately from Theorem 1.2, for (ii) states that if $\Delta(\mathsf{A}, \mathscr{D}) = \mathbb{Z}(\mathsf{A}, \mathscr{D})$ then $\mathbb{Z}(\mathsf{A}, \mathscr{D})$ is a tiling set, while if $\Delta(\mathsf{A}, \mathscr{D}) \neq \mathbb{Z}(\mathsf{A}, \mathscr{D})$ then (i) and (2.10) produce a tiling $\mathscr{S}$ which is a strict subset of $\mathbb{Z}(\mathsf{A}, \mathscr{D})$.

The case that $T(\mathsf{A}, \mathscr{D})$ has a lattice tiling with $\mathbb{Z}(\mathsf{A}, \mathscr{D})$ is exactly what Vince [38] calls a *replicating tesselation*, and he calls $(\mathsf{A}, \mathscr{D})$ a *rep-tiling pair*. He gives an algorithm which decides for a given pair $(\mathsf{A}, \mathscr{D})$ whether or not $\Delta(\mathsf{A}, \mathscr{D}) = \mathbb{Z}(\mathsf{A}, \mathscr{D})$ holds. Example 2.3 below gives a case where $\Delta(\mathsf{A}, \mathscr{D}) \neq \mathbb{Z}(\mathsf{A}, \mathscr{D})$.

We can reduce lattice self-affine tiles to a simpler form using the invertible linear map $\mathsf{L}\colon \mathbb{R}^n \to \mathbb{R}^n$ that maps the lattice $\mathbb{Z}(\mathsf{A}, \mathscr{D})$ to $\mathbb{Z}^n$. In that case set $\tilde{\mathsf{A}} := \mathsf{L}\mathsf{A}\mathsf{L}^{-1}$ and $\tilde{\mathscr{D}} = \mathsf{L}\mathscr{D}\mathsf{L}^{-1}$. From (2.2) it is easy to see that

$$T(\tilde{\mathsf{A}}, \tilde{\mathscr{D}}) = L(T(\mathsf{A}, \mathscr{D}))\,\mathsf{L}^{-1},$$

hence the measure and tiling properties of $T(\mathsf{A}, \mathscr{D})$ are recoverable from $T(\tilde{\mathsf{A}}, \tilde{\mathscr{D}})$. In particular $T(\tilde{\mathsf{A}}, \tilde{\mathscr{D}})$ is a lattice self-affine tile with $\mathbb{Z}(\tilde{\mathsf{A}}, \tilde{\mathscr{D}}) = \mathbb{Z}^n$.

The only matrices $\tilde{\mathsf{A}}$ having $\mathbb{Z}^n$ as an $\tilde{\mathsf{A}}$-invariant lattice are integer matrices, i.e. $\mathsf{A} \in M_n(\mathbb{Z})$. We call a self-affine tile $T(\mathsf{A}, \mathscr{D})$ with $\mathsf{A} \in M_n(\mathbb{Z})$ and $\mathscr{D} \subseteq \mathbb{Z}^n$ an *integral self-affine tile.*

The difference set $\Delta(\mathsf{A}, \mathscr{D})$ is not always uniformly discrete, see Example 2.1. However, for a lattice self-affine tile, the difference set $\Delta(\mathsf{A}, \mathscr{D})$ is contained in the lattice $\mathbb{Z}(\mathsf{A}, \mathscr{D})$, hence is uniformly discrete.

The third theorem concerns quasiperiodic self-replicating tilings. An important result, due to Kenyon [19], stated below, is that if a self-affine tile $T(\mathsf{A}, \mathscr{D})$ gives a quasiperiodic self-replicating tiling of $\mathbb{R}^n$, then $T(\mathsf{A}, \mathscr{D})$ must be a lattice self-affine tile. By the remarks above, the set of tilings of $\mathbb{R}^n$ obtainable using $T(\mathsf{A}, \mathscr{D})$ are then structurally the same as those using some integral self-affine tile $T(\tilde{\mathsf{A}}, \tilde{\mathscr{D}})$.

THEOREM 1.3 (Rigidity Theorem). (i) *If the self-affine tile $T(\mathsf{A}, \mathscr{D})$ gives a quasiperiodic self-replicating tiling of $\mathbb{R}^n$ for the expanding real matrix A, then A is similar to an integer matrix and the set of vectors $\{\mathsf{A}^k(\mathscr{D} - \mathscr{D}) : k \geqslant 0\}$ generates a full rank A-invariant lattice Λ in $\mathbb{R}^n$, and $T(\mathsf{A}, \mathscr{D})$ is a lattice self-affine tile.*

(ii) *Conversely, if A is similar to an integer matrix and the digit set $\mathscr{D}$ has $|\mathscr{D}| = |\det(\mathsf{A})|$ and $\{\mathsf{A}^k(\mathscr{D} - \mathscr{D}) : k \geqslant 0\}$ generates a full rank A-invariant lattice, and if the Lebesgue measure $\mu(T(\mathsf{A}, \mathscr{D})) > 0$, then $T(\mathsf{A}, \mathscr{D})$ tiles $\mathbb{R}^n$ with a quasiperiodic self-replicating tiling with matrix A^k for some $k \geqslant 1$, which moreover is atomic.*

The difficult part (i) of this theorem is due to Kenyon [19, Theorem 7]. We prove here only the easier converse result (ii). A good deal more can be said about the tilings of $\mathbb{R}^n$ possible using integral self-affine tiles; in particular, a large subclass of them have lattice tilings, cf. Lagarias and Wang [22], [23].

We give four examples that illustrate various possibilities for tilings and self-replicating tilings with self-affine tiles, with proofs deferred to Sect. 4. Examples 2.1 and 2.2 are essentially due to Kenyon [19], but we derive stronger properties of these examples than Kenyon does. Recall that to describe an SRT it suffices to specify the finite set of tiles $\mathscr{C}_0$ that touch $\mathbf{0}$, since the tiling is then uniquely determined by repeated inflation of $\mathscr{C}_0$ using (2.6).

EXAMPLE 2.1. For $\mathsf{A} = \left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right]$ and $\mathscr{D} = \{\left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1+\varepsilon \\ -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} \varepsilon \\ -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1+\varepsilon \\ -1 \end{smallmatrix}\right]\}$ with $\varepsilon = \frac{1}{4}\sqrt{2}$, the tile $T(\mathsf{A}, \mathscr{D})$ is a non-integral self-affine tile in $\mathbb{R}^2$. Then $\mathbf{0} \in T^{\circ}$, hence there is a self-replicating tiling with matrix A, with $\mathscr{C}_0 = T$. This tiling has the local isomorphism property but not the local finiteness property, so it is not a quasiperiodic tiling. There is also a lattice tiling of $\mathbb{R}^2$ using the tile T with lattice $\mathbb{Z}^2$, which however is not an SRT. The set $\Delta(\mathscr{D}_{\mathsf{A}, \infty})$ is not uniformly discrete.

Example 2.2. For $\mathsf{A}=\left[\begin{smallmatrix}2&0\\0&2\end{smallmatrix}\right]$ and $\mathscr{D}=\{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\}$, the tile $T(\mathsf{A}, \mathscr{D})$ is the unit square, so is an integral self-affine tile in $\mathbb{R}^2$. There is a (non-atomic) self-replicating tiling with matrix $\mathsf{B}=\mathsf{A}^2=\left[\begin{smallmatrix}4&0\\0&4\end{smallmatrix}\right]$ using the tile T which is generated by the configuration $\mathscr{C}_0=(T+\left[\begin{smallmatrix}-2/3\\0\end{smallmatrix}\right])\cup(T+\left[\begin{smallmatrix}-1/3\\-1\end{smallmatrix}\right])$. This tiling has the local finiteness property but not the local isomorphism property, so it is not a quasiperiodic tiling. There is also a lattice tiling of $\mathbb{R}^2$ with lattice $\mathbb{Z}^2$, which is a (non-atomic) SRT with matrix A and with $\mathscr{C}_0=T\cup(T+\left[\begin{smallmatrix}-1\\0\end{smallmatrix}\right])\cup(T+\left[\begin{smallmatrix}0\\-1\end{smallmatrix}\right])\cup(T+\left[\begin{smallmatrix}-1\\-1\end{smallmatrix}\right])$. There are no atomic SRT's for T having the inflation matrix A.

Example 2.3. For $\mathsf{A}=\left[\begin{smallmatrix}2&1\\0&2\end{smallmatrix}\right]$ and $\mathscr{D}=\{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}3\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}3\\1\end{smallmatrix}\right]\}$, the tile $T(\mathsf{A}, \mathscr{D})$ is an integral self-affine tile. All self-replicating tilings using this tile have matrix $\mathsf{B}=\mathsf{A}^k$ for some $k\geqslant 1$, and all *self-replicating tilings using T are non-periodic tilings which are $\frac{1}{2}$-periodic*. However, there is also a lattice tiling of $\mathbb{R}^2$ by T, using the lattice $3\mathbb{Z}\oplus\mathbb{Z}$.

Example 2.4. For $\mathsf{A}=[4]$ and $\mathscr{D}=\{0, 1, 8, 9\}$, the tile $T(\mathsf{A}, \mathscr{D})$ is $[0, 1]\cup[2, 3]$, so is an integral self-affine tile in $\mathbb{R}$. There is a periodic tiling of $\mathbb{R}$ using T with period lattice $4\mathbb{Z}$, but there is no lattice tiling of $\mathbb{R}$ using T.

We prove the properties stated above for the first three of these examples in §4, and a proof for example 2.4 appears in Lagarias and Wang [22]. In example 2.3 the digit set $\mathscr{D}$ forms a complete set of coset representatives of $\mathbb{Z}^2/\mathsf{A}(\mathbb{Z}^2)$, and $\mathscr{D}$ is not contained in any A-invariant proper sublattice of $\mathbb{Z}^n$, therefore Theorem 12 of Kenyon [19] is not correct.

3. Proofs of Main Results

In the remainder of this paper $\mu(T)$ denotes the Lebesgue measure of the set T, and $\|\mathbf{x}\|$ denotes the Euclidean norm of a vector $\mathbf{x}$.

Before beginning the proofs, we recall facts concerning the existence of the attractor $T=T(\mathsf{A}, \mathscr{D})$. We first construct a metric on $\mathbb{R}^n$ with respect to which all the maps

$$\phi_i(\mathbf{x})=\mathsf{A}^{-1}(\mathbf{x}+\mathbf{d}_i), \qquad 1\leqslant i\leqslant m, \tag{3.1}$$

are strict contractions, following Lind [25]. Since A is expanding, we may choose a constant ρ with

$$1<\rho<\min|\lambda_i|,$$

where $\{\lambda_i\}$ denote the eigenvalues of A. We define

$$\|\mathbf{x}\|' = \sum_{k=0}^{\infty} \rho^k \, \|\mathsf{A}^{-k}\mathbf{x}\|. \tag{3.2}$$

Since the eigenvalues of $\rho\mathsf{A}^{-1}$ are strictly smaller than 1, this series converges and defines a norm $\|\cdot\|'$ on $\mathbb{R}^n$, and it satisfies

$$\|\mathsf{A}^{-1}\mathbf{x}\|' = \rho^{-1} \sum_{k=1}^{\infty} \rho^k \, \|\mathsf{A}^{-k}\mathbf{x}\| \leqslant \rho^{-1} \, \|\mathbf{x}\|'. \tag{3.3}$$

All the maps $\phi_i(x)$ are strict contractions in $\mathbb{R}^n$ with respect to the metric $d(\mathbf{x}_1, \mathbf{x}_2) = \|\mathbf{x}_1 - \mathbf{x}_2\|'$, with the contractivity factor ρ^{-1}. Thus the mappings $\{\phi_i : 1 \leqslant i \leqslant m\}$ acting on the complete metric space $(\mathbb{R}^n, \|\cdot\|')$ form a (*hyperbolic*) *iterated function* system (IFS) in the terminology of Barnsley [3]. A basic result of [17] is that the set-valued operator

$$\phi(Y) := \bigcup_{i=1}^{m} \mathsf{A}^{-1}(Y + \mathbf{d}_i) \tag{3.4}$$

is also a strict contraction with contractivity factor ρ^{-1} when acting on the metric space $\mathscr{H}(\mathbb{R}^n)$ of all compact subsets of $\mathbb{R}^n$ taken with the Hausdorff metric $d_H(\cdot, \cdot)$ induced from the metric $\|\cdot\|'$ on $\mathbb{R}^n$. It therefore has a unique fixed point $T = T(\mathsf{A}, \mathscr{D})$, which is called the *attractor* of the IFS. Furthermore, starting with any nonempty compact set W the iterates $\phi^{(n)}(W)$ converge to the attractor T in this metric on $\mathscr{H}(\mathbb{R}^n)$, see Barnsley [3, Theorem 1, p. 82]. The set defined by the right side of (2.2) is the attractor T, for it is compact and satisfies the functional equation

$$\mathsf{A}(T) = \bigcup_{i=1}^{m} (T + \mathbf{d}_i), \tag{3.5}$$

hence it also satisfies

$$T = \bigcup_{i=1}^{m} \mathsf{A}^{-1}(T + \mathbf{d}_i), \tag{3.6}$$

so T is a fixed point of (3.4).

Proof of Theorem 1.1. It is immediate that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).

We begin by verifying (ii) $\Rightarrow$ (iii). Suppose that T has nonempty interior, and let $\overline{T^\circ}$ denote the closure of T°. Now (3.6) gives

$$\bigcup_{i=1}^{m} \mathsf{A}^{-1}(T^\circ + \mathbf{d}_i) \subseteq T^\circ,$$

because the left side is an open set contained in T. Taking the closure of both sides yields

$$\bigcup_{i=1}^{m} \mathsf{A}^{-1}(\overline{T^\circ} + \mathbf{d}_i) \subseteq \overline{T^\circ}, \tag{3.7}$$

where we used $\overline{\mathsf{A}^{-1}(T^\circ + \mathbf{d}_i)} = \mathsf{A}^{-1}(\overline{T^\circ} + \mathbf{d}_i)$, since A^{-1} is a homeomorphism. Thus (3.7) says that

$$\phi(\overline{T^\circ}) \subseteq \overline{T^\circ}, \tag{3.8}$$

where ϕ is the operator (3.4) acting on the space $\mathscr{H}(\mathbb{R}^n)$. Now applying the k-fold iterated map $\phi^{(k)}$ yields $\phi^{(k+1)}(\overline{T^\circ}) \subseteq \overline{\phi^{(k)}(T^\circ)}$, whence

$$\phi^{(k+1)}(\overline{T^\circ}) \subseteq \overline{T^\circ}.$$

However the basic fact about hyperbolic IFS's is that the iterates $\phi^{(n)}(W)$ of any compact W converge to the attractor T in the Hausdorff metric. Thus every point of the attractor is a limit point of a sequence in $\phi^{(n)}(\overline{T^\circ})$ whence

$$T \subseteq \overline{T^\circ},$$

and this yields $T = \overline{T^\circ}$ as required.

To prove that $\mu(\partial T) = 0$, we note first that iterating (3.5) gives

$$\mathsf{A}^k(T) = \bigcup_{\mathbf{d} \in \mathscr{D}_{\mathsf{A},\,k}} (T + \mathbf{d}). \tag{3.9}$$

The Lebesgue measure $\mu(\mathsf{A}^k(T))$ satisfies

$$\begin{aligned} \mu(\mathsf{A}^k(T)) &= |\det(\mathsf{A})|^k \, \mu(T) \\ &\leqslant \sum_{d \in \mathscr{D}_{\mathsf{A},\,k}} \mu(T + \mathbf{d}) \\ &= |\det(\mathsf{A})|^k \, \mu(T). \end{aligned} \tag{3.10}$$

If $\mu(T) > 0$, equality can hold in (3.10) only if all of the $|\det \mathsf{A}|^k$ digit sequences $\{\sum_{j=0}^{k-1} \mathsf{A}^j \mathbf{d}_{i_j} \colon \text{all } \mathbf{d}_{i_j} \in \mathscr{D}\}$ in $\mathscr{D}_{\mathsf{A},\,k}$ are distinct, and in addition

$$\mu((T + \mathbf{d}) \cap (T + \mathbf{d}')) = 0 \tag{3.11}$$

whenever $\mathbf{d}, \mathbf{d}'$ are distinct elements of $\mathscr{D}_{\mathsf{A},\,k}$, since

$$\mu\Bigg(\bigcup_{\mathbf{d} \in \mathscr{D}_{\mathsf{A},\,k}} (T + \mathbf{d})\Bigg) \leqslant \Bigg(\sum_{\mathbf{d} \in \mathscr{D}_{\mathsf{A},\,k}} \mu(T + \mathbf{d})\Bigg) - \mu((T + \mathbf{d}) \cap (T + \mathbf{d}')).$$

Since T contains an open set, some $\mathsf{A}^k(T)$ contains an open ball of diameter $2\operatorname{diam}(T)$, hence it necessarily contains an entire copy $T+\mathbf{d}$ in (3.9). Then the boundary $\partial T+\mathbf{d}$ is entirely covered[2] by boundaries $\partial T+\mathbf{d}'$ of the other tiles in $\mathscr{D}_{\mathsf{A},k}$. Hence

$$\mu(\partial T+\mathbf{d}) \leqslant \sum_{\mathbf{d}' \in \mathscr{D}_{\mathsf{A},k}-\{\mathbf{d}\}} \mu((\partial T+\mathbf{d}) \cap (\partial T+\mathbf{d}')) = 0,$$

and $\mu(\partial T) = 0$.

We next prove (i) $\Rightarrow$ (iv). Since $\mu(T) > 0$, the argument above shows that (3.11) holds. Suppose (iv) were false. Then there exists a sequence $\{(\mathbf{f}_l^{(1)}, \mathbf{f}_l^{(2)}): l \geqslant 1\}$ where $\mathbf{f}_l^{(1)}$ and $\mathbf{f}_l^{(2)}$ are distinct elements of some $\mathscr{D}_{\mathsf{A},k_l}$ with

$$\lim_{m \to \infty} \|\mathbf{f}_l^{(1)} - \mathbf{f}_l^{(2)}\| = 0.$$

Now $T(\mathsf{A}, \mathscr{D})$ is closed, hence measurable, and by hypothesis $\mu(T(\mathsf{A}, \mathscr{D})) > 0$. We claim that for all $\mathbf{y} \in \mathbb{R}^n$ sufficiently close to 0

$$\mu(T \cap (T+\mathbf{y})) > 0. \tag{3.12}$$

Indeed the characteristic function χ_T is in $L^1(\mathbb{R})$, hence there is a point $\mathbf{x}^* \in T$ with

$$\lim_{r \to 0} \frac{1}{\mu(B_r(\mathbf{x}^*))} \int_{B_r(\mathbf{x}^*)} \chi_T(\mathbf{y})\, d\mathbf{y} = \chi_T(\mathbf{x}^*) = 1,$$

where $B_r(\mathbf{x}) = \{\mathbf{y}: \|\mathbf{x}-\mathbf{y}\| \leqslant r\}$, cf. Stein [32, p. 5]. That is, $\mathbf{x}^*$ is a Lebesgue point of χ_T, and for each $\varepsilon > 0$ one has

$$\mu(B_r(\mathbf{x}^*) \cap T) \geqslant (1-\varepsilon)\, \mu(B_r(\mathbf{x}^*)) \tag{3.13}$$

for all sufficiently small r. Now for $\|\mathbf{y}\| < \varepsilon'$, the ball $B_{r-\varepsilon'}(\mathbf{x}^*+\mathbf{y}) \subseteq B_r(\mathbf{x}^*)$, hence

$$\begin{aligned} \mu(B_r(\mathbf{x}^*) \cap (T+\mathbf{y})) &\geqslant \mu(B_{r-\varepsilon'}(\mathbf{x}^*+\mathbf{y}) \cap (T+\mathbf{y})) \\ &= \mu(B_{r-\varepsilon'}(\mathbf{x}^*) \cap T) \\ &> (1-\varepsilon) \left(\frac{r-\varepsilon'}{r}\right)^n \mu(B_r(\mathbf{x}^*)) \\ &> (1-\varepsilon'')\, \mu(B_r(\mathbf{x}^*)). \end{aligned}$$

[2] The closed set $\bigcup_{\mathbf{d}' \in \mathscr{D}_{\mathsf{A},k}-\{\mathbf{d}\}} (T+\mathbf{d}')$ covers the open ball minus $T+\mathbf{d}$, so it covers ∂T.

By inclusion-exclusion

$$\mu((T+\mathbf{y})\cap T) \geqslant \mu(B_r(\mathbf{x}^*)\cap T)+\mu(B_r(\mathbf{x}^*)\cap(T+\mathbf{y}))-\mu(B_r(\mathbf{x}^*))$$
$$>(1-\varepsilon-\varepsilon'')\,\mu(B_r(\mathbf{x}^*))>0,$$

proving (3.12).

Applying (3.12) yields

$$\mu((T+\mathbf{f}_l^{(1)})\cap(T+\mathbf{f}_l^{(2)}))=\mu(T\cap(T+\mathbf{f}_l^{(1)}-\mathbf{f}_l^{(2)}))>0$$

for all sufficiently large l, which contradicts (3.11). Thus (i) $\Rightarrow$ (iv).

Next we prove (iv) $\Rightarrow$ (i). Again consider the mapping ϕ given by (3.4), which is a strictly contractive map with factor ρ^{-1} on $(\mathscr{H}(\mathbb{R}^n), d)$. Consider the closed ball in the $\|\cdot\|'$-norm,

$$B'_r=B'_r(\mathbf{0})=\{\mathbf{x}\in\mathbb{R}^n\colon\ \|\mathbf{x}\|'\leqslant r\}, \tag{3.14}$$

where $\|\cdot\|'$ is the norm (3.2). We claim that

$$\phi(B'_r)\subseteq B'_r \tag{3.15}$$

for all sufficiently large r. This follows from (3.3) since $\mathbf{x}\in B'_r$ gives

$$\begin{aligned}\|\mathsf{A}^{-1}(\mathbf{x}+\mathbf{d}_i)\|' &\leqslant \|\mathsf{A}^{-1}\mathbf{x}\|'+\|\mathsf{A}^{-1}\mathbf{d}_i\|'\\ &\leqslant \rho^{-1}\,\|\mathbf{x}\|'+\rho^{-1}\,\|\mathbf{d}_i\|'\\ &\leqslant \rho^{-1}(r+\|\mathbf{d}_i\|')\leqslant r,\end{aligned}$$

provided

$$r\geqslant\frac{\rho}{\rho-1}\max_{1\leqslant i\leqslant m}\,(\|\mathbf{d}_i\|').$$

Now (3.15) yields $\phi^{(k+1)}(B'_r)\subseteq\phi^{(k)}(B'_r)$. However, the sequence $\{\phi^{(n)}(B'_r)\}$ converges to the attractor in the Hausdorff metric, and since it forms a nested sequence of compact sets, we have

$$T(\mathsf{A},\mathscr{D})=\bigcap_{k=1}^{\infty}\phi^{(k)}(B'_r).$$

Consequently (by Lebesgue's dominated convergence theorem)

$$\mu(T(\mathsf{A},\mathscr{D}))=\lim_{k\to\infty}\mu(\phi^{(k)}(\mathsf{B}'_r)). \tag{3.16}$$

The uniform discreteness property of $\mathscr{D}_{\mathsf{A},\infty}$ guarantees that there is a positive constant δ such that for all $\mathbf{f}_1, \mathbf{f}_2 \in \mathscr{D}_{\mathsf{A},\infty}$,

$$\|\mathbf{f}_1 - \mathbf{f}_2\|' > \delta \qquad \text{if} \quad \mathbf{f}_1 \neq \mathbf{f}_2. \tag{3.17}$$

Consider a ball B'_ε with $0 < \varepsilon < \frac{1}{2}\delta$. By (3.4)

$$\phi^{(k)}(Y) := \bigcup_{\mathbf{d} \in \mathscr{D}_{\mathsf{A},k}} \mathsf{A}^{-k}(Y + \mathbf{d}),$$

hence

$$\mathsf{A}^k \phi^{(k)}(B'_\varepsilon) = \bigcup_{\mathbf{d} \in \mathscr{D}_{\mathsf{A},k}} (B'_\varepsilon + \mathbf{d}). \tag{3.18}$$

However, $\mathscr{D}_{\mathsf{A},k}$ has $|\det(\mathsf{A})|^k$ distinct elements by hypothesis (iv), and all sets on the right side of (3.18) are disjoint by (3.17). Thus the measure of both sides satisfies

$$|\det(\mathsf{A})|^k \, \mu(\phi^{(k)}(B'_\varepsilon)) = |\det(\mathsf{A})|^k \mu(B'_\varepsilon).$$

Choose $0 < \varepsilon < r$ such that $B_\varepsilon \subseteq B_r$, and we then have

$$\mu(\phi^{(k)}(B_r)) \geqslant \mu(\phi^{(k)}(B'_\varepsilon)) = \mu(B'_\varepsilon),$$

which with (3.14) yields

$$\mu(T(\mathsf{A}, \mathscr{D})) \geqslant \mu(B'_\varepsilon) > 0,$$

which proves (i).

It remains to prove (i) $\Rightarrow$ (ii). This is essentially a result of Kenyon [19, Theorem 10]. For completeness we give a proof. Now (i) implies that $T(\mathsf{A}, \mathscr{D})$ has a Lebesgue point $\mathbf{x}^*$, i.e. there is a sequence $r_k \to 0$ and $\varepsilon_k \to 0$ with

$$\mu(B_{r_k}(\mathbf{x}^*) \cap T) \geqslant (1 - \varepsilon_k)\, \mu(B_{r_k}(\mathbf{x}^*)). \tag{3.19}$$

We already showed (i) $\Rightarrow$ (iv), hence the set $\mathscr{D}_{\mathsf{A},\infty}$ is uniformly discrete with a constant δ, say.

Claim. *There exist positive constants r_0, c_0 and δ_0 such that for each $m \geqslant 1$ there exists in $\mathbb{R}^n$ a finite set $\mathscr{E}_m \subseteq B_{r_0}(\mathbf{0})$, of cardinality at most c_0, with $\|\mathbf{e} - \mathbf{e}'\| \geqslant \delta_0$ for any distinct $\mathbf{e}, \mathbf{e}' \in \mathscr{E}_m$, such that*

$$\mu(B_1(\mathbf{0}) \cap (T + \mathscr{E}_m)) \geqslant (1 - 5^{n+1}\varepsilon_m)\, \mu(B_1(\mathbf{0})), \tag{3.20}$$

where $B_1(\mathbf{0})$ is the Euclidean unit ball centered at the origin.

To prove the claim, (3.19) gives

$$\mu(\mathsf{A}^l(B_{r_m}(\mathbf{x}^*)\cap T))\geqslant(1-\varepsilon_m)\,\mu(\mathsf{A}^l B_{r_m}(\mathbf{x}^*)),\qquad \text{all}\quad l\geqslant 0. \tag{3.21}$$

We first show that for sufficiently large l, there exists a unit ball $B_1(\mathbf{y})\subseteq\mathsf{A}^l(B_{r_m}(\mathbf{x}^*))$ with

$$\mu(B_1(\mathbf{y})\cap\mathsf{A}^l(T))\geqslant(1-5^{n+1}\varepsilon_m)\,\mu(B_1(\mathbf{0})). \tag{3.22}$$

Indeed, since A is expanding, $\mathsf{A}^l(B_{r_m}(\mathbf{x}^*))$ is an ellipsoid $E_{l,m}$ whose shortest axis goes to infinity as $l\to\infty$. Let $E'_{l,m}$ be the homothetically shrunk ellipsoid with shortest axis decreased in length by 2, so that all points in $E'_{l,m}$ are at distance at least 1 from the boundary of $E_{l,m}$. By a standard covering lemma (Stein [32, p. 9]) applied to $E'_{l,m}$ there is a set $\{B_1(\mathbf{y}')\}$ of disjoint unit balls with centers in $E_{l,m}$ that cover volume at least $5^{-n}\mu(E'_{l,m})$. Also $5^{-n}\mu(E'_{l,m})\geqslant 5^{-n-1}\mu(E_{l,m})$ provided that the shortest axis of $E_{l,m}$ is of length at least $2(n+1)$. All these balls lie inside $E_{l,m}$. Now (3.21) allows at most $\varepsilon_m\mu(\mathsf{A}^l B_{r_m}(\mathbf{x}^*))$ of the volume of $\mathsf{A}^l B_{r_m}(\mathbf{x}^*)$ to be uncovered by $\mathsf{A}^l(B_{r_m}(\mathbf{x}^*)\cap T)$, so even if this entire uncovered volume is distributed into the disjoint balls $\{B_1(\mathbf{y}')\}$, at least one of them must satisfy (3.22).

Next we use the inflation property (3.9) to rewrite (3.22) as

$$\mu\left(B_1(\mathbf{y})\cap\left(\bigcup_{\mathbf{d}\in\mathscr{D}_{\mathsf{A},l}}(T+\mathbf{d})\right)\right)\geqslant(1-5^{n+1}\varepsilon_m)\,\mu(B_1(\mathbf{y})),$$

hence

$$\mu\left(B_1(\mathbf{0})\cap\left(\bigcup_{\mathbf{d}\in\mathscr{D}_{\mathsf{A},l}}(T+\mathbf{d}-\mathbf{y})\right)\right)\geqslant(1-5^{n+1}\varepsilon_m)\,\mu(B_1(\mathbf{y})).$$

This shows that if we choose

$$\mathscr{E}_m=\{\mathbf{e}=\mathbf{d}-\mathbf{y}:\mathbf{d}\in\mathscr{D}_{\mathsf{A},l}\qquad\text{with}\quad(T+\mathbf{d}-\mathbf{y})\cap B_1(\mathbf{0})\neq\varnothing\}.$$

then (3.21) holds. Also $\mathscr{D}_{\mathsf{A},l}\subseteq\mathscr{D}_{\mathsf{A},\infty}$ is uniformly discrete with constant δ, hence

$$\|\mathbf{e}-\mathbf{e}'\|\geqslant\delta$$

for all $\mathbf{e},\mathbf{e}'\in\mathscr{E}_m$. Since T is compact, all possible $\mathbf{e}$ lie inside the ball $B_{r_0}(\mathbf{0})$ with $r_0=1+\max\{\|\mathbf{x}\|:\mathbf{x}\in T\}$. The ball $B_{r_0}(\mathbf{0})$ can be packed with disjoint balls of radius $\frac{1}{2}\delta$ centered at the points of $\mathscr{E}_m$, hence there is an upper bound $c_0=(2r_0/\delta)^n$ on the cardinality of $\mathscr{E}_m$, which proves the claim.

To finish the proof of (i) $\Rightarrow$ (ii), we apply the claim and choose a convergent subsequence of $\{\mathscr{E}_m\}$, call it $\mathscr{E}_{m_k}$, with $m_k \to \infty$ and we set

$$\mathscr{E}^* := \lim_{k \to \infty} \mathscr{E}_{m_k}.$$

Clearly $\mathscr{E}^*$ has cardinality at most c_0. Now

$$\begin{aligned}\mu(B_1(\mathbf{0}) \cap (T + \mathscr{E}^*)) &\geqslant \liminf_{k \to \infty} \mu(B_1(\mathbf{0}) \cap (T + \mathscr{E}_{m_k}))\\ &\geqslant \liminf_{k \to \infty} (1 - 5^{n-1}\varepsilon_{m_k})\, \mu(B_1(\mathbf{0}))\\ &= \mu(B_1(\mathbf{0})).\end{aligned}$$

Since $T + \mathscr{E}^*$ is a closed set, this forces

$$B_1(\mathbf{0}) \cap (T + \mathscr{E}^*) = B_1(\mathbf{0}).$$

Now $T + \mathscr{E}^*$ is a finite union of translates of T, hence at least one of them must have nonempty interior, so $T^\circ \neq \emptyset$. ∎

The method of proof of (i) $\Rightarrow$ (ii) given above generalizes in a straightforward fashion to give the following result.

THEOREM 3.1. *Let* $\mathsf{A} \in M_n(\mathbb{R})$ *be an expanding matrix and let* $\mathscr{D} = \{\mathbf{d}_i : 1 \leqslant i \leqslant l\} \subseteq \mathbb{R}^n$ *be any finite set containing* $\mathbf{0}$. *Suppose that the set* $\mathscr{D}_{\mathsf{A},\infty}$ *is uniformly discrete, but different expansions in* $\mathscr{D}_{\mathsf{A},\infty}$ *are permitted to be equal. If* $T(\mathsf{A}, \mathscr{D})$ *is the attractor of the hyperbolic iterated function system* $\{\phi_j(\mathbf{x}) = A^{-1}(\mathbf{x} + \mathbf{d}_j) : 1 \leqslant j \leqslant l\}$ *then* $\mu(T(\mathsf{A}, \mathscr{D})) > 0$ *implies that* $T(\mathsf{A}, \mathscr{D})$ *has nonempty interior.*

This immediately yields:

COROLLARY 3.1. *If* $\mathsf{A} \in M_n(\mathbb{Z})$ *is an expanding matrix and* $\mathscr{D} \subseteq \mathbb{Z}^n$ *is any finite set, then* $\mu(T(\mathsf{A}, \mathscr{D})) > 0$ *implies that* $T(\mathsf{A}, \mathscr{D})$ *has nonempty interior.*

We reduce to the case $\mathbf{0} \in \mathscr{D}$ and apply Theorem 3.1.

Proof of Theorem 1.2. We may without loss of generality suppose that $\mathbf{0} \in \mathscr{D}$ so $\Delta(\mathsf{A}, \mathscr{D}) = \Delta(\mathscr{D}_{\mathsf{A},\infty})$. Indeed, $\Delta(A, \mathscr{D}) = \Delta(\mathsf{A}, \mathscr{D} + \mathbf{y})$, so if (i), (ii) are proved for any $(\mathsf{A}, \mathscr{D} + \mathbf{y})$ they hold also for $(\mathsf{A}, \mathscr{D})$.

Result (i) is proved by repeatedly applying the inflation property

$$\mathsf{A}^k(T) = \bigcup_{\mathbf{d} \in \mathscr{D}_{\mathsf{A},k}} (T + \mathbf{d}).$$

Since $\mathbf{0} \in \mathscr{D}$ we have $\mathbf{0} \in \mathscr{D}_{\mathsf{A},k} \subseteq \mathscr{D}_{A,k+1}$, and these sets give consistent tilings of larger and larger "patches" $\mathsf{A}^k(T)$ of $\mathbb{R}^n$. Now $\mathbf{0} \in \mathscr{D}$ implies $\mathbf{0} \in T$, and we treat two cases, depending on whether $\mathbf{0}$ is in the interior T° of T or not.

Suppose first that $\mathbf{0} \in T^\circ$. Then, since A is expanding, $\bigcup_{i=1}^{\infty} \mathsf{A}^k(T) = \mathbb{R}^n$, hence T tiles $\mathbb{R}^n$ with the tiling set $\mathscr{S} = \mathscr{D}_{\mathsf{A},\,\infty}$. The criterion (2.6) for a self-replicating tiling with matrix B is equivalent, when $\mathsf{B} = \mathsf{A}$, to

$$\mathsf{A}(\mathscr{S}) + \mathscr{D} \subseteq \mathscr{S}, \tag{3.23}$$

as can be seen using (1.1). It is then immediate that $\mathscr{S} = \mathscr{D}_{\mathsf{A},\,\infty}$ is an atomic self-replicating tiling with matrix A, because $\mathsf{A}(\mathscr{D}_{\mathsf{A},\,k}) + \mathscr{D} = \mathscr{D}_{\mathsf{A},\,k+1}$, whence $\mathsf{A}(\mathscr{D}_{\mathsf{A},\,\infty}) + \mathscr{D} \subseteq \mathscr{D}_{\mathsf{A},\,\infty}$. Finally $\mathscr{D}_{\mathsf{A},\,\infty} \subseteq \Delta(\mathscr{D}_{\mathsf{A},\,\infty})$, because $\mathbf{0} \in \mathscr{D}_{\mathsf{A},\,\infty}$.

We turn to the harder case where $\mathbf{0}$ is on the boundary of T. We use the fact that, for all $k \geqslant 1$,

$$T(\mathsf{A}, \mathscr{D}) = T(\mathsf{A}^k, \mathscr{D}_{\mathsf{A},\,k}), \tag{3.24}$$

which follows from iterating (2.1). The basic idea is that for large enough k, we can find some digit $\mathbf{d}^* \in \mathscr{D}_{\mathsf{A},\,k}$ such that the translated digit set $\mathscr{D}' = \mathscr{D}_{\mathsf{A},\,k} - \mathbf{d}^*$ has the tile

$$\begin{aligned} T(\mathsf{A}^k, \mathscr{D}') &= T(\mathsf{A}^k, \mathscr{D}_{\mathsf{A},\,k}) - \left(\sum_{j=1}^{\infty} \mathsf{A}^{-jk} \right) \mathbf{d}^* \\ &= T(\mathsf{A}, \mathscr{D}) - \left(\sum_{j=1}^{\infty} \mathsf{A}^{-jk} \right) \mathbf{d}^*, \end{aligned} \tag{3.25}$$

which contains $\mathbf{0}$ in its interior. If so, then since $\mathbf{0} \in \mathscr{D}'$, the proof above applies to show that the tile $T(\mathsf{A}^k, \mathscr{D}')$ gives an atomic self-replicating tiling of $\mathbb{R}^n$ with $\mathsf{B} = \mathsf{A}^k$ and $\mathscr{S}' = \mathscr{D}'_{\mathsf{A}^k,\,\infty}$. Thus $T(\mathsf{A}, \mathscr{D})$ also gives an atomic self-replicating tiling of $\mathbb{R}^n$ with matrix A^k and translation set

$$\mathscr{S}'' = \mathscr{S}' + \left(\sum_{j=1}^{\infty} \mathsf{A}^{-jk} \right) \mathbf{d}^*.$$

Furthermore $T(\mathsf{A}, \mathscr{D})$ also tiles $\mathbb{R}^n$ using the translation set $\mathscr{S}'$, and it is easy to check that

$$\mathscr{S}' = \mathscr{D}'_{\mathsf{A}^k,\,\infty} \subseteq \Delta(\mathscr{D}_{\mathsf{A},\,\infty}). \tag{3.26}$$

The tiling $T + \mathscr{S}'$ is not guaranteed to be a self-replicating tiling.

It remains to find a large k and a digit $\mathbf{d}^*$ as above. By Theorem 1.1, T° is nonempty, and since the set

$$\left\{ \sum_{j=1}^{k} \mathsf{A}^{-j}\mathbf{d}_{i_j} : \quad k \geqslant 1 \quad \text{and all} \quad \mathbf{d}_{i_j} \in \mathscr{D} \right\}$$

is dense in T, we can find some point

$$\mathbf{x}^* = \sum_{j=1}^{k_0} \mathsf{A}^{-j}\mathbf{d}_{i_j}, \qquad \text{with} \quad \mathbf{x}^* \in T^\circ. \tag{3.27}$$

Some open ball $B(\mathbf{x}^*, \delta) = \{\mathbf{y} : \|\mathbf{x}^* - \mathbf{y}\| < \delta\}$ is contained in T°, and by (3.25) it suffices to find k and $\mathbf{d}^* \in \mathscr{D}_{\mathsf{A},k}$ so that

$$\left\| \left(\sum_{j=1}^{\infty} \mathsf{A}^{-jk} \right) \mathbf{d}^* - \mathbf{x}^* \right\| < \delta. \tag{3.28}$$

We take $k \geqslant k_0$ and set $\mathbf{d}^* = \mathsf{A}^k \mathbf{x}^*$, so

$$\mathbf{d}^* = \mathsf{A}^k \mathbf{x}^* = \sum_{j=1}^{k_0} \mathsf{A}^{k-j} \mathbf{d}_{i_j} \in \mathscr{D}_{\mathsf{A},k}.$$

Now

$$\left\| \left(\sum_{j=1}^{\infty} A^{-jk} \right) \mathbf{d}^* - \mathbf{x}^* \right\| = \left\| \left(\sum_{j=1}^{\infty} \mathsf{A}^{-jk} \right) \mathbf{x}^* \right\| \leqslant \|\mathbf{x}^*\| \left(\sum_{j=1}^{\infty} \|\mathsf{A}^{-k}\|^j \right),$$

where $\|\mathsf{A}^{-k}\|$ is the Euclidean operator norm. Since A is expanding, $\|\mathsf{A}^{-k}\| \to 0$ as $k \to \infty$, hence (3.28) holds for all large enough k. This proves (i).

To prove (ii), suppose $\mathbf{0} \in \mathscr{D}$ and that $\Delta(\mathscr{D}_{\mathsf{A},\infty})$ is a lattice Λ. Since $\mathsf{A}(\Delta(\mathscr{D}_{\mathsf{A},\infty})) \subseteq \Delta(\mathscr{D}_{\mathsf{A},\infty})$, this gives $\mathsf{A}(\Lambda) \subseteq \Lambda$, and this case can only occur if A is similar to a matrix in $M_n(\mathbb{Z})$. It suffices to show that if $\mathbf{f}$ and $\mathbf{f}'$ are distinct points in $\Delta(\mathscr{D}_{\mathsf{A},\infty})$, then

$$\mu((T + \mathbf{f}) \cap (T + \mathbf{f}')) = 0. \tag{3.29}$$

For if so then all tiles in $\{T + \mathbf{f} : \mathbf{f} \in \Delta(\mathscr{D}_{\mathsf{A},\infty})\}$ have disjoint interiors, while (i) shows that the union of some subset $\mathscr{S}$ of them tiles $\mathbb{R}^n$, and we must have $\mathscr{S} = \Delta(\mathscr{D}_{\mathsf{A},\infty})$. To prove (3.29) we note that it asserts that

$$\mu(T \cap (T + (\mathbf{f}' - \mathbf{f}))) = 0. \tag{3.30}$$

However $\mathbf{f}' - \mathbf{f} \in \Lambda(\mathscr{D}_{\mathsf{A},\infty})$ since $\Lambda(\mathscr{D}_{\mathsf{A},\infty})$ is a lattice. Thus $\mathbf{f}' - \mathbf{f} = \mathbf{d} - \mathbf{d}'$ with $\mathbf{d}, \mathbf{d}' \in \mathscr{D}_{\mathsf{A},\infty}$, so (3.30) is equivalent to

$$\mu((T + \mathbf{d}) \cap (T + \mathbf{d}')) = 0. \tag{3.31}$$

However, $\mathbf{d}, \mathbf{d}' \in \mathscr{D}_{\mathsf{A},k}$ for sufficiently large k, and $\mathbf{d} \neq \mathbf{d}'$ since $\mathbf{f} \neq \mathbf{f}'$, so (3.31) holds by the measure disjointness of all tiles in the subdivision (3.9) of $\mathsf{A}^k(T)$. ∎

Proof of Theorem 1.3. Part (i) is a result of Kenyon [19, Theorem 7]. To show part (ii), we use the self-replicating tiling with matrix A^k constructed in the proof of Theorem 1.2 (ii). This tiling was constructed by inflating a translate $T' = T + \mathbf{u}$ of the original tile $T = T(\mathsf{A}, \mathscr{D})$ such that $T' = T(\mathsf{A}^k, \mathscr{D}')$ contains $\mathbf{0}$ in its interior, and the digit set $\mathscr{D}'$ contains $\mathbf{0}$. We prove that this tiling is quasiperiodic. The assumption that $\mathsf{A}(\Lambda) \subseteq \Lambda$ and $\mathscr{D} \subseteq \Lambda$ implies that $\mathscr{D}' \subseteq \Lambda$ also. In consequence the tile T' tiles with tiling set $\mathscr{S} := \mathscr{D}'_{\mathsf{A}^k,\infty} \subseteq \Lambda$. Consequently any finite "patch" of tiles has all translates at lattice points, so the local finiteness property is satisfied. To verify the local isomorphism property, note that $\bigcup_{j=0}^{\infty} (\mathsf{A}^k)^j (T') = \mathbb{R}^n$ since $\mathbf{0}$ is in the interior of T'. Thus any finite patch $\mathscr{C}$ of tiles lies inside some $\mathsf{A}^{jk}(T')$. However inflation by A^{jk}, shows that the tiles in the tiling $T' + \mathscr{S}$ combine to give a tiling of $\mathbb{R}^n$ with translates of tiles $T'' = \mathsf{A}^{jk}(T')$, and each of these tiles contains a copy of $\mathscr{C}$. Furthermore every ball of radius twice the diameter of T'' in $\mathbb{R}^n$ contains a copy of T'', so the local isomorphism property is established. ∎

Remarks. It seems conceivable that the converse condition (ii) of Theorem 1.3 may remain true with the stronger conclusion that using the tile $T(\mathsf{A}, \mathscr{D})$ there exists a (possibly non-atomic) quasiperiodic SRT with matrix A. If this is so, a nontrivial modification of the proof above is needed to prove it, see Example 2.2.

4. Proofs for Examples

We justify the assertions made about the examples in §2.

Proof for Example 2.1. To show that the tile T is self-affine, we verify property (iv) of Theorem 1.1. The set $\mathscr{D}_{\mathsf{A},\infty}$ consists of certain vectors of the form $\left[\begin{smallmatrix} m_0 + m_1 \varepsilon \\ m_2 \end{smallmatrix}\right]$ with $m_0, m_1, m_2 \in \mathbb{Z}$. Recall that every $l \in \mathbb{Z}$ has a unique finite expansion

$$l = \sum_{j=0}^{k-1} a_j 3^j, \qquad a_j \in \{-1, 0, 1\}; \tag{4.1}$$

this is called the *balanced ternary expansion* of m, cf. Knuth [21, p. 190]. It is easy to see that the 3^{2k} elements of $\mathscr{D}_{\mathsf{A},k}$ are distinct, in which m_0 and m_2 can be arbitrary integers of the form (4.1), while m_1 is completely determined from the expansion (4.1) of m_2 by

$$m_1 = \sum_{a_j=-1} 3^j, \qquad \text{where} \quad m_2 = \sum_{j=0}^{k} a_j 3^j. \tag{4.2}$$

We show that $\mathscr{D}_{\mathsf{A},\infty}$ is uniformly discrete with $\delta = 1$. To see this, note that the centers of two elements of $\mathscr{D}_{\mathsf{A},k}$ can be at Euclidean distance less than 1 only if they have the same value of m_2. Then they must have the same value of m_1 by (4.2), and since their values of m_0 differ they are at distance at least 1. Thus (iv) is verified.

Using the formula for $\mathscr{D}_{\mathsf{A},\infty}$ it is easy to check that

$$\Delta(\mathscr{D}_{\mathsf{A},\infty}) = \left\{ \begin{bmatrix} m_0 + m_1 \varepsilon \\ m_2 \end{bmatrix} : m_0, m_1, m_2 \in \mathbb{Z} \right\}.$$

This set is not uniformly discrete since $\varepsilon = \frac{1}{4}\sqrt{2}$ is irrational.

We next verify that $\mathbf{0} \in T^\circ$, by showing that T includes all points (x, y) with $|x|, |y| \leqslant \frac{1}{4}$. The set of balanced ternary expansions

$$y = \sum_{j=1}^{\infty} b_j 3^{-j}, \qquad b_j \in \{-1, 0, 1\}$$

covers $[-\frac{1}{2}, \frac{1}{2}]$. Hence y with $|y| \leqslant \frac{1}{4}$ has such an expansion, and associated to it is a unique value

$$\tilde{x} := \frac{1}{4}\sqrt{2} \left(\sum_{b_j=-1} 3^{-j} \right),$$

for which

$$0 \leqslant \tilde{x} \leqslant \frac{1}{4}\sqrt{2} \left(\sum_{j=1}^{\infty} 3^{-j} \right) < \frac{1}{4}.$$

For any x with $|x| \leqslant \frac{1}{4}$ set $\tilde{x}^* := x - \tilde{x}$, so $|\tilde{x}| \leqslant \frac{1}{2}$, and $\tilde{x}^*$ has a balanced ternary expansion $\tilde{x}^* = \sum_{j=1}^{\infty} a_j 3^{-j}$. Then the sequence of digits

$$\mathbf{d}_j = \begin{bmatrix} a_j + \hat{b}_j \varepsilon \\ b_j \end{bmatrix} \in \mathscr{D} \tag{4.3}$$

with $\hat{b}_j = 1$ if $b_j = -1$ and $\hat{b}_j = 0$ otherwise, has

$$\begin{bmatrix} x \\ y \end{bmatrix} = \sum_{j=1}^{\infty} \mathsf{A}^{-j} \mathbf{d}_j, \tag{4.4}$$

as required. The tile T is pictured in Figure 4.1.

Since $\mathbf{0} \in T^\circ$ and $\mathbf{0} \in \mathscr{D}$ the choice $\mathscr{C}_0 = T(\mathsf{A}, \mathscr{D})$ generates a SRT with tiling set $\mathscr{S} = \mathscr{D}_{\mathsf{A}, \infty}$ (by the proof of Theorem 1.2). The description of $\mathscr{D}_{\mathsf{A}, \infty}$ shows that this tiling is invariant under the translation $[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]$. We claim that it is a $\frac{1}{2}$-periodic tiling. If it were a periodic tiling, then it would automatically satisfy the local finiteness property, and we now show it does not. Compare all tiles with centers at $m_2 = \frac{1}{2}(3^{k-1} - 1)$ and $m_2' = \frac{1}{2}(3^{k-1} + 1)$. The associated values of m_1 are $m_1 = 0$ and $m_1' = \frac{1}{2}(3^{k-1} - 1)$. These tiles lie in two strips parallel to the x-axis invariant under the translation $[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}]$, hence there are tiles from the two strips with center distance less than 2 and offset by $(m_1'/4)\varepsilon \pmod 1$ in the x-direction. Since $\varepsilon = \frac{1}{4}\sqrt{2}$ is irrational, all values $\{l\varepsilon \pmod 1) : l \in \mathbb{Z}\}$ are distinct, hence we have produced infinitely many different local neighborhoods with radius $R = 2$, which violates the local finiteness property.

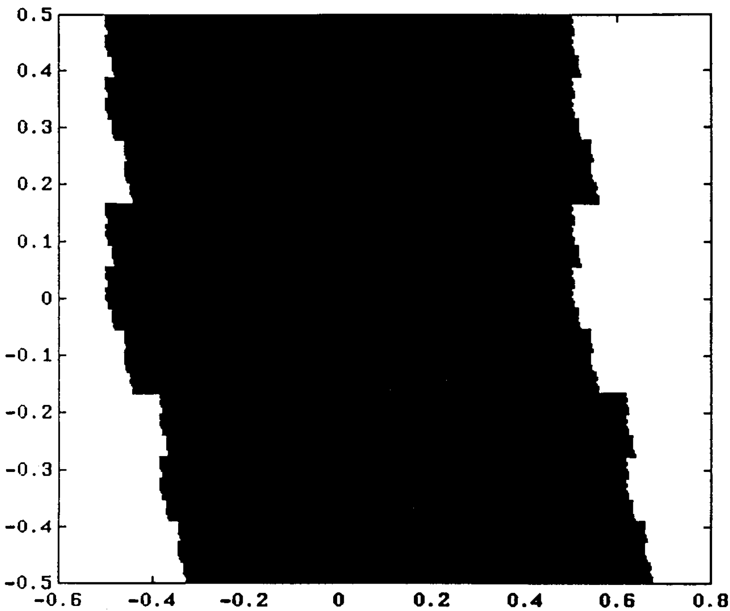

Fig. 4.1. $T(\mathsf{A}, \mathscr{D})$ for $\mathsf{A} = [\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}]$ and $\mathscr{D} = \{[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}], [\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}], [\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}], [\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}], [\begin{smallmatrix} -1+\varepsilon \\ -1 \end{smallmatrix}], [\begin{smallmatrix} \varepsilon \\ -1 \end{smallmatrix}], [\begin{smallmatrix} 1+\varepsilon \\ -1 \end{smallmatrix}]\}$ with $\varepsilon = \frac{1}{4}\sqrt{2}$.

Finally, the explicit form of expansions (4.4) for elements in $T(\mathsf{A}, \mathscr{D})$ allows one to show that the tile T has flat horizontal top and bottom boundaries at $y = -\frac{1}{2}$ and $y = \frac{1}{2}$ respectively, and that the SRT above tiles all the horizontal strips $\{y\colon m - \frac{1}{2} \leqslant y \leqslant m + \frac{1}{2}\}$ for $m \in \mathbb{Z}$. It is periodic in each strip under the translation $\begin{bmatrix}0\\1\end{bmatrix}$. We can slide strips horizontally to obtain a lattice tiling with lattice $\Lambda = \mathbb{Z}^2$. ▮

Proof for Example 2.2. It is clear that the tile $T(\mathsf{A}, \mathscr{D}) = [0, 1] \times [0, 1]$, so is self-affine. Certainly $\mathscr{C}_0 = (T + \begin{bmatrix}-2/3\\0\end{bmatrix}) \cup (T + \begin{bmatrix}1/3\\-1\end{bmatrix})$ includes $\mathbf{0}$ in its interior, and to show that it gives a SRT with matrix $\mathsf{B} = \begin{bmatrix}4&0\\0&4\end{bmatrix}$ it suffices to check that the original tiles $T + \begin{bmatrix}-2/3\\0\end{bmatrix}$ and $T + \begin{bmatrix}1/3\\-1\end{bmatrix}$ appear in the inflated tiling $\mathsf{B}(\mathscr{C}_0)$; see Fig. 4.2.

The resulting SRT clearly tiles the upper and lower half-planes each with a checkerboard tiling, and these two tilings are displaced by $\begin{bmatrix}1/3\\0\end{bmatrix}$ along the x-axis. Thus tiles touching the line $y = 0$ have different local neighborhoods than tiles elsewhere (taking radius $r = 2$), so the local isomorphism property does not hold for this SRT.

The existence of a non-atomic $\mathbb{Z}^2$ lattice SRT using T is obvious. Finally, there can be no atomic SRT by T with the inflation matrix A. For, if there were, it would have $\mathscr{C}_0 := T' = T + \begin{bmatrix}x\\y\end{bmatrix}$, with $-1 < y < 0$. Then the inflated tile $\mathsf{A}(T')$ would contain four translates of T, which have y-coordinates $2y$ and $2y + 1$, so that T' is not a sub-tile of $\mathsf{A}(T')$, a contradiction. ▮

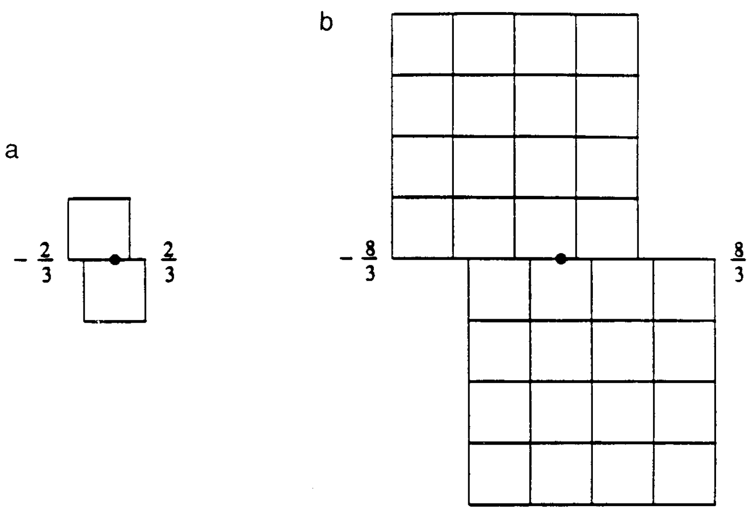

Fig. 4.2. A non-atomic SRT using the unit square with $\mathsf{B} = \begin{bmatrix}4&0\\0&4\end{bmatrix}$. (a) Region $\mathscr{C}_0$. (b) Inflated region $\mathsf{B}(\mathscr{C}_0)$.

Proof for Example 2.3. Since the digit set $\mathscr{D}$ is a complete set of representatives of $\mathbb{Z}^2/\mathsf{A}(\mathbb{Z}^2)$, $T(\mathsf{A}, \mathscr{D})$ is a self-affine tile by Corollary 1.1 or Bandt [1]. It is pictured in Fig. 4.3.

Now, since $\mathsf{A}^{-j} = \left[\begin{smallmatrix} 2^{-j} & -j2^{-j-1} \\ 0 & 2^{-j} \end{smallmatrix}\right]$ for $j \geqslant 1$, we have $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in T(\mathsf{A}, \mathscr{D})$ if

$$\begin{bmatrix} x \\ y \end{bmatrix} = \sum_{j=1}^{\infty} \mathsf{A}^{-j} \begin{bmatrix} 3d_{j,1} \\ d_{j,2} \end{bmatrix}, \tag{4.5}$$

where all $d_{j,1}, d_{j,2} \in \{0, 1\}$ may be chosen arbitrarily, whence

$$y = \sum_{j=0}^{\infty} 2^{-j} d_{j,2}, \tag{4.6}$$

$$x = 3\left(\sum_{j=0}^{\infty} 2^{-j} d_{j,1}\right) - g(y), \tag{4.7}$$

where

$$g(y) = \sum_{j=1}^{\infty} j2^{-j-1} d_{j,2}. \tag{4.8}$$

From these formulae one sees first that $0 \leqslant y \leqslant 1$ and also that for each fixed value of y the allowed values of x form an interval of length 3. The

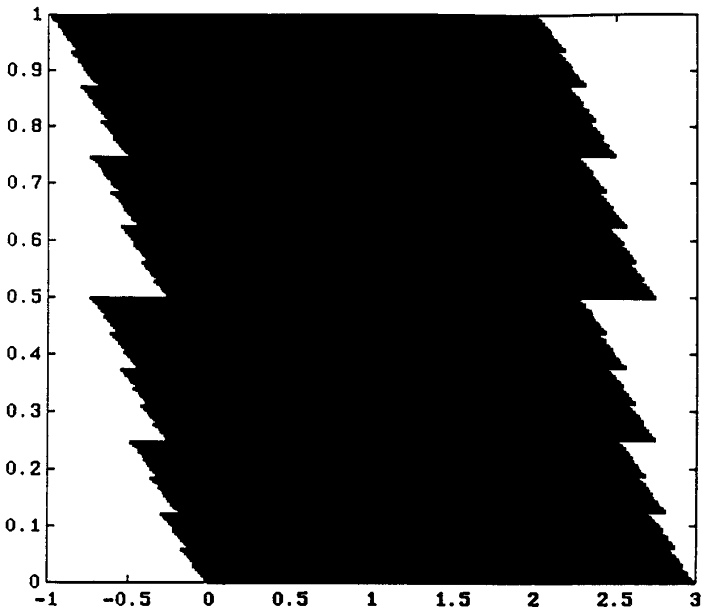

Fig. 4.3. $T(\mathsf{A}, \mathscr{D})$ for $\mathsf{A} = \left[\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right]$ and $\mathscr{D} = \left\{\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix}\right]\right\}$.

tile T thus has horizontal top and bottom sides at $y=0$ and $y=1$, respectively. Its other two sides consist of parallel fractal-like boundaries given by $\{(g(y), y): 0 \leqslant y \leqslant 1\}$ and $\{(g(y)+3, y): 0 \leqslant y \leqslant 1\}$. The function $g(y)$ is discontinuous at every dyadic rational[3] $l/2^k$, where it has a jump of size

$$k2^{-k} - 1 - \sum_{j=k+1}^{\infty} j2^{-j-1} = -2^{-k} \tag{4.9}$$

as y increases. This gives a serrated appearance to the sides of the tile T. The largest jumps are of size 1/2, and are exactly at half-integer values of y, and the tile T has a horizontal edge from $(-\frac{3}{4}, \frac{1}{2})$ to $(-\frac{1}{4}, \frac{1}{2})$; see Fig. 4.3. This has the following consequence. Any horizontal neighbor of any tile in a tiling $\mathscr{S}$ must either have the same lower boundary as this tile or else have its lower boundary translated by $\frac{1}{2}$, for otherwise the largest jump in the serration will not match. Now we show:

Claim 0. *Every tiling of $\mathbb{R}^2$ with translates of T is at least $\frac{1}{2}$-periodic. The period lattice always includes one of the periods $\left[\begin{smallmatrix} -1/2 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$. If there are two tiles T, T' in the tiling with $T' - T = \left[\begin{smallmatrix} x \\ 1/2 \end{smallmatrix}\right]$ for some x, then the period lattice includes $\left[\begin{smallmatrix} -1/2 \\ 1 \end{smallmatrix}\right]$, while if there are two tiles with $T' - T = \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$ for some real $-1 < x < 2$, then the period lattice includes $\left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$.*

To prove Claim 0, for a given tile T, look at a neighboring tile T' adjacent to its right serrated edge. By the remark above, T' either has a common horizontal edge with T, so $T' = T + \left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$, or else T' is shifted vertically up or down by $\frac{1}{2}$, the possibilities being $T' = T + \left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right] \pm \left[\begin{smallmatrix} -1/4 \\ 1/2 \end{smallmatrix}\right]$.

Suppose that there are two tiles where $\left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right] \pm \left[\begin{smallmatrix} -1/4 \\ 1/2 \end{smallmatrix}\right]$ occurs. Then the two tiles create two corners, which can only be filled by tiles placed at $T \pm \left[\begin{smallmatrix} 1/2 \\ 1 \end{smallmatrix}\right]$ and $T' \mp \left[\begin{smallmatrix} -1/2 \\ 1 \end{smallmatrix}\right]$, respectively. These create new corners, and by induction the tiling must contain an infinite strip two tiles wide:

$$\left\{T + j\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} : j \in \mathbb{Z}\right\} \cup \left\{T' + j\begin{bmatrix} -1/2 \\ 1 \end{bmatrix} : j \in \mathbb{Z}\right\}.$$

Now the jumps in the serrated edges of this strip force all neighboring tiles in either side to be in similar strips, with each strip invariant under the translation $\left[\begin{smallmatrix} -1/2 \\ 1 \end{smallmatrix}\right]$. By induction one establishes that the whole tiling is periodic with period $\left[\begin{smallmatrix} -1/2 \\ 1 \end{smallmatrix}\right]$.

If there are no such neighboring tiles, then every tile has a single right neighbor translated by $\left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$, so the tiling has $\left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$ as a period.

[3] Here l is odd, and $y = l/2^k$ has two dyadic expansions (4.6), one having $d_{j,2} = 1$ for all $j \geqslant k+1$, the other $d_{j,2} = 0$ for all $j \geqslant k+1$. These produce the jump. We *define* $g(l/2^k)$ to be the value after the jump, i.e. we take the terminating dyadic expansion.

Finally, suppose T has a top neighbor $T' = T + \left[\begin{smallmatrix} x \\ 1 \end{smallmatrix}\right]$, with $-1 < x < 2$. This creates two corners which can only be filled with tiles placed at $T + \left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$ and $T' - \left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$, respectively. As in the argument above, new corners are created, forcing a perfect tiling of the horizontal strip $\{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] : 0 \leqslant y \leqslant 2, x \in \mathbb{R}\}$ by translates of T invariant under translation by $\left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$. Then neighboring strips of height one must also be perfectly tiled, with tilings invariant under $\left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$, so the whole tiling has $\left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$ as a period. This proves Claim 0.

The proof above did not establish the existence of any tilings of $\mathbb{R}^2$ by T. To show existence of the various tilings above, one must show that the serrated edges of T and $T' = T + \mathbf{d}$ fit perfectly together, for $\mathbf{d} = \left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 11/4 \\ 1/2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 13/4 \\ -1/2 \end{smallmatrix}\right]$, respectively. For $\left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$ this is guaranteed by the inflation rule, since $\left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right] \in \mathscr{D}$. In fact it is also true for $\left[\begin{smallmatrix} 11/4 \\ 1/2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 13/4 \\ -1/2 \end{smallmatrix}\right]$, but we do not prove it here, since we will not need it in the sequel.

The tile T clearly lattice tiles $\mathbb{R}^2$ using the lattice $3\mathbb{Z} \oplus \mathbb{Z} = \{\left[\begin{smallmatrix} 3m \\ n \end{smallmatrix}\right]\} : m, n \in \mathbb{Z}$. By the remarks above the set $\{T + m\left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right] : m \in \mathbb{Z}\}$ perfectly tiles the horizontal strip $\{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] : x \in \mathbb{R}$ and $0 \leqslant y \leqslant 1\}$; the rest is obvious.

Now we study SRT's by T. We will show that the "serration function" $g(y)$ contains an aperiodic pattern which shows up under inflation and makes all SRT's non-periodic. We proceed by a series of claims.

Claim 1. *If $\mathsf{B} \in M_2(\mathbb{R})$ is an expanding matrix such that $\mathsf{B}(T)$ can be tiled with translates of T, then necessarily $\mathsf{B} = \mathsf{A}^k$ for some $k \geqslant 1$. The tiling of $\mathsf{B}(T)$ by copies of T is unique.*

To prove this claim, observe that since the tiles T have a horizontal side, so must $\mathsf{B}(T)$, hence it preserves the x-axis, so must have the form $\mathsf{B} = \left[\begin{smallmatrix} \alpha & \beta \\ 0 & \gamma \end{smallmatrix}\right]$. Now $\mathsf{B}(T)$ has a unique tiling by translates of T, for its bottom horizontal edge must have tiles uniquely packed in a row along it, filling it perfectly. The partially packed region $\mathsf{B}(T)$ still has a horizontal bottom edge, so a second row of tiles packs uniquely, and so on. Next, since there are an integral number of tiles in each row, α is an integer, and since there must be an integral number of rows, so is γ. To continue, we examine the effect of B on the serrations (4.9). We have

$$\mathsf{B}\begin{bmatrix} g(y) \\ y \end{bmatrix} = \begin{bmatrix} \alpha g(y) + \beta y \\ \gamma y \end{bmatrix}. \tag{4.10}$$

Since the length of the discontinuities are inflated by the factor α, we must have $\alpha = 2^k$ or else the inflated jump discontinuities fit no tile. The inflation factor γ stretches the vertical spacing between jump discontinuities, and it will not preserve the correct spacing pattern unless

$\gamma = \alpha = 2^k$. Finally, since T and $\mathsf{B}(T)$ both have lower left corner at $\mathbf{0}$, the leftmost serrated edge $\left[\begin{smallmatrix} 2^k g(y) + \beta y \\ 2^k y \end{smallmatrix}\right]$ of $\mathsf{B}(T)$ must coincide with that of T for $0 \leqslant y \leqslant 1/2^k$. This range of y makes $d_{j,2} = 0$ for $0 \leqslant j \leqslant k$, and (4.8) now implies that $\beta = k2^{k-1}$. Thus $\mathsf{B} = \mathsf{A}^k$ and Claim 1 is proved.

CLAIM 2. *Let $f(j)$ (mod 3) denote the x-coordinates of the bottom corners of any tile in the j-th horizontal row of the unique tiling of $\mathsf{A}^k(T)$ with copies of T. Then $f(0) = 0$ and $f(j)$ satisfies the recursions*

$$f(2j) \equiv 2f(j) + j \pmod 3, \tag{4.11a}$$

$$f(2j+1) \equiv f(2j) \pmod 3. \tag{4.11b}$$

To prove Claim 2, note that $f(j)$ (mod 3) is well-defined. Now $\mathsf{A}^k(T)$ is derived by repeated inflation under A, and we proceed by induction on k. Suppose $\left[\begin{smallmatrix} f(j) + 3l \\ j \end{smallmatrix}\right]$ is the corner of some tile in $\mathsf{A}^{k-1}(T)$. Then, by (1.1),

$$\mathsf{A}\left(T + \begin{bmatrix} f(j) + 3l \\ j \end{bmatrix}\right) = \bigcup_{\mathbf{d} \in \mathscr{D}} \left(T + \begin{bmatrix} 2f(j) + j + 6l \\ 2j \end{bmatrix} + \mathbf{d}\right).$$

Taking $\mathbf{d} = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ on the right, there is a tile at height $2j$, so

$$f(2j) \equiv 2f(j) + j + 6l \pmod 3,$$

which is (4.11a). Taking $\mathbf{d} = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ we obtain (4.11b), proving Claim 2.

CLAIM 3. *If j has the binary expansion $\sum_{i=0}^{k-1} b_i 2^i$, then*

$$f(j) \equiv \sum_{i=0}^{k-1} i b_i 2^i \pmod 3. \tag{4.12}$$

The function $f(j)$ is aperiodic. That is, for any positive integer m, there exist positive integers j, j′ with

$$f(j+m) - f(j) \not\equiv f(j'+m) - f(j') \pmod 3. \tag{4.13}$$

To prove Claim 3, (4.12) is verified by checking that the right side satisfies the recursions (4.11) and the initial condition $f(0) = 0$. To verify (4.13), suppose m has binary expansion $\sum_{i=0}^{l} c_i 2^i$, with $c_l = 1$, and choose $j = 2^{l+1}$ and either take $j' = 2^l$ if $l \not\equiv 1$ (mod 3) or else take $j' = 2^l + 2^{l+1}$ if $l \equiv 1$ (mod 3). Then (4.13) follows by residue calculations using (4.12), proving Claim 3.

We now show that all self-replicating tilings using T are non-periodic. We begin by observing that $\left[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}\right]$ is always a period of any SRT for T. Indeed, Claim 2 gives $f(0) = 0$, $f(1) = 0$, hence the tiling of $\mathsf{B}(T)$ by copies of T contains in its bottom row and next row upward tiles T and T', respectively, with $T' - T = [\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}]$, whence Claim 0 shows that $[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}]$ must be a period of the tiling. Thus the SRT must tile in horizontal strips.

We now argue by contradiction. Suppose there were a periodic SRT with period lattice Λ. Since $[\begin{smallmatrix} 3 \\ 0 \end{smallmatrix}]$ is a period, it is in Λ. Also, since the tiling $\mathscr{S}$ lies in horizontal strips of height one, there is an independent period $[\begin{smallmatrix} l \\ m \end{smallmatrix}]$ in Λ, in which $l \in \mathbb{R}$ and $m \neq 0$ is an integer. We show that l is rational. To do this it suffices to establish that all tiles $\mathscr{C}_0$ touching $\mathbf{0}$ are of the form $T + \mathbf{v}$ for rational vectors $\mathbf{v}$, because then all tiles in $\mathscr{S}$ are rational translates of T, so the periods must be also. Suppose $\mathbf{v} = [\begin{smallmatrix} x \\ y \end{smallmatrix}]$, and $\mathsf{B} = \mathsf{A}^k$, and we have

$$\mathsf{A}^k \left(T + \begin{bmatrix} x \\ y \end{bmatrix} \right) = \bigcup_{\mathbf{d} \in \mathscr{D}_{\mathsf{A},k}} \left(T + \begin{bmatrix} 2^k x + k 2^{k-1} y \\ 2^k y \end{bmatrix} + \mathbf{d} \right).$$

The tiling is in horizontal strips; hence we have $y \equiv 2^k y \pmod 1$ so that $y \in \mathbb{Q}$. Now if $\mathbf{0}$ is in the interior of $T + [\begin{smallmatrix} x \\ y \end{smallmatrix}]$ then some tile on the right must coincide with $T + [\begin{smallmatrix} x \\ y \end{smallmatrix}]$, hence

$$x \equiv 2^k x + k 2^{k-1} y + d_1 \qquad (\mathrm{mod}\ 3), \tag{4.14}$$

which forces $x \in \mathbb{Q}$. If $\mathbf{0}$ lies on the boundary of $T + [\begin{smallmatrix} x \\ y \end{smallmatrix}]$, on a serrated edge, then $[\begin{smallmatrix} -x \\ -y \end{smallmatrix}]$ lies on a serrated edge, so $x = g(-y)$ or $g(-y) + 3$, and the fact that $g(y)$ is rational whenever y is rational forces $x \in \mathbb{Q}$. Finally if $\mathbf{0}$ is on a horizontal edge, i.e. $[\begin{smallmatrix} x \\ y \end{smallmatrix}] = [\begin{smallmatrix} x \\ 0 \end{smallmatrix}]$, then (4.14) still applies, so $x \in \mathbb{Q}$. Now we have shown $l = p/q$ is rational, so $3q[\begin{smallmatrix} l \\ m \end{smallmatrix}] = [\begin{smallmatrix} 3p \\ 3qm \end{smallmatrix}] \in \Lambda$, consequently there is a vertical period $[\begin{smallmatrix} 0 \\ 3qm \end{smallmatrix}]$ in Λ. But this contradicts the three claims. Claim 1 says that this SRT has inflation by A^k for some k, so we can inflate one tile in $\mathscr{S}$ to a large enough size using some A^{jk} that its unique tiling with copies of T excludes vertical period $3qm$ using Claims 2 and 3. This contradiction proves the SRT is non-periodic, hence it must be $\frac{1}{2}$-periodic. ∎

5. Open Problems

The most fundamental unsolved problem concerns restrictions on the form of the matrix A in a self-affine tile.

Conjecture 1. If $T(\mathsf{A}, \mathscr{D})$ is a self-affine tile, then A is similar to an integer matrix.

This conjecture may be viewed in light of Theorem 1.1 (iv), which puts a severe restriction on the digit set $\mathscr{D}$.

Another class of unsolved problems concerns how regular are the tilings possible with an arbitrary self-affine tile. The weakest of these is the following:

Conjecture 2. For each self-affine tile T there is a quasiperiodic tiling of $\mathbb{R}^n$ using T as a prototile.

This tiling need not be self-replicating. Radin and Woolf [30] show that any tile T that tiles by translation has a tiling satisfying the local isomorphism property, but not necessarily the local finiteness property. They also show that if a tile T can have its boundary completely covered in only finitely many (translation-inequivalent) ways by translates of itself with disjoint interiors, and if T tiles $\mathbb{R}^n$ by translation, then it has a quasiperiodic tiling.

Related to Conjecture 2 is the question of whether there is a self-affine tile T that tiles $\mathbb{R}^n$ by translation but only aperiodically. No such tile, of any kind, is known to exist in $\mathbb{R}^n$ for any $n \geqslant 1$. Tiles in $\mathbb{R}^2$ that are topological disks always have a periodic tiling, see Girault–Beauquier and Nivat [13]. Work of Schmitt [31] recently led to the discovery of a convex polyhedron with eight faces that tiles $\mathbb{R}^3$ using *translations and rotations*, for which all tilings are aperiodic, see Danzer [6].

Note that the truth of Conjecture 1 implies the truth of Conjecture 2 in all cases where $\mathscr{D}$ generates a lattice for A, by Theorem 1.3 (ii).

A strengthening of Conjecture 2, which is open even for the special case of integral self-affine tiles, is:

Conjecture 3. For each self-affine tile in $\mathbb{R}^n$ there is a periodic tiling of $\mathbb{R}^n$ using translates of T.

Finally we formulate a conjecture concerning the regularity of self-replicating tilings.

Conjecture 4. Every self-replicating tiling in $\mathbb{R}^n$ is at least $1/n$-periodic.

Example 2.2 shows that a self-replicating tiling need not be quasiperiodic. This conjecture is proved for $n = 1$ and 2, with the case $n = 2$ settled in Kenyon [19], [20].

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