On the far field patterns for electromagnetic scattering by a chiral obstacle in a chiral environment

Christodoulos Athanasiadis

Department of Mathematics, University of Athens, Panepistemiopolis, GR 15784 Athens, Greece

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Abstract

A time-harmonic plane electromagnetic wave is scattered by a chiral body in a chiral environment. The body is either a perfect conductor, or a dielectric, or a scatterer with an impedance surface. Using the Huygens’s principle, we construct in closed forms both the left-circularly polarized and right-circularly polarized electric far field patterns for such chiral media. We prove reciprocity relations and general scattering theorems for chiral materials which are a generalization of those obtained by Twersky for achiral electromagnetic scattering. In the special case when the directions of incidence and observation are the same we prove the associated forward scattering theorems.

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1. Introduction

In this work we study the far field patterns corresponding to the scattering of time-harmonic plane electromagnetic waves by a homogeneous chiral obstacle surrounded by an infinite homogeneous isotropic chiral medium. In particular, we prove reciprocity, general
scattering theorems and optical theorems. We consider the cases where the obstacle is either a dielectric (penetrable scatterer) or a perfect conductor (impenetrable scatterer) or a scatterer with an impedance surface. For all three problems we construct in closed forms the far field patterns.

In a homogeneous isotropic chiral medium the electromagnetic fields are composed of left-circularly polarized (LCP) and right-circularly polarized (RCP) components, which have different wave numbers and independent directions of propagation. When either an LCP or an RCP (or a linear combination of LCP and RCP) electromagnetic wave is incident upon a chiral scatterer then the scattered field is composed of both LCP and RCP components. Hence, both LCP and RCP far field patterns are derived. For details on the properties of the chiral media we refer to the books [12,15].

Scattering relations for a chiral scatterer in an achiral environment have been proved in [3]. For plane-wave scattering from an achiral object in a chiral environment we refer to [16]. Lakhtakia in [13,14] and in his book [15] has proved reciprocity and optical theorems for a chiral scatterer in a chiral environment, extending the corresponding results for the achiral case obtained by de Hoop [8,9]. In [1], [2] and [5] there exist various scattering relations for acoustic, electromagnetic and elastic waves, respectively. Some details about optical theorems for electromagnetic scattering in achiral media can be found in [10,11]. In [6] Dassios et al. have proved the most general scattering theorems for complete dyadic fields, with two different wave numbers but the same direction of propagation. In this paper we prove for chiral materials a general scattering theorem, in the sense of Kleinman and Dassios [7].

In Section 2, decoupling of electric and magnetic fields in equations and boundary conditions, we formulate three electric scattering problems for which the far field patterns will be studied. In Section 3, the LCP and RCP electric far field patterns are defined via the asymptotic behavior of the scattered electric field. Using the Huygens’s principle we construct in closed forms the electric far field patterns. In Section 4, we prove general scattering theorems which are a generalization of those obtained by Twersky in [18] for achiral electromagnetic scattering. These theorems are useful in determining low frequency expansions of the far field patterns [7] and in studying the spectrum of the far field operator [1–3]. Also we prove various reciprocity relations. In Section 5, we consider either LCP or RCP incidence. Specializing to the same direction of incidence and observation in the general scattering theorems we obtain closed form expressions for the scattering cross sections in terms of the forward LCP or RCP far field pattern. Finally, in Section 6 we recover as special cases, the basic scattering relations for electromagnetic waves in an achiral medium, while in Section 7 we conclude to some interesting comments.

2. Formulation

Let \( \Omega^- \) be a bounded and closed subset of \( \mathbb{R}^3 \) having a \( C^2 \)-boundary \( S \). The set \( \Omega^- \) will be referred to as the scatterer. The exterior \( \Omega \) of the scatterer is an infinite homogeneous isotropic chiral medium with electric permittivity \( \varepsilon \) magnetic permeability \( \mu \) and chirality measure \( \beta \). The scatterer is filled with a homogeneous isotropic chiral medium with corresponding physical parameters \( \varepsilon^- \), \( \mu^- \) and \( \beta^- \). All the physical parameters are assumed to be real positive constants.
Let \((E_i, H_i)\) be a time-harmonic plane electromagnetic wave incident upon the scatterer \(\Omega^-\) and \((E_s, H_s)\) be the corresponding scattered field. Then the total electromagnetic field \((E_t, H_t)\) in \(\Omega\) is given by
\[
E_t = E_i + E_s, \quad H_t = H_i + H_s \quad \text{in} \quad \Omega.
\] (2.1)

We assume that the scattered field satisfies the Silver–Müller radiation condition
\[
E_s(r) + \eta \hat{r} \times H_s(r) = o\left(\frac{1}{r}\right), \quad r \to \infty,
\] (2.2)
uniformly in all directions, \(\hat{r} \in S^2\), where \(S^2\) is the unit sphere in \(\mathbb{R}^3\), \(\hat{r} = r/r, r = |r|\) and \(\eta = (\mu/\varepsilon)^{1/2}\) is the intrinsic impedance of the chiral medium in \(\Omega\). In view of the Drude–Born–Fedorov constitutive relations [15], the total exterior electromagnetic field satisfies in the source-free region \(\Omega\) the modified Maxwell equations
\[
\nabla \times E_t = \beta \gamma^2 E_t + i \omega \mu \gamma \kappa E_t \quad \text{in} \quad \Omega,
\] (2.3)
\[
\nabla \times H_t = \beta \gamma^2 H_t - i \omega \epsilon \gamma \kappa E_t \quad \text{in} \quad \Omega,
\] (2.4)
where \(\omega\) is the angular frequency and
\[
\kappa^2 = \omega^2 \varepsilon \mu, \quad \gamma^2 = \kappa^2 (1 - \beta^2 \kappa^2)^{-1},
\] (2.5)
with \(|\beta \kappa| < 1\) [15, p. 87].

We note that \(E_t\) and \(H_t\) are divergence-free fields. In addition, \(\kappa\) is not a wave number, but it is a shorthand notation without any particular physical significance. The incident field \((E_i, H_i)\) satisfies Eqs. (2.3), (2.4) and hence the scattered field also satisfies (2.3), (2.4).

In order to prove electromagnetic scattering theorems in a chiral medium, it is more convenient to consider the decoupling of electric and magnetic field in the system (2.2)–(2.4), by eliminating the magnetic field. The vector equation thus arising is
\[
\nabla \times \nabla \times E_t - 2 \beta \gamma^2 \nabla \times E_t - \gamma^2 E_t = 0 \quad \text{in} \quad \Omega.
\] (2.6)

The Silver–Müller radiation condition is modified as follows:
\[
\hat{r} \times \nabla \times E_t(r) - 2 \beta \gamma^2 \hat{r} \times E_t(r) + \frac{i \gamma^2}{\kappa} E_t(r) = o\left(\frac{1}{r}\right), \quad r \to \infty,
\] (2.7)
uniformly in all directions \(\hat{r} \in S^2\). When the scatterer is impenetrable, i.e., no fields exist in \(\Omega^-\), one of the following boundary conditions will be imposed on the total field on \(S\).

**Perfectly conducting surface:**
\[
\hat{n} \times E_t(r) = 0, \quad r \in S.
\] (2.8)

**Impedance surface:**
\[
\hat{n} \times \nabla \times E_t(r) = \beta \gamma^2 \hat{n} \times E_t(r) - \frac{i \gamma^2}{\kappa z_s} \hat{n} \times (\hat{n} \times E_t(r)), \quad r \in S,
\] (2.9)
where \(z_s\) is a complex function on \(S\) which denotes the surface impedance of the scatterer and \(\hat{n}\) is the outward normal unit vector on \(S\). When the scatterer is a dielectric, i.e., penetrable, we have
Transmission conditions:
\[
\hat{n} \times \mathbf{E}'(r) = \hat{n} \times \mathbf{E}^-(r), \quad r \in S,
\]
(2.10)
\[
\hat{n} \times \nabla \times \mathbf{E}'(r) = \frac{\varepsilon^-}{\varepsilon} \left( \frac{\gamma}{\gamma^+} \right)^2 \hat{n} \times \nabla \times \mathbf{E}^-(r) + \gamma^2 \left( \beta - \frac{\varepsilon^-}{\varepsilon} \beta^+ \right) \hat{n} \times \mathbf{E}^-(r),
\]
(2.11)
where
\[
\kappa^- = \omega (\varepsilon^- \mu^-)^{1/2}, \quad \gamma^- = \kappa^- (1 - (\beta^- \kappa^-)^2)^{-1/2}
\]
(2.12)
and \( \mathbf{E}^- \) is the electric field in the interior of the scatterer satisfying the equation
\[
\nabla \times \nabla \times \mathbf{E}^- - \left( \nabla \times \mathbf{E}^- \right)(r) - 2 \beta^- (\gamma^-)^2 \nabla \times \mathbf{E}^- - (\gamma^-)^2 \mathbf{E}^- = 0 \quad \text{in } \Omega^-.
\]
(2.13)

3. Far field patterns

A time-harmonic electromagnetic plane wave propagating in a homogeneous chiral medium is, in general, a linear combination of a left-circularly polarized (LCP) plane wave and a right-circularly polarized (RCP) plane wave [15]. In particular, for the electric wave \( \mathbf{E}'(r) \) we have
\[
\mathbf{E}'(r) = \mathbf{E}_L'(r; \hat{d}_L, \mathbf{p}_L) + \mathbf{E}_R'(r; \hat{d}_R, \mathbf{p}_R),
\]
(3.1)
where
\[
\mathbf{E}_L'(r; \hat{d}_L, \mathbf{p}_L) = \mathbf{p}_L e^{i \gamma_L \hat{d}_L \cdot r}
\]
(3.2)
is an LCP plane electric wave and
\[
\mathbf{E}_R'(r; \hat{d}_R, \mathbf{p}_R) = \mathbf{p}_R e^{i \gamma_R \hat{d}_R \cdot r}
\]
(3.3)
is an RCP plane electric wave. The unit vectors \( \hat{d}_L, \hat{d}_R \in S^2 \) describe the directions of propagation and the complex vectors \( \mathbf{p}_L, \mathbf{p}_R \) give the polarizations of the LCP and RCP plane electric wave, respectively. The real wave numbers \( \gamma_L \) and \( \gamma_R \) are given by
\[
\gamma_L = \frac{\kappa}{1 - \kappa \beta}, \quad \gamma_R = \frac{\kappa}{1 + \kappa \beta}
\]
(3.4)
and satisfy the relations [15]
\[
\gamma_L - \gamma_R = 2 \beta \gamma^2, \quad \gamma_L \gamma_R = \gamma^2.
\]
(3.5)
The electric wave \( \mathbf{E}' \) is incident upon the scatterer \( \Omega^- \) and the scattered electric field \( \mathbf{E}' \) is derived. The asymptotic behavior of \( \mathbf{E}' \) has been studied by Lakhtakia et al. via the Beltrami fields (see [12,15]). Here we define the LCP and RCP electric far field patterns using the Huygens’s principle for chiral media [12].

**Theorem 3.1.** Let \( \mathbf{E}' \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) be a solution of (2.6) satisfying the radiation condition (2.7). Then \( \mathbf{E}' \) has the asymptotic form
\[
\mathbf{E}'(r) = h(\gamma_L r) g_L^r(\hat{r}) + h(\gamma_R r) g_R^r(\hat{r}) + O \left( \frac{1}{r^2} \right), \quad r \to \infty,
\]
(3.6)
uniformly in all directions \( \mathbf{\hat{r}} \in S^2 \), where \( h(x) = e^{ix}/(ix) \) is the zeroth order spherical Hankel function of the first kind. The vector fields \( \mathbf{g}_L^\ast \) and \( \mathbf{g}_R^\ast \) are the electric LCP far field pattern and RCP far field pattern, respectively. They are given by

\[
\mathbf{g}_L^\ast(\mathbf{\hat{r}}) = \frac{i\kappa \gamma_L}{8\pi \gamma^2} \tilde{K}_L(\mathbf{\hat{r}}) \cdot \int_{S} \mathbf{\hat{n}} \times \left[ \gamma_L \nabla \times \mathbf{E}'(\mathbf{r}') + \gamma^2 \mathbf{E}'(\mathbf{r}') \right] e^{-i\gamma L \mathbf{\hat{r}} \cdot \mathbf{r}'} d\mathbf{s}(\mathbf{r}'),
\]

\[
\mathbf{g}_R^\ast(\mathbf{\hat{r}}) = \frac{i\kappa \gamma_R}{8\pi \gamma^2} \tilde{K}_R(\mathbf{\hat{r}}) \cdot \int_{S} \mathbf{\hat{n}} \times \left[ \gamma_R \nabla \times \mathbf{E}'(\mathbf{r}') - \gamma^2 \mathbf{E}'(\mathbf{r}') \right] e^{-i\gamma R \mathbf{\hat{r}} \cdot \mathbf{r}'} d\mathbf{s}(\mathbf{r}')
\]

and satisfy

\[
\mathbf{\hat{r}} \cdot \mathbf{g}_L^\ast(\mathbf{\hat{r}}) = \mathbf{\hat{r}} \cdot \mathbf{g}_R^\ast(\mathbf{\hat{r}}) = 0,
\]

\[
\mathbf{\hat{r}} \times \mathbf{g}_L^\ast(\mathbf{\hat{r}}) = -i\mathbf{g}_R^\ast(\mathbf{\hat{r}}), \quad \mathbf{\hat{r}} \times \mathbf{g}_R^\ast(\mathbf{\hat{r}}) = i\mathbf{g}_L^\ast(\mathbf{\hat{r}}).
\]

The dyadics \( \tilde{K}_L(\mathbf{\hat{r}}) \) and \( \tilde{K}_R(\mathbf{\hat{r}}) \) are given by

\[
\tilde{K}_L(\mathbf{\hat{r}}) = \mathbf{\hat{I}} - \mathbf{\hat{n}} \mathbf{\hat{n}} \times \mathbf{\hat{I}}, \quad \tilde{K}_R(\mathbf{\hat{r}}) = \mathbf{\hat{I}} - \mathbf{\hat{n}} \mathbf{\hat{n}} \times \mathbf{\hat{I}},
\]

where \( \mathbf{\hat{I}} = \mathbf{\hat{x}} \mathbf{\hat{x}} + \mathbf{\hat{y}} \mathbf{\hat{y}} + \mathbf{\hat{z}} \mathbf{\hat{z}} \) is the identity dyadic.

**Proof.** In accordance with the Huygens’s principle for homogeneous chiral media, the scattered electric field has the following integral representation [12]:

\[
\mathbf{E}'(\mathbf{r}) = -2\beta \gamma^2 \int_{S} \tilde{B}(\mathbf{r}, \mathbf{r}') \cdot \left[ \mathbf{\hat{n}} \times \mathbf{E}'(\mathbf{r}') \right] d\mathbf{s}(\mathbf{r}')
\]

\[
+ \int_{S} \left[ \tilde{B}(\mathbf{r}, \mathbf{r}') \cdot \left[ \mathbf{\hat{n}} \times \left( \nabla \times \mathbf{E}'(\mathbf{r}') \right) \right] \right.
\]

\[
+ \left[ \nabla_{\mathbf{r}} \times \tilde{B}(\mathbf{r}, \mathbf{r}') \cdot \left[ \mathbf{\hat{n}} \times \mathbf{E}'(\mathbf{r}') \right] \right] d\mathbf{s}(\mathbf{r}'),
\]

where \( \tilde{B}(\mathbf{r}, \mathbf{r}') \) is the infinite medium Green’s dyadic, given by

\[
\tilde{B}(\mathbf{r}, \mathbf{r}') = \tilde{B}_L(\mathbf{r}, \mathbf{r}') + \tilde{B}_R(\mathbf{r}, \mathbf{r}'),
\]

where

\[
\tilde{B}_L(\mathbf{r}, \mathbf{r}') = \frac{i\kappa \gamma_L}{8\pi \gamma^2} \left[ \gamma_L \mathbf{\hat{I}} + \frac{1}{\gamma_L} \nabla \nabla + \nabla \times \mathbf{\hat{I}} \right] h(\gamma_L |\mathbf{r} - \mathbf{r}'|),
\]

\[
\tilde{B}_R(\mathbf{r}, \mathbf{r}') = \frac{i\kappa \gamma_R}{8\pi \gamma^2} \left[ \gamma_R \mathbf{\hat{I}} + \frac{1}{\gamma_R} \nabla \nabla - \nabla \times \mathbf{\hat{I}} \right] h(\gamma_R |\mathbf{r} - \mathbf{r}'|).
\]

Using the asymptotic relations

\[
|\mathbf{r} - \mathbf{r}'| = r - \mathbf{\hat{r}} \cdot \mathbf{r}' + O\left( \frac{1}{r} \right), \quad r \to \infty,
\]

\[
\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} = \mathbf{\hat{r}} + O\left( \frac{1}{r} \right), \quad r \to \infty,
\]

in (3.13)–(3.15) we obtain
\[ \tilde{\mathbf{B}}(\mathbf{r}, \mathbf{r}') = \frac{i k \gamma_L^2}{8 \pi \gamma^2} h(\gamma_L r) e^{-i \gamma_L \hat{r} \cdot \mathbf{r}'} \tilde{K}_L(\hat{r}) + \frac{i k \gamma_R^2}{8 \pi \gamma^2} h(\gamma_R r) e^{-i \gamma_R \hat{r} \cdot \mathbf{r}'} \tilde{K}_R(\hat{r}) + O\left(\frac{1}{r^2}\right), \quad r \to \infty, \]  

\[ \nabla_{\mathbf{r}} \times \tilde{\mathbf{B}}(\mathbf{r}, \mathbf{r}') = \frac{i k \gamma_L^3}{8 \pi \gamma^2} h(\gamma_L r) e^{-i \gamma_L \hat{r} \cdot \mathbf{r}'} \tilde{K}_L(\hat{r}) - \frac{i k \gamma_R^3}{8 \pi \gamma^2} h(\gamma_R r) e^{-i \gamma_R \hat{r} \cdot \mathbf{r}'} \tilde{K}_R(\hat{r}) + O\left(\frac{1}{r^2}\right), \quad r \to \infty, \]  

where the relations

\[ \hat{\mathbf{r}} \times \tilde{K}_L(\hat{\mathbf{r}}) = -i \tilde{K}_L(\hat{\mathbf{r}}), \quad \hat{\mathbf{r}} \times \tilde{K}_R(\hat{\mathbf{r}}) = i \tilde{K}_R(\hat{\mathbf{r}}) \]  

have been used.

Introducing (3.18) and (3.19) into (3.12) we obtain

\[ g_e^L(\hat{\mathbf{r}}) = \frac{i k \gamma_L^2}{8 \pi \gamma^2} \tilde{K}_L(\hat{\mathbf{r}}) \cdot \int_S \hat{\mathbf{n}} \times \left[ \nabla \times \mathbf{E}^t(\mathbf{r}') + (\gamma_L - 2 \beta \gamma^2) \mathbf{E}^t(\mathbf{r}') \right] e^{-i \gamma_L \hat{r} \cdot \mathbf{r}'} \, ds(\mathbf{r}'), \]  

\[ g_e^R(\hat{\mathbf{r}}) = \frac{i k \gamma_R^2}{8 \pi \gamma^2} \tilde{K}_R(\hat{\mathbf{r}}) \cdot \int_S \hat{\mathbf{n}} \times \left[ \nabla \times \mathbf{E}^t(\mathbf{r}') - (\gamma_R + 2 \beta \gamma^2) \mathbf{E}^t(\mathbf{r}') \right] e^{-i \gamma_R \hat{r} \cdot \mathbf{r}'} \, ds(\mathbf{r}'). \]  

and using the relations (3.5) we conclude to (3.7) and (3.8). The formulae (3.9) and (3.10) result by straightforward calculations taking into account (3.20) and the fact that

\[ \hat{\mathbf{r}} \cdot \tilde{K}_L(\hat{\mathbf{r}}) = \hat{\mathbf{r}} \cdot \tilde{K}_R(\hat{\mathbf{r}}) = 0. \]  

\[ \Box \]

**Remark 3.1.** Setting \( \mathbf{E}^t = \mathbf{E}^t - \mathbf{E}^i \) in the integrals of (3.12), taking into account that \( \mathbf{E}^i \) is an entire solution of (2.6) and using again the asymptotic forms (3.18) and (3.19) we obtain

\[ g_e^L(\hat{\mathbf{r}}) = \frac{i k \gamma_L^2}{8 \pi \gamma^2} \tilde{K}_L(\hat{\mathbf{r}}) \cdot \int_S \hat{\mathbf{n}} \times \left[ \nabla \times \mathbf{E}^t(\mathbf{r}') + \gamma_L^2 \mathbf{E}^t(\mathbf{r}') \right] e^{-i \gamma_L \hat{\mathbf{r}} \cdot \mathbf{r}'} \, ds(\mathbf{r}'), \]  

\[ g_e^R(\hat{\mathbf{r}}) = \frac{i k \gamma_R^2}{8 \pi \gamma^2} \tilde{K}_R(\hat{\mathbf{r}}) \cdot \int_S \hat{\mathbf{n}} \times \left[ \nabla \times \mathbf{E}^t(\mathbf{r}') - \gamma_R^2 \mathbf{E}^t(\mathbf{r}') \right] e^{-i \gamma_R \hat{\mathbf{r}} \cdot \mathbf{r}'} \, ds(\mathbf{r}'). \]  

Using the expressions (3.24) and (3.25) and applying the boundary conditions (2.8)–(2.11), we obtain the electric far field patterns for various kinds of scatterers.

**Theorem 3.2.** The electric far field patterns are given by the formulae

\[ g_e^A(\hat{\mathbf{r}}) = \frac{i k \gamma_A^2}{8 \pi \gamma^2} \tilde{K}_A(\hat{\mathbf{r}}) \cdot \int_S \hat{\mathbf{n}} \times \nabla \times \mathbf{E}^t(\mathbf{r}') e^{-i \gamma_A \hat{\mathbf{r}} \cdot \mathbf{r}'} \, ds(\mathbf{r}'), \quad A = L, R, \]
for a perfectly conducting surface,

\[ g_A^s(\hat{r}) = \frac{i\gamma^2}{8\pi} K_A(\hat{r}) \cdot \int_S \left[ \hat{n} \times \mathbf{E}'(\mathbf{r}') - \frac{i}{\varepsilon} \hat{n} \times (\hat{n} \times \mathbf{E}'(\mathbf{r}')) \right] e^{-i\gamma_A L \mathbf{r}' \cdot \mathbf{r}} \, ds(\mathbf{r}'), \]

\[ A = L, R, \quad (3.27) \]

for an impedance surface and

\[ g_A^s(\hat{r}) = \frac{i\gamma^2}{8\pi} K_A(\hat{r}) \cdot \int_{\Omega^2} \left[ a_A \nabla \times \mathbf{E}^-(\mathbf{r}') + b_A \mathbf{E}^-(\mathbf{r}') \right] e^{-i\gamma_A L \mathbf{r}' \cdot \mathbf{r}} \, ds(\mathbf{r}'), \]

\[ A = L, R, \quad (3.28) \]

where

\[
\begin{align*}
a_L &= \frac{1}{\kappa} + \left( \beta - \frac{\gamma L}{\gamma L^2} \right) \left( \frac{\gamma}{\varepsilon} \right) \varepsilon, \\
b_L &= -\gamma_L \left( \frac{1}{\kappa} + \left( \beta - \frac{1}{\gamma L} \right) \left( \frac{\gamma}{\varepsilon} \right) \varepsilon \right) \\
a_R &= \frac{1}{\kappa} + \left( \beta + \frac{\gamma R}{(\gamma R)^2} \right) \left( \frac{\gamma}{\varepsilon} \right) \varepsilon, \\
b_R &= -\gamma_R \left( \frac{1}{\kappa} - \left( \beta - \frac{1}{\gamma R} \right) \left( \frac{\gamma}{\varepsilon} \right) \varepsilon \right),
\end{align*}
\]

\[ (3.29)-(3.30) \]

for a dielectric.

**Remark 3.2.** The scattered magnetic field \( \mathbf{H}^s \) satisfies Eq. (2.6) and has the integral representation (3.12) where \( \mathbf{E}^s \) has been replaced by \( \mathbf{H}^s \) [12]. Consequently, the magnetic LCP and RCP far field patterns \( g_L^m \) and \( g_R^m \), respectively, are given by

\[ g_L^m(\hat{r}) = \frac{i\gamma L}{8\pi\gamma^2} K_L(\hat{r}) \cdot \int_S \hat{n} \times \left[ \gamma L \nabla \times \mathbf{H}'(\mathbf{r}') + \gamma^2 \mathbf{H}'(\mathbf{r}') \right] e^{-i\gamma L \mathbf{r}' \cdot \mathbf{r}} \, ds(\mathbf{r}'), \]

\[ (3.31) \]

\[ g_R^m(\hat{r}) = \frac{i\gamma R}{8\pi\gamma^2} K_R(\hat{r}) \cdot \int_S \hat{n} \times \left[ \gamma R \nabla \times \mathbf{H}'(\mathbf{r}') - \gamma^2 \mathbf{H}'(\mathbf{r}') \right] e^{-i\gamma R \mathbf{r}' \cdot \mathbf{r}} \, ds(\mathbf{r}'). \]

\[ (3.32) \]

Also, \( g_L^m \) and \( g_R^m \) satisfy the relations

\[ \hat{r} \cdot g_L^m(\hat{r}) = \hat{r} \cdot g_R^m(\hat{r}) = 0, \]

\[ \hat{r} \times g_L^m(\hat{r}) = -i\hat{n} g_L^m(\hat{r}), \quad \hat{r} \times g_R^m(\hat{r}) = i\hat{n} g_R^m(\hat{r}). \]

\[ (3.33)-(3.34) \]

The magnetic far field patterns are connected with the electric far field patterns via the following formulae.

**Theorem 3.3.** In a chiral medium the far field patterns satisfy the relations

\[ g_L^s(\hat{r}) = i\eta g_L^m(\hat{r}), \quad g_R^s(\hat{r}) = -i\eta g_R^m(\hat{r}), \]

\[ \hat{r} \times g_L^s(\hat{r}) = \eta g_L^m(\hat{r}), \quad \hat{r} \times g_R^s(\hat{r}) = \eta g_R^m(\hat{r}), \]

\[ \hat{r} \times g_L^m(\hat{r}) = -\frac{1}{\eta} g_L^s(\hat{r}), \quad \hat{r} \times g_R^m(\hat{r}) = \frac{1}{\eta} g_R^s(\hat{r}). \]

\[ (3.35)-(3.37) \]
Proof. Using the formulae (3.7), (3.8), (3.31) and (3.32) and taking into account Eqs. (2.3) and (2.4) we conclude to (3.35). From (3.10), (3.34) and (3.35) we take (3.36) and (3.37).

Remark 3.3. Lakhtakia in [15], using the Bohren transform has derived the asymptotic form

$$E^s(r) = \frac{e^{i\gamma_L r}}{r} F_1(\hat{r}) - i \eta \frac{e^{i\gamma_R r}}{r} F_2(\hat{r}) + O\left(\frac{1}{r^2}\right), \quad r \to \infty,$$

(3.38)

where $$F_1$$ and $$F_2$$ are the LCP and RCP Beltrami far field patterns, respectively (see [15, p. 184]). The electric far field patterns $$g^e_L$$ and $$g^e_R$$ given by (3.7) and (3.8) are connected with $$F_1$$ and $$F_2$$ via the relations

$$g^e_L(\hat{r}) = \frac{i\gamma_L}{2} F_1(\hat{r}), \quad g^e_R(\hat{r}) = \frac{\eta\gamma_R}{2} F_2(\hat{r}).$$

(3.39)

4. Scattering theorems

We consider two time-harmonic plane electric waves

$$E^i_j(r) = E^i_L(r; \hat{d}_{Lj}, p_{Lj}) + E^i_R(r; \hat{d}_{Rj}, p_{Rj}), \quad j = 1, 2,$$

(4.1)

incident upon the scatterer $$\Omega^-$$, $$E^i_L(r; \hat{d}_{Lj}, p_{Lj})$$, $$j = 1, 2$$, are LCP electric waves given by (3.2) and $$E^i_R(r; \hat{d}_{Rj}, p_{Rj})$$, $$j = 1, 2$$, are RCP electric waves given by (3.3). The unit vectors $$\hat{d}_{Lj}, \hat{d}_{Rj}$$ (directions of propagation) are connected to the complex vectors $$p_{Lj}, p_{Rj}$$ (polarizations) via the relations [15]

$$p_{Lj} \cdot \hat{d}_{Lj} = 0, \quad \hat{d}_{Lj} \times p_{Lj} = -i p_{Lj},$$

(4.2)

$$p_{Rj} \cdot \hat{d}_{Rj} = 0, \quad \hat{d}_{Rj} \times p_{Rj} = i p_{Rj}.$$  

(4.3)

The corresponding scattered fields $$E^s_j$$, $$j = 1, 2$$, satisfy the asymptotic relations

$$E^s_j(r) = h(\gamma_L r) g^s_L(\hat{r}) + h(\gamma_R r) g^s_R(\hat{r}) + O\left(\frac{1}{r^2}\right), \quad r \to \infty,$$

(4.4)

uniformly in all directions $$\hat{r} \in S^2$$, where $$g^s_L(\hat{r})$$ and $$g^s_R(\hat{r})$$ are the LCP and RCP far field patterns given by (3.7) and (3.8), respectively.

Remark 4.1. The far field patterns are dependent on the directions of propagation and polarization of the incident plane electric waves. In general, if $$E^s_j(r) = E^s_L(r; \hat{d}_{Lj}, p_{Lj}) + E^s_R(r; \hat{d}_{Rj}, p_{Rj})$$, then we shall indicate this dependence by writing

$$g^s_L(\hat{r}) = g^s_L(\hat{r}; \hat{d}_{Lj}, p_{Lj}) + g^s_L(\hat{r}; \hat{d}_{Rj}, p_{Rj}),$$  

(4.5)

$$g^s_R(\hat{r}) = g^s_R(\hat{r}; \hat{d}_{Lj}, p_{Lj}) + g^s_R(\hat{r}; \hat{d}_{Rj}, p_{Rj}).$$  

(4.6)

In particular, when we have LCP incidence, then

$$g^s_L(\hat{r}) = g^s_L(\hat{r}; \hat{d}_{Lj}, p_{Lj}), \quad g^s_R(\hat{r}) = g^s_R(\hat{r}; \hat{d}_{Lj}, p_{Lj})$$

(4.7)
and when we have RCP incidence, then
\[ g^e_{Lj}(\mathbf{r}) = g^e_L(\mathbf{r}; \mathbf{d}_{Rj}, \mathbf{p}_{Rj}), \quad g^e_R(\mathbf{r}) = g^e_R(\mathbf{r}; \mathbf{d}_{Rj}, \mathbf{p}_{Rj}). \] (4.8)

For two vector functions \( \mathbf{u} \) and \( \mathbf{v} \) we introduce the notation
\[ I_S(\mathbf{u}, \mathbf{v}) = \int_S \mathbf{n} \cdot (\mathbf{u} \times \nabla \times \mathbf{v} - \mathbf{v} \times \nabla \times \mathbf{u}) \, ds - 2\beta \gamma^2 \lambda \int_S \mathbf{n} \cdot (\mathbf{u} \times \mathbf{v}) \, ds, \] (4.9)
where \( S \) is the surface of the scatterer \( \Omega^- \) and \( \mathbf{n} \) is the outward normal unit vector on \( S \). In what follows, \( \mathbf{w} \) denotes the complex conjugation of \( \mathbf{w} \). Some relations between \( I_S(\cdot, \cdot) \) and the far field patterns are proved in the following lemmas.

**Lemma 4.1.** Let \( \mathbf{E}^i_1, \mathbf{E}^i_2 \) be two plane electric waves incident upon the scatter \( \Omega^- \) and \( \mathbf{E}^e_1, \mathbf{E}^e_2 \) be the corresponding scattered fields. Then we have
\[ I_S(\mathbf{E}^i_1, \mathbf{E}^i_2) = 0, \] (4.10)
\[ I_S(\mathbf{E}^i_1, \mathbf{E}^e_2) = \frac{2i\gamma^2}{\kappa} \int_{S} \left[ \frac{1}{\gamma^2} \mathbf{g}^e_{L1}(\mathbf{r}) \cdot \mathbf{g}^e_{L2}(\mathbf{r}) + \frac{1}{\gamma^2} \mathbf{g}^e_{R1}(\mathbf{r}) \cdot \mathbf{g}^e_{R2}(\mathbf{r}) \right] \, ds(\mathbf{r}). \] (4.11)

**Proof.** We consider a sphere \( S_r \) centered at the origin with radius \( r \) large enough to include the scatterer in its interior. The scattered fields \( \mathbf{E}^e_1, \mathbf{E}^e_2 \) are regular solutions of (2.6) in the region exterior to \( S \) and interior to the surface \( S_r \), with \( r \gg 1 \). Hence we can use the vector second Green’s theorem on the first integral and Gauss’ theorem on the second integral to transform \( I_S(\mathbf{E}^i_1, \mathbf{E}^e_2) \), over the surface \( S \) of the scatterer, to \( I_S(\mathbf{E}^e_1, \mathbf{E}^e_2) \) over the surface of the sphere \( S_r \), i.e.,
\[ I_S(\mathbf{E}^i_1, \mathbf{E}^e_2) = I_{S_r}(\mathbf{E}^e_1, \mathbf{E}^e_2), \] (4.12)
\[ I_S(\mathbf{E}^i_1, \mathbf{E}^e_2) = I_{S_r}(\mathbf{E}^i_1, \mathbf{E}^e_2). \] (4.13)

Letting \( r \to \infty \), we pass to the radiation zone and thus we can use the asymptotic form (4.4). Then, in view of the formulae (3.4) and (3.9) we conclude to (4.10). Using again (3.9) and the relations
\[ \mathbf{g}^e_{Lj}(\mathbf{r}) \cdot \mathbf{g}^e_{Rj}(\mathbf{r}) = 0, \quad j = 1, 2, \] (4.14)
we obtain (4.11). \( \square \)

**Lemma 4.2.** Under the hypotheses of Lemma 4.1 the following relations are valid:
\[ I_S(\mathbf{E}^i_1(\cdot; \mathbf{d}_{A1}, \mathbf{p}_{A1}), \mathbf{E}^e_2) = \frac{4i\pi \gamma^2}{\kappa \gamma A} \mathbf{p}_{A1} \cdot \mathbf{g}^e_{A2}(-\mathbf{d}_{A1}), \] (4.15)
\[ I_S(\mathbf{E}^i_1(\cdot; \mathbf{d}_{A2}, \mathbf{p}_{A2}), \mathbf{E}^e_1) = \frac{4i\pi \gamma^2}{\kappa \gamma A} \mathbf{p}_{A2} \cdot \mathbf{g}^e_{A1}(-\mathbf{d}_{A2}), \] (4.16)
\[ I_S(\mathbf{E}^i_1(\cdot; \mathbf{d}_{A1}, \mathbf{p}_{A1}), \mathbf{E}^e_2) = \frac{4i\pi \gamma^2}{\kappa \gamma A} \mathbf{p}_{A1} \cdot \mathbf{g}^e_{A2}(\mathbf{d}_{A1}), \] (4.17)
\[ I_S(\mathbf{E}_1^i, \mathbf{E}_A^{i+}(\hat{\mathbf{d}}_{A2}, \mathbf{p}_{A2})) = \frac{4i\pi}{\kappa} \gamma \mathbf{p}_{A2} \cdot \mathbf{g}_{A1}(\hat{\mathbf{d}}_{A2}). \]  \tag{4.18}

for \( A = L, R \).

**Proof.** The above relations result from (3.21) and (3.22) taking into account the formulae

\[ \mathbf{p}_{Aj} \cdot \hat{\mathbf{K}}_A(\mathbf{d}_{A2}) = 2\mathbf{p}_{Aj}, \quad A = L, R, \quad j = 1, 2, \tag{4.19} \]

and the equations

\[ \nabla \times \mathbf{E}_L^i = \gamma_L \mathbf{E}_L^i, \quad \nabla \times \mathbf{E}_R^i = -\gamma_R \mathbf{E}_R^i. \tag{4.20} \]

Also we use the fact that

\[ \mathbf{E}_A^i(\mathbf{r}'; -\hat{\mathbf{d}}_{Aj}, \mathbf{p}_{Aj}) = \mathbf{E}_A^i(\mathbf{r}'; \hat{\mathbf{d}}_{Aj}, \mathbf{p}_{Aj}), \quad A = L, R, \quad j = 1, 2. \tag{4.21} \]

**Lemma 4.3.** Under the hypotheses of Lemma 4.1 we have

\[ I_S(\mathbf{E}_1^i, \mathbf{E}_2^s) = I_S(\mathbf{E}_2^i, \mathbf{E}_1^s). \tag{4.22} \]

**Proof.** In view of (2.1) and the bilinearity of \( I_S(\cdot, \cdot) \), we have

\[ I_S(\mathbf{E}_1^i, \mathbf{E}_2^s) = I_S(\mathbf{E}_1^i, \mathbf{E}_2^s) + I_S(\mathbf{E}_1^i, \mathbf{E}_2^s) + I_S(\mathbf{E}_1^i, \mathbf{E}_2^s) + I_S(\mathbf{E}_1^i, \mathbf{E}_2^s). \tag{4.23} \]

For the boundary conditions (2.8) and (2.9) it is clear that

\[ I_S(\mathbf{E}_1^i, \mathbf{E}_2^s) = 0. \tag{4.24} \]

For the transmission conditions (2.10) and (2.11) we have

\[ I_S(\mathbf{E}_1^i, \mathbf{E}_2^s) = \frac{\varepsilon_-}{\varepsilon} \left( \frac{\gamma}{\gamma_-} \right)^2 I_S(\mathbf{E}_1^i, \mathbf{E}_2^-). \tag{4.25} \]

Now, applying in \( \Omega^- \) the vector second Green’s theorem for the first integral and the Gauss’ theorem for the second integral of \( I_S(\cdot, \cdot) \) and taking into account that both \( \mathbf{E}_1^i, \mathbf{E}_2^- \) are solutions of (2.6) in \( \Omega^- \), we conclude to (4.24). On the other hand, since the incident plane waves \( \mathbf{E}_1^i, \mathbf{E}_2^- \) are solutions of (2.6) without singularities in \( \Omega^- \), working as before, we take

\[ I_S(\mathbf{E}_1^i, \mathbf{E}_2^-) = 0. \tag{4.26} \]

Consequently, due to (4.10), (4.24) and (4.26), the relation (4.23) becomes

\[ I_S(\mathbf{E}_1^i, \mathbf{E}_2^s) = -I_S(\mathbf{E}_1^i, \mathbf{E}_2^s) = I_S(\mathbf{E}_2^i, \mathbf{E}_1^s), \tag{4.27} \]

which completes the proof of lemma. \( \square \)

Let us note that all the previous lemmas are valid independently of the boundary conditions that are imposed on the surface of the scatterer \( \Omega^- \). A simple consequence of Lemmas 4.2 and 4.3 and the relation (4.1) is the known reciprocity principle [15] for chiral media.
Theorem 4.1. Under the hypotheses of Lemma 4.1 the following reciprocity relation holds true:
\[
\frac{1}{\gamma_L} p_{L1} \cdot g_{L1}^e (-\hat{d}_{L2}) + \frac{1}{\gamma_R} p_{R2} \cdot g_{R1}^e (-\hat{d}_{R2}) = \frac{1}{\gamma_L} p_{L1} \cdot g_{L2}^e (-\hat{d}_{L1}) + \frac{1}{\gamma_R} p_{R1} \cdot g_{R2}^e (-\hat{d}_{R1}). \tag{4.28}
\]

Applying Lemmas 4.2 and 4.3 for either LCP or RCP incidence and taking into account Remark 4.1 we obtain the following more specific reciprocity relations.

Corollary 4.1. Under the hypotheses of Lemma 4.1 the following reciprocity relations are valid:

- If both $\mathbf{E}_i^1$ and $\mathbf{E}_i^2$ are LCP or RCP incident plane electric waves, then
\[
p_{A2} \cdot g_{A}^e (-\hat{d}_{A2}; \hat{d}_{A1}, \mathbf{p}_{A1}) = p_{A1} \cdot g_{A}^e (-\hat{d}_{A1}; \hat{d}_{A2}, \mathbf{p}_{A2}), \quad A = L, R. \tag{4.29}
\]

- If $\mathbf{E}_i^1$ is an LCP and $\mathbf{E}_i^2$ is an RCP incident plane electric wave, then
\[
\frac{1}{\gamma_L} p_{L1} \cdot g_{L2}^e (-\hat{d}_{L1}; \hat{d}_{R2}, \mathbf{p}_{R2}) = \frac{1}{\gamma_R} p_{R2} \cdot g_{R1}^e (-\hat{d}_{R2}; \hat{d}_{L1}, \mathbf{p}_{L1}). \tag{4.30}
\]

Now, we prove a “general scattering theorem” which is a connection of the electric far field patterns for two LCP and two RCP directions and an integral over directions of both LCP and RCP electric far field patterns. This provides a generalization to chiral materials of results obtained in [7] for achiral materials.

Theorem 4.2. Under the hypotheses of Lemma 4.1 and the assumption that the boundary condition (2.8) or the transmission conditions (2.10), (2.11) are satisfied on $S$, the following relation is valid:
\[
\frac{1}{\gamma_L} \left[ p_{L1} \cdot g_{L2}^e (\hat{d}_{L1}) + p_{L2} \cdot g_{L1}^e (\hat{d}_{L2}) \right] + \frac{1}{\gamma_R} \left[ p_{R1} \cdot g_{R2}^e (\hat{d}_{R1}) + p_{R2} \cdot g_{R1}^e (\hat{d}_{R2}) \right] = -\frac{1}{2\pi} \int_{S^2} \left[ \frac{1}{\gamma_L} \overline{g_{L1}^e (\hat{r}) \cdot g_{L2}^e (\hat{r})} + \frac{1}{\gamma_R} \overline{g_{R1}^e (\hat{r}) \cdot g_{R2}^e (\hat{r})} \right] ds(\hat{r}). \tag{4.31}
\]

Proof. As in Lemma 4.3 we have the analysis
\[
I_5(\overline{\mathbf{E}_1^T}, \mathbf{E}_2^T) = I_5(\mathbf{E}_1^T, \mathbf{E}_2^T) + I_5(\mathbf{E}_1^T, \mathbf{E}_2^T) + I_5(\mathbf{E}_1^T, \mathbf{E}_2^T) + I_5(\mathbf{E}_1^T, \mathbf{E}_2^T). \tag{4.32}
\]

For the boundary condition (2.8) and transmission conditions (2.10), (2.11) again we have
\[
I_5(\overline{\mathbf{E}_1^T}, \mathbf{E}_2^T) = 0. \tag{4.33}
\]

Also, for the incident plane waves $\overline{\mathbf{E}_1^T}$ and $\mathbf{E}_2^T$, as in Lemma 4.3, it is valid
\[
I_5(\mathbf{E}_1^T, \mathbf{E}_2^T) = 0. \tag{4.34}
\]
In view of (4.1), (4.33) and (4.34), relation (4.32) implies
\[
I_S(E_L^i(:d_L1, p_L1), E_L^2) + I_S(E_R^i(:d_R1, p_R1), E_R^2) + I_S(E_L^i(:d_L2, p_L2))
\]
\[
+ I_S(E_R^i(:d_R2, p_R2)) = -I_S(E_L^1, E_L^2).
\]
(4.35)

Inserting (4.11) and (4.17), (4.18) for \(A = L, R\), into (4.35), the relation (4.31) is derived.

For the impedance boundary condition the following general scattering theorem is valid.

**Theorem 4.3.** Under the hypotheses of Lemma 4.1 and the assumption that the impedance boundary condition (2.9) is satisfied on \(S\), the following relation holds true:

\[
\frac{1}{\gamma_L} [\hat{p}_{L1} \cdot \hat{g}_{L2}(\hat{d}_{L1}) + \hat{p}_{L2} \cdot \hat{g}_{L1}(\hat{d}_{L2})] + \frac{1}{\gamma_R} [\hat{p}_{R1} \cdot \hat{g}_{R2}(\hat{d}_{R1}) + \hat{p}_{R2} \cdot \hat{g}_{R1}(\hat{d}_{R2})]
\]
\[
= -\frac{1}{2\pi} \int_{S^2} \frac{\text{Re}(z_{e})}{|z_{e}|^2} \left( \hat{n} \times \hat{E}_{2}^{r} (r') \right) \cdot \left( \hat{n} \times \hat{E}_{1}^{i} (r') \right) ds(r').
\]
(4.36)

**Proof.** As in the proof of Theorem 4.2, we use the analysis (4.32). In view of the impedance boundary condition (2.9) we have

\[
I_S(E_L^1, E_L^2) = \frac{-2i\gamma^2}{\kappa} \int_{S} \frac{\text{Re}(z_{e})}{|z_{e}|^2} \left( \hat{n} \times \hat{E}_{2}^{r} (r') \right) \cdot \left( \hat{n} \times \hat{E}_{1}^{i} (r') \right) ds(r').
\]
(4.37)

We replace (4.33) by (4.37) and the theorem is proved. \(\square\)

Now, we consider either LCP or RCP incidence. In this case we will use the notation

\[
I_{S^2}(\hat{a}_1, b_1; \hat{a}_2, b_2) = \int_{S^2} \left[ \frac{1}{\gamma_L} \hat{g}_{L}^{r}(\hat{r}; \hat{a}_1, b_1) \cdot \hat{g}_{L}^{i}(\hat{r}; \hat{a}_2, b_2) 
\right.
\]
\[
\left. + \frac{1}{\gamma_R} \hat{g}_{R}^{r}(\hat{r}; \hat{a}_1, b_1) \cdot \hat{g}_{R}^{i}(\hat{r}; \hat{a}_2, b_2) \right] ds(\hat{r}).
\]
(4.38)

where \(\hat{a}_j, j = 1, 2\), represent the propagation vectors of \(E_j^{i}\) and \(b_j, j = 1, 2\), their polarizations. Then from Theorems 4.2 and 4.3 the following general scattering relations are derived.

**Corollary 4.2.** Under the hypotheses of Theorem 4.2 we have:
– If both $E_{1}^{i}$ and $E_{2}^{i}$ are LCP or RCP incident plane electric waves, then for $A = L, R$ we have
\[
\frac{1}{\gamma^{A}} \left[ \hat{p}_{A1} \cdot g^{A}_{A}(\hat{d}_{A1}; \hat{d}_{A2}, p_{A2}) + p_{A2} \cdot g^{A}_{A}(\hat{d}_{A2}; \hat{d}_{A1}, p_{A1}) \right] \nonumber \\
= - \frac{1}{2\pi} I_{S^{2}}(\hat{d}_{A1}, p_{A1}; \hat{d}_{A2}, p_{A2}). \tag{4.39}
\]

– If $E_{1}^{i}$ is an LCP and $E_{2}^{i}$ is an RCP incident plane electric wave, then
\[
\frac{1}{\gamma^{L}} \hat{p}_{L1} \cdot g^{L}_{L}(\hat{d}_{L1}; \hat{d}_{R2}, p_{R2}) + \frac{1}{\gamma^{R}} p_{R2} \cdot g^{R}_{R}(\hat{d}_{R2}; \hat{d}_{L1}, p_{L1}) \nonumber \\
= - \frac{1}{2\pi} I_{S^{2}}(\hat{d}_{L1}, p_{L1}; \hat{d}_{R2}, p_{R2}). \tag{4.40}
\]

**Corollary 4.3.** Under the hypotheses of Theorem 4.3 we have:

– If both $E_{1}^{i}$ and $E_{2}^{i}$ are LCP or RCP incident plane electric waves, then
\[
\frac{1}{\gamma^{A}} \left[ \hat{p}_{A1} \cdot g^{A}_{A}(\hat{d}_{A1}; \hat{d}_{A2}, p_{A2}) + p_{A2} \cdot g^{A}_{A}(\hat{d}_{A2}; \hat{d}_{A1}, p_{A1}) \right] \nonumber \\
= - \frac{1}{2\pi} I_{S^{2}}(\hat{d}_{A1}, p_{A1}; \hat{d}_{A2}, p_{A2}) \\
- \frac{1}{2\pi} \int_{S} \Re (\hat{\nu} \times E_{1}^{i}(\hat{r}')) \cdot (\hat{\nu} \times E_{2}^{i}(\hat{r}')) \, ds(\hat{r}'). \tag{4.41}
\]

– If $E_{1}^{i}$ is an LCP and $E_{2}^{i}$ is an RCP incident plane electric wave, then
\[
\frac{1}{\gamma^{L}} \hat{p}_{L1} \cdot g^{L}_{L}(\hat{d}_{L1}; \hat{d}_{R2}, p_{R2}) + \frac{1}{\gamma^{R}} p_{R2} \cdot g^{R}_{R}(\hat{d}_{R2}; \hat{d}_{L1}, p_{L1}) \nonumber \\
= - \frac{1}{2\pi} I_{S^{2}}(\hat{d}_{L1}, p_{L1}; \hat{d}_{R2}, p_{R2}) \\
- \frac{1}{2\pi} \int_{S} \Re (\hat{\nu} \times E_{1}^{i}(\hat{r}')) \cdot (\hat{\nu} \times E_{2}^{i}(\hat{r}')) \, ds(\hat{r}'). \tag{4.42}
\]

5. Cross sections

We now consider either an LCP or an RCP plane electric wave $E_{A}^{i}(r, \hat{d}_{A}, p_{A})$, $A = L, R$, incident upon the scatterer $\Omega^{-}$. The scattering cross section $\sigma_{A}^{s}$ expresses the scattered power and is defined by [15, p. 481]
\[
\sigma_{A}^{s} = \frac{2\sqrt{\mu}}{|p_{A}|^{2} \sqrt{\varepsilon}} (P^{s}), \tag{5.1}
\]
where
\[
\langle P_s \rangle = \frac{1}{2} \text{Re} \int_{S} \hat{n} \cdot (E^s \times H^s) \, ds
\]
(5.2)
is the time-averaged scattered power [15, p. 194]. Taking into account that \( E^s, H^s \) satisfy (2.3), after some calculations, we find
\[
\sigma_s^A = -\frac{\kappa}{\gamma^2 |p_A|^2} \text{Im} \int_{S} (\hat{n} \times E^s) \cdot (\nabla \times E^s - \beta \gamma^2 E^s) \, ds.
\]
(5.3)
As in Lemma 4.1, we consider a sphere \( S_r \) centered at the origin with radius \( r \gg 1 \). Applying Gauss’ theorem in the region between \( S \) and \( S_r \) we take
\[
\sigma_s^A = -\frac{\kappa}{\gamma^2 |p_A|^2} \text{Im} \int_{S_r} (\hat{n} \times E^s) \cdot (\nabla \times E^s - \beta \gamma^2 E^s) \, ds.
\]
(5.4)
For \( r \to \infty \) we can use the asymptotic form (3.6) and obtain
\[
\sigma_s^A = \frac{1}{|p_A|^2} \int_{S} \left[ \frac{1}{\gamma_L^2} |\mathbf{g}_L^e(\hat{r}_A, \hat{d}_A, p_A)|^2 + \frac{1}{\gamma_R^2} |\mathbf{g}_R^e(\hat{r}_A, \hat{d}_A, p_A)|^2 \right] \, ds(\hat{r}).
\]
(5.5)
The absorption cross section \( \sigma_a^A \) defines the total energy absorbed by a lossy scatterer and is given by
\[
\sigma_a^A = 2 \sqrt{\mu |p_A|^2} \langle P_a \rangle,
\]
(5.6)
where
\[
\langle P_a \rangle = -\frac{1}{2} \text{Re} \int_{S} \hat{n} \cdot (E^t \times H^t) \, ds
\]
(5.7)
is the time-average absorbed power. As in the scattering cross section, we find
\[
\sigma_a^A = -\frac{\kappa}{\gamma^2 |p_A|^2} \text{Im} \int_{S} (\hat{n} \times E^t) \cdot (\nabla \times E^t - \beta \gamma^2 E^t) \, ds.
\]
(5.8)
The extinction cross section \( \sigma_e^A \) is given by
\[
\sigma_e^A = \sigma_s^A + \sigma_a^A
\]
(5.9)
and describes the total energy that the scatterer extracts from the incident wave either by radiation or by absorption.

For a perfect conductor it is clear that \( \sigma_a^A = 0 \). Also, for a dielectric, inserting the transmission conditions (2.10) and (2.11) into (5.8) and applying the divergence theorem in the region \( \Omega^- \) we take \( \sigma_a^A = 0 \). In the case of an impedance boundary condition the surface of the scatterer absorbs energy. Inserting the boundary condition (2.9) into (5.8) we conclude that
\[
\sigma_a^A = \frac{1}{|p_A|^2} \int_{S} \frac{\text{Re}(z_s)}{|z_s|^2} |\hat{n} \times E^t|^2 \, ds.
\]
(5.10)
Now, we can prove the following optical theorems.

**Theorem 5.1.** If \( E^i_A(\mathbf{r}; \hat{d}_A, \mathbf{p}_A) \), \( A = L, R \), is a plane electric wave incident upon a perfect conductor or a dielectric, then

\[
\sigma^e_A = -\frac{4\pi}{\gamma^2_A |\mathbf{p}_A|^2} \text{Re}\{\mathbf{p}_A \cdot \mathbf{g}^e_A(\hat{d}_A; \hat{d}_A, \mathbf{p}_A)\}. \tag{5.11}
\]

**Proof.** Applying (4.39) for \( \hat{d}_{A1} = \hat{d}_{A2} = \hat{d}_A \) and \( \mathbf{p}_{A1} = \mathbf{p}_{A2} = \mathbf{p}_A \) and taking into account the formula (5.5) the theorem is proved. \( \square \)

**Theorem 5.2.** If \( E^i_A(\mathbf{r}; \hat{d}_A, \mathbf{p}_A) \), \( A = L, R \), is a plane electric wave incident upon a scatterer with an impedance boundary condition, then

\[
\sigma^e_A = -\frac{4\pi}{\gamma^2_A |\mathbf{p}_A|^2} \text{Re}\{\mathbf{p}_A \cdot \mathbf{g}^e_A(\hat{d}_A; \hat{d}_A, \mathbf{p}_A)\}. \tag{5.12}
\]

**Proof.** The relation (4.41) for \( \hat{d}_{A1} = \hat{d}_{A2} = \hat{d}_A \) and \( \mathbf{p}_{A1} = \mathbf{p}_{A2} = \mathbf{p}_A \) with (5.9) and (5.10) prove the theorem. \( \square \)

### 6. Reduction to achiral case

When the scatterer lies in an achiral environment, i.e., \( \beta = 0 \) in \( \Omega \), then we have \( \gamma_L = \gamma_R = \gamma = \kappa \) and the covering equations (2.3), (2.4) become

\[
\nabla \times \mathbf{E}' = i\omega\mu \mathbf{H}', \quad \nabla \times \mathbf{H}' = -i\omega \varepsilon \mathbf{E}' \quad \text{in} \quad \Omega \tag{6.1}
\]

where now \( \kappa = \omega \sqrt{\varepsilon \mu} \) is a wavenumber. The asymptotic relation (3.6) takes the form

\[
\mathbf{E}'(\mathbf{r}) = h(\kappa r)\left[\mathbf{g}^e_L(\hat{\mathbf{r}}) + \mathbf{g}^e_R(\hat{\mathbf{r}})\right] + O\left(\frac{1}{r^2}\right), \quad r \to \infty. \tag{6.2}
\]

In (6.2), by definition, \( \mathbf{g}^e_L + \mathbf{g}^e_R \) is the electric far field pattern for electromagnetic scattering in an achiral environment [7]. In fact, from (3.7) and (3.8) for \( \beta = 0 \) we have

\[
\mathbf{g}^e_L(\hat{\mathbf{r}}) + \mathbf{g}^e_R(\hat{\mathbf{r}}) = \frac{i\kappa}{4\pi} \left[\hat{\mathbf{I}} - \hat{\mathbf{r}}\hat{\mathbf{r}}\right] \cdot \int_{S} \left[\hat{\mathbf{n}} \times \nabla \times \mathbf{E}'(\mathbf{r}') + i\kappa \hat{\mathbf{r}} \times \left(\hat{\mathbf{n}} \times \mathbf{E}'(\mathbf{r}')\right)\right] e^{-i\kappa \hat{\mathbf{r}} \cdot \mathbf{r}'} ds(\mathbf{r'}). \tag{6.3}
\]

The right-hand side of (6.3) is the achiral electric far field pattern \( \mathbf{g}^e \) [7, p. 59]. Hence

\[
\mathbf{g}^e_L(\hat{\mathbf{r}}) + \mathbf{g}^e_R(\hat{\mathbf{r}}) = \mathbf{g}^e(\hat{\mathbf{r}}). \tag{6.4}
\]

Similarly, for the achiral magnetic far field pattern \( \mathbf{g}^m \) we have

\[
\mathbf{g}^m_L(\hat{\mathbf{r}}) + \mathbf{g}^m_R(\hat{\mathbf{r}}) = \mathbf{g}^m(\hat{\mathbf{r}}). \tag{6.5}
\]
Also, from (3.35), (6.4) and (6.5) we obtain
\[ g_L(\hat{r}) - g_R(\hat{r}) = i\eta g^m(\hat{r}), \quad g_L^m(\hat{r}) - g_R^m(\hat{r}) = \frac{1}{i\eta} g^e(\hat{r}). \] (6.6)

From (3.9), (3.33), (3.36), (3.37), (6.4) and (6.5) we take the well-known relations [4,7]
\[ \hat{r} \cdot g^e(\hat{r}) = \hat{r} \cdot g^m(\hat{r}) = 0, \] (6.7)
\[ \hat{r} \times g^e(\hat{r}) = i\eta g^m(\hat{r}), \quad \hat{r} \times g^m(\hat{r}) = -\frac{1}{i\eta} g^e(\hat{r}). \] (6.8)

When \( \beta = 0 \), the incident electric waves are \( E'_i(r) = E'_i(\hat{d}_j, p_j), j = 1, 2, \) where \( p_j = p_{Lj} = p_{Rj} \) and \( \hat{d}_j = \hat{d}_{Lj} = \hat{d}_{Rj} \). Theorem 4.1 becomes
\[ p_2 \cdot g^e(-\hat{d}_2; \hat{d}_1, p_1) = p_1 \cdot g^e(-\hat{d}_1; \hat{d}_2, p_2), \] (6.9)
which is the known reciprocity principle for achiral electromagnetic scattering [4,7]. Similarly, Theorem 4.2 becomes
\[ \bar{p}_1 \cdot g^e(\hat{d}_1; \hat{d}_2, p_2) + p_2 \cdot g^e(\hat{d}_2; \hat{d}_1, p_1) = -\frac{1}{2\pi} \int_{S^2} g^e(\hat{r}; \hat{d}_1, p_1) \cdot g^e(\hat{r}; \hat{d}_2, p_2) ds(\hat{r}), \] (6.10)
which is the general scattering theorem for electromagnetic scattering in an achiral medium [2,7].

7. Conclusion

We have considered the scattering of electromagnetic waves in a chiral medium by various kind of scatterers. Using the integral representation for the scattered electric field in accordance with the Huygens’s principle for chiral media we have constructed the electric far field patterns and applying the optical theorem we have evaluated the corresponding scattering cross sections for LCP or RCP incidence. We note that the far field patterns are the most important functions in scattering theory and they play a significant role in solving inverse scattering problems. The main results of this work are the general scattering theorems which are generalizations of the corresponding theorems for achiral media [1,2,7,17,18]. These theorems are very useful in scattering theory.

We refer to some known applications for achiral cases which we think that can be extended to chiral cases. For scatterers with inversion symmetry, that is, scatterers for which \( r \in S \) implies \( -r \in S \), general scattering theorems are used in low frequency theory to approximate the far field patterns [7, p. 87]. Another application is in the study of the far field operator which plays a central role in the dual space method for solving inverse electromagnetic scattering problems [2], [4, p. 275]. In particular, assuming that the incident waves in the general scattering theorem are Herglotz functions we conclude that the far field operator has a countable number of eigenvalues which lie on a suitable circle. As
we have also seen, the optical theorem is a corollary of the general scattering theorem for appropriate directions of incidence and polarizations.

Finally, we note that the results obtained were checked against known results by reducing the chiral problem to the achiral one as the chirality measure becomes zero.

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