



Isomorphic properties of intersection bodies

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Abstract

We study isomorphic properties of two generalizations of intersection bodies – the class I_k^n of k -intersection bodies in \mathbb{R}^n and the class \mathcal{BP}_k^n of generalized k -intersection bodies in \mathbb{R}^n . In particular, we show that all convex bodies can be in a certain sense approximated by intersection bodies, namely, if K is any symmetric convex body in \mathbb{R}^n and $1 \leq k \leq n - 1$ then the outer volume ratio distance from K to the class \mathcal{BP}_k^n can be estimated by

$$\text{o.v.r.}(K, \mathcal{BP}_k^n) := \inf \left\{ \left(\frac{|C|}{|K|} \right)^{\frac{1}{n}} : C \in \mathcal{BP}_k^n, K \subseteq C \right\} \leq c \sqrt{\frac{n}{k} \log \frac{en}{k}},$$

where $c > 0$ is an absolute constant. Next we prove that if K is a symmetric convex body in \mathbb{R}^n , $1 \leq k \leq n - 1$ and its k -intersection body $I_k(K)$ exists and is convex, then

$$d_{BM}(I_k(K), B_2^n) \leq c(k),$$

where $c(k)$ is a constant depending only on k , d_{BM} is the Banach–Mazur distance, and B_2^n is the unit Euclidean ball in \mathbb{R}^n . This generalizes a well-known result of Hensley and Borell. We conclude the paper with volumetric estimates for k -intersection bodies.

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1. Introduction

Let K be a symmetric star body in \mathbb{R}^n . Following Lutwak [35], we say that a body $I(K)$ is the *intersection body of K* if the radius of $I(K)$ in every direction is equal to the volume of the central hyperplane section of K perpendicular to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\rho_{I(K)}(\xi) = |K \cap \xi^\perp|,$$

where ξ^\perp is the central hyperplane perpendicular to ξ and $|\cdot|$ stands for the volume. A more general class of *intersection bodies* can be defined as the closure in the radial metric of the class of intersection bodies of star bodies (see [47, Theorem 5.9] for a direct proof and [12,31] for more on the subject).

Intersection bodies play an important role in the solution of the Busemann–Petty problem posed in [9] in 1956: suppose that K and L are origin symmetric convex bodies in \mathbb{R}^n so that, for every $\xi \in S^{n-1}$,

$$|K \cap \xi^\perp| \leq |L \cap \xi^\perp|.$$

Does it follow that $|K| \leq |L|$? The problem was completely solved in the end of the 90s, and the answer is affirmative if $n \leq 4$ and negative if $n \geq 5$. The solution has appeared as a result of work of many mathematicians (see [12, Chapter 8] or [31, Chapter 5] for details). A connection between intersection bodies and the Busemann–Petty problem was established by Lutwak [35]: the answer to the Busemann–Petty problem in \mathbb{R}^n is affirmative if and only if every symmetric convex body in \mathbb{R}^n is an intersection body.

A more general concept of a k -intersection body was introduced in [28,30]. For an integer k , $1 \leq k < n$ and star bodies K, D in \mathbb{R}^n , we say that D is the k -*intersection body of K* if for every $(n-k)$ -dimensional subspace H of \mathbb{R}^n ,

$$|D \cap H^\perp| = |K \cap H|.$$

The k -intersection body of K is unique, but for $k > 1$ it does not always exist. If the k -intersection body of K exists, we denote it by $I_k(K)$. Taking the closure in the radial metric of the class of all D s that appear as k -intersection bodies of star bodies, we define the class of k -*intersection bodies*. The class of k -intersection bodies is related to a certain generalization of the Busemann–Petty problem in the same way as intersection bodies are related to the original problem (see [28] for details; this generalization offers a condition on the volume of sections that allows us to compare the volumes of two bodies in arbitrary dimensions). We denote the class of k -intersection bodies in \mathbb{R}^n by \mathcal{I}_k^n . In [30] the first named author also gave a Fourier characterization of \mathcal{I}_k^n : $K \in \mathcal{I}_k^n$ if and only if $\|\cdot\|_K^{-k}$ is a positive definite distribution in \mathbb{R}^n .

Another generalization of intersection bodies was introduced by Zhang [52]. Let $1 \leq k \leq n - 1$. The spherical Radon transform R_{n-k} is an operator acting from $C(S^{n-1})$ to the space $C(G_{n,n-k})$ of continuous functions on the Grassmannian:

$$R_{n-k}f(H) = \int_{S^{n-1} \cap H} f(x) dx, \quad \forall f \in C(S^{n-1}), H \in G_{n,n-k}.$$

We say that a star body K is a *generalized k -intersection body* if there exists a linear positive functional μ on $R_{n-k}(C(S^{n-1}))$ such that for every $f \in C(S^{n-1})$,

$$\int_{S^{n-1}} \|x\|_K^{-k} f(x) dx = \mu(R_{n-k}(f)).$$

Following the notation of [39] we denote the class of all generalized k -intersection bodies in \mathbb{R}^n by \mathcal{BP}_k^n . A characterization of the class \mathcal{BP}_k^n was obtained by Grinberg and Zhang [18] as a generalization of the corresponding result of Goodey and Weil for the original intersection bodies [16]: a star body K belongs to the class \mathcal{BP}_k^n if and only if it is the limit (in the radial metric) of k -radial sums of ellipsoids. The class of generalized intersection bodies is related to the so-called lower dimensional Busemann–Petty problem (LDBP-problem). Suppose that $1 \leq k < n$ and symmetric convex bodies K, L in \mathbb{R}^n satisfy

$$|K \cap H| \leq |L \cap H|,$$

for every $(n - k)$ -dimensional subspace H of \mathbb{R}^n . Does it follow that $|K| \leq |L|$? It was proved in [52] that the answer to this question is affirmative if and only if every symmetric convex body in \mathbb{R}^n is a generalized k -intersection body. Using this, Bourgain and Zhang [6] (see also [48]) proved that for the dimensions of sections $n - k > 3$ the answer to the LDBP-problem is negative. Another proof of this result was given later in [30]. The problem is still open in the cases where the dimension of sections $n - k = 2, 3$.

In the case $k = 1$ the classes $\mathcal{I}_1^n = \mathcal{BP}_1^n$ coincide with the class of original intersection bodies. Also in the case $k = n - 1$ the two classes contain all symmetric star bodies in \mathbb{R}^n . It was proved in [29] (see also [31, Corollary 4.9]) that the class \mathcal{I}_{n-3}^n contains all symmetric convex bodies in \mathbb{R}^n , and this is no longer true for the classes \mathcal{I}_k^n with $k < n - 3$ ([28] or [31, Theorem 4.13]). It is known that $\mathcal{BP}_k^n \subseteq \mathcal{I}_k^n$ ([30]; see also [39] or [31, p. 92] for simpler proofs). The latter two results immediately imply the negative answer to the LDBP-problem with the dimension of sections greater than 3. The answer to the problem with two- or three-dimensional sections would be positive, if the classes \mathcal{BP}_k^n and \mathcal{I}_k^n with $k = n - 2$ or $k = n - 3$ were equal. However, Milman [40] proved that \mathcal{BP}_k^n is a proper subclass of \mathcal{I}_k^n for $2 \leq k \leq n - 2$. The example of Milman is not convex, so LDBP-problem is still open for two- and three-dimensional sections. Another open problem is whether the classes \mathcal{I}_k^n increase with k . It was proved by Yaskin [51] that for $k_1 < k_2$ there exists a symmetric convex body that belongs to $\mathcal{I}_{k_2}^n$ but not to $\mathcal{I}_{k_1}^n$. However, the inclusion $\mathcal{I}_{k_1}^n \subset \mathcal{I}_{k_2}^n$ is known only in the case where k_1 divides k_2 (see [39]). For more results on these classes of bodies see [32] and references there.

In spite of all these results, the isomorphic properties of intersection bodies are not very well understood. The first result of this kind was established by Hensley and Borell (see [20,4]): if K is symmetric and convex, then the Banach–Mazur distance $d_{BM}(I(K), B_2^n) \leq c$, where $c > 0$ is

an absolute constant, which means that intersection bodies of convex bodies are isomorphic to ellipsoids (note that if K is a symmetric convex body, then the classical result of Busemann [7] guaranties that $I(K)$ is also symmetric and convex) (see [23] for extension of the Hensley and Borell results to p -convex bodies). Busemann [8] also showed an “isoperimetric” type inequality: if K is symmetric convex and $|K| = 1$, then $|I(K)| \leq |I(D_n)|$, where D_n is the Euclidean ball with volume 1. This result can be extended to a class more general than convex bodies (even for Borel measurable sets) (see [17,13]). However, intersection bodies of convex bodies form only a small part of the class of convex intersection bodies.

As it was proved by the first named author [27] (using the Fourier characterization of the intersection bodies), the unit ball of any finite dimensional subspace of L_p , $p \in (0, 2]$, is a k -intersection body for every k , and in particular all polar projection bodies (unit balls of subspaces of L_1) are intersection bodies. The class of intersection bodies is strictly larger than the class of polar of projection bodies; see [26] for examples. A long standing question is if the two classes are “isomorphic”, i.e. whether for every intersection body I there exists a projection body Π , such that $d_{BM}(I, \Pi^\circ) \leq c$, where $c > 0$ is an absolute constant (this question is related to the 1970 problem of Kwapien [33] from the Banach space theory through the connection between intersection bodies and L_p -spaces; see [22]). A closely related result was proved in [22]: for any $q \in (0, 1)$ and any $k < n$, there exists a constant $c(q, k)$ such that for any convex k -intersection body K in \mathbb{R}^n there exists a subspace of L_q whose unit ball D satisfies $d_{BM}(K, D) < c(q, k)$. Note that the constant $c(q, k)$ goes to infinity when q tends to 1, and, if not for that, the case $q = 1$ would imply the desired result for polar projection bodies.

One of the most important open problems in convex geometry – the Hyperplane conjecture – also has isomorphic nature. One of the many formulations of the conjecture is the isomorphic Busemann–Petty problem which asks whether there exists an absolute constant c so that, for any symmetric convex bodies K, L in \mathbb{R}^n satisfying $|K \cap H| \leq |L \cap H|$, for all central hyperplanes H , we necessary have

$$|K| \leq c|L|.$$

Equivalently, the conjecture asks if there exists a uniform upper bound for the isotropic constant L_K (see Section 2 for definition). The best known bound is due to Klartag [24]: $L_K < cn^{1/4}$, (see also [25]). We refer to [41] and [14] for more information on the subject.

In this article we prove several isomorphic results for intersection bodies and their generalizations. We have already mentioned the fact that the class \mathcal{I}_{n-3}^n contains all symmetric convex bodies, but this is no longer the case for the classes \mathcal{I}_k^n with $k < n - 3$. We start with a result showing that k -intersection bodies with $k < n - 3$ are still in some sense dense in the class of all convex bodies.

Let K be a symmetric convex body in \mathbb{R}^n and let \mathcal{A} be a class of star bodies. We define the outer volume ratio distance by

$$\text{o.v.r.}(K, \mathcal{A}) := \inf\{\text{o.v.r.}(K, L), L \in \mathcal{A}\},$$

where

$$\text{o.v.r.}(K, L) := \inf\left\{\left(\frac{|TL|}{|K|}\right)^{\frac{1}{n}} : K \subseteq TL, T \in GL_n\right\}.$$

Theorem 1.1. *Let K be a symmetric convex body in \mathbb{R}^n and $1 \leq k \leq n - 1$. Then*

$$\text{o.v.r.}(K, \mathcal{BP}_k^n) \leq c \sqrt{\frac{n \log \frac{en}{k}}{k}}, \tag{1.1}$$

where $c > 0$ is an absolute constant.

Recall that $\mathcal{BP}_k^n \subseteq \mathcal{I}_k^n$, so the result also applies to k -intersection bodies.

Our second result extends to the class of k -intersection bodies the classical result of Hensley and Borell that an intersection body of a convex body is isomorphic to an ellipsoid. Here one faces two additional difficulties. First the k -intersection body of a convex body does not necessarily exist and, secondly, even if it exists it may not be convex. So any result must take into account these two conditions as additional assumptions. We prove

Theorem 1.2. *Let K be a symmetric convex body in \mathbb{R}^n , $1 \leq k \leq n - 1$, and assume that $I_k(K)$ exists and it is convex. Then*

$$d_{BM}(I_k(K), B_2^n) \leq c(k), \tag{1.2}$$

where $c(k)$ depends only on k .

Finally, we get some estimates for the volume radius of k -intersection bodies:

Theorem 1.3. *Let K be a symmetric star body in \mathbb{R}^n with $|K| = |B_2^n|$. Assume that the k -intersection body of K exists. Then*

$$\left(\frac{|I_k(K)|}{|I_k(B_2^n)|} \right)^{\frac{1}{n}} \geq \frac{L_{B_2^n}}{L_K}, \tag{1.3}$$

with equality if and only if K is a symmetric ellipsoid. Here L_K stands for the isotropic constant of K .

Moreover, if $I_k(K)$ is a convex body, we have that

$$\left(\frac{|I_k(K)|}{|I_k(B_2^n)|} \right)^{\frac{1}{n}} \leq c \log(1 + d_{BM}(I_k(K), B_2^n)) \leq c \min\{\log n, k \log k\}, \tag{1.4}$$

where $c > 0$ is a universal constant.

The paper is organized as follows: In Section 2 we introduce basic definitions and notation. In Sections 3, 4 and 5 we give the proof of Theorems 1.1, 1.2 and 1.3 respectively. We provide some final remarks in Section 6.

2. Notation and definitions

We work in \mathbb{R}^n , which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\| \cdot \|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1} . We will write D_n for the Euclidean ball of volume 1 ($D_n := |B_2^n|^{-\frac{1}{n}} B_2^n$). The Grassmannian manifold $G_{n,k}$ of k -dimensional subspaces of \mathbb{R}^n is equipped with the Haar probability measure $\mu_{n,k}$. We denote by GL_n the set of linear invertible transformations and SL_n for the measure preserving linear transformations.

A compact set K in \mathbb{R}^n is called a star body if the origin is an interior point of K , every straight line passing through the origin crosses the boundary of K at exactly two points and the Minkowski functional of K defined by

$$\|x\|_K = \min\{a \geq 0: x \in aK\}, \quad \forall x \in \mathbb{R}^n,$$

is a continuous function on \mathbb{R}^n . The radial function of K is given by

$$\rho_K(x) = \max\{a > 0: ax \in K\}.$$

Throughout this article we say that K is symmetric if $K = -K$. The radial metric on the class of star bodies is defined by

$$d_r(K, L) = \max_{\xi \in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)|.$$

If K, L are star bodies and $k \in \mathbb{N}$, then the k -radial sum of K and L is a star body D whose radial function is equal to

$$\rho_D = (\rho_K^k + \rho_L^k)^{1/k}.$$

If K is a convex body, we write h_K for the support function of K :

$$h_K(x) = \max_{\xi \in K} \langle x, \xi \rangle, \quad \forall x \in \mathbb{R}^n.$$

We define the geometric distance between two bodies K_1 and K_2 , $d_G(K_1, K_2)$, as the infimum of positive numbers r such that there exists some $a > 0$ so that

$$K_1 \subseteq aK_2 \subseteq rK_1.$$

The Banach–Mazur distance is defined as

$$d_{BM}(K_1, K_2) := \inf_{T \in GL_n} d_G(K_1, TK_2).$$

Let K be a star body in \mathbb{R}^n of volume 1. We define the isotropic constant of K, L_K as

$$L_K^2 := \frac{1}{n} \inf \left\{ \int_{TK} \|x\|_2^2 dx : T \in SL_n \right\}.$$

For two convex bodies K and L in \mathbb{R}^n , the covering number of K by L , denoted by $N(K, L)$, is defined as the minimal number of translates of L with their centers in K , needed to cover K .

The notation $a \simeq b$ means that there exist universal constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. We refer to the books [12,31,42,46,49] for basic facts from the Brunn–Minkowski theory, the asymptotic theory of finite dimensional normed spaces and intersection bodies.

3. On the outer volume ratio of convex bodies with respect to the class \mathcal{BP}_k^n

The main idea of the proof of Theorem 1.1 is to first cover a given convex body with Euclidean balls and then show that, if the number of balls is not too big, one can approximate the union of the balls by a body in \mathcal{BP}_k^n , where k will be related to the covering number of K .

We will use the following theorem for covering numbers of Pisier (see [46, p. 116]).

Theorem 3.1. *Let K be a symmetric convex body in \mathbb{R}^n and $\alpha \in (0, 2)$. Then there exists a $T \in GL_n$ such that if $K_1 = TK$ then*

- (1) $|K_1| = |B_2^n|$,
- (2) $\log N(K_1, tB_2^n) \leq \frac{cn}{t^\alpha(2-\alpha)}, \forall t \geq 1$, where $c > 0$ is a universal constant.

It is known that the position of the body K , constructed in the previous theorem, satisfies the reverse Brunn–Minkowski inequality of V. Milman [38]. For completeness we provide a proof.

Corollary 3.2. *Let K be a symmetric convex body with the properties of the body K_1 from the previous theorem. Then, for $t \geq 1$*

$$|K + tB_2^n|^{\frac{1}{n}} \leq ct|K|^{\frac{1}{n}}, \tag{3.1}$$

where $c > 0$ is an absolute constant.

Proof. Choose $\alpha = 1$. Then using Theorem 3.1, we have

$$\begin{aligned} \frac{|K + tB_2^n|^{\frac{1}{n}}}{t|K|^{\frac{1}{n}}} &= \frac{|K + tB_2^n|^{\frac{1}{n}}}{t|B_2^n|^{\frac{1}{n}}} \leq N(K + tB_2^n, 2tB_2^n)^{\frac{1}{n}} \\ &\leq N(K, tB_2^n)^{\frac{1}{n}} \leq c. \quad \square \end{aligned}$$

We will need the following elementary statement:

Lemma 3.3. *Let $z \in \mathbb{R}^n$ and $t > 0$. Then there exists a centered ellipsoid \mathcal{E} such that*

$$z + tB_2^n \subseteq \mathcal{E} \subseteq \text{co}\{2z + 2\sqrt{2}tB_2^n, -2z + 2\sqrt{2}tB_2^n\}. \tag{3.2}$$

Proof. We may assume that $z := z_1 e_1$. Let C be the “cylinder” defined as

$$C := \{(s, y) \in \mathbb{R}^n : |s| \leq z_1 + t, \|y\|_2 \leq t\}.$$

Then one can check that

$$z + tB_2^n \subseteq C \subseteq \text{co}\{z + \sqrt{2}tB_2^n, -z + \sqrt{2}tB_2^n\} =: K. \tag{3.3}$$

Let $Q := \{(s, y) \in \mathbb{R}^n : |s| \leq 1, \|y\|_2 \leq 1\}$. Then $C := TQ$, where $T \in GL_n$. Define $\mathcal{E}_1 := TB_2^n$. Then

$$B_2^n \subseteq Q \subseteq \sqrt{2}B_2^n \quad \text{or} \quad \mathcal{E}_1 \subseteq C \subseteq \sqrt{2}\mathcal{E}_1. \tag{3.4}$$

Then, by (3.3) we get that

$$z + tB_2^n \subseteq C \subseteq \sqrt{2}\mathcal{E}_1 \subseteq 2C \subseteq 2K.$$

We set $\mathcal{E} := \sqrt{2}\mathcal{E}_1$ and the proof is complete. \square

Note that the class \mathcal{BP}_k^n is closed under k -radial sums and that ellipsoids belong to this class for all $1 \leq k \leq n - 1$.

Let V_1, V_2 be two star bodies in \mathbb{R}^n . We define the distance between V_1 and V_2 as

$$d(V_1, V_2) := \sup_{x \in \mathbb{R}^n, x \neq 0} \left\{ \frac{\rho_{V_1}(x)}{\rho_{V_2}(x)}, \frac{\rho_{V_2}(x)}{\rho_{V_1}(x)} \right\}. \tag{3.5}$$

Observe that the definition implies that

$$\frac{1}{d}V_1 \subseteq V_2 \subseteq dV_1, \tag{3.6}$$

where $d := d(V_1, V_2)$.

Lemma 3.4. *Let $1 \leq k \leq n - 1$ and $\log N \leq k$. Let $V_i, i \leq N$, be symmetric star bodies in \mathcal{BP}_k^n and let $V = \bigcup_{i=1}^N V_i$. Then there exists $D \in \mathcal{BP}_k^n$ such that*

$$d(V, D) \leq e. \tag{3.7}$$

Proof. Let ρ_{V_i} be the corresponding radial functions of V_i . Then

$$\rho_V := \max_{i \leq N} \rho_{V_i}(x), \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{3.8}$$

For any $a \in \mathbb{R}^N$ we have that

$$\|a\|_\infty \leq \|a\|_k \leq N^{\frac{1}{k}} \|a\|_\infty \leq e \|a\|_\infty. \tag{3.9}$$

Let D be the star body defined by

$$\rho_D^k(x) := \sum_{i=1}^N \rho_{V_i}^k(x), \quad 0 \neq x \in \mathbb{R}^n. \tag{3.10}$$

Note that $D \in \mathcal{BP}_k^n$ and by (3.9) we also have that $d(V, D) \leq e$. \square

We can now give a proof of Theorem 1.1:

Proof of Theorem 1.1. Let $\alpha := 2 - \frac{1}{\log e \frac{n}{k}}$. Note that $\text{o.v.r.}(TK, \mathcal{BP}_k^n) = \text{o.v.r.}(K, \mathcal{BP}_k^n)$ for any $T \in GL_n$. So we may assume that K is in the position described in Theorem 3.1. We have that for every $t \geq 1$, there exists N , such that $\log N \leq \frac{cn}{t^\alpha(2-\alpha)}$, and $z_1, \dots, z_N \in K$ such that

$$K \subseteq \bigcup_{i=1}^N z_i + tB_2^n.$$

Given any z_i , let \mathcal{E}_i be as in Lemma 3.3. Then we have that

$$K \subseteq \bigcup_{i=1}^N \mathcal{E}_i \subseteq 2(\text{co}\{z_1, \dots, z_N\} + \sqrt{2}tB_2^n) \subseteq 2(K + \sqrt{2}tB_2^n).$$

Choose t such that $k = \frac{cn}{t^\alpha(2-\alpha)}$. Then $\log N \leq k$. So, by Lemma 3.4, there exists a $D \in \mathcal{BP}_k^n$ such that $d(D, V) \leq e$, where $V = \bigcup_{i=1}^N \mathcal{E}_i$. So,

$$\frac{1}{e}K \subseteq \frac{1}{e}V \subseteq D \subseteq eV \subseteq 2e(K + \sqrt{2}tB_2^n). \tag{3.11}$$

Recall that $t := (\frac{n}{k(2-\alpha)})^{\frac{1}{\alpha}} \simeq \sqrt{\frac{n}{k}} \sqrt{\log e \frac{n}{k}}$. Then (3.11) becomes

$$\frac{1}{e}K \subseteq D \subseteq 2e \left(K + c \sqrt{\frac{n}{k}} \sqrt{\log e \frac{n}{k}} B_2^n \right). \tag{3.12}$$

Moreover, by Corollary 3.2, we have that

$$|D|^{\frac{1}{n}} \leq 2e |K + tB_2^n|^{\frac{1}{n}} \leq ct |K|^{\frac{1}{n}}.$$

Let $D_1 := eD$. We have that $D_1 \in \mathcal{BP}_k^n$, $K \subseteq D_1$ and

$$\frac{|D_1|^{\frac{1}{n}}}{|K|^{\frac{1}{n}}} \leq c't \leq c'' \sqrt{\frac{n}{k}} \sqrt{\log e \frac{n}{k}}. \tag{3.13}$$

This finishes the proof. \square

4. Distances for k -intersection bodies

Let K be a star body in \mathbb{R}^n . Recall that the k -intersection body of K (if it exists) is a star body that satisfies

$$|I_k(K) \cap F| = |K \cap F^\perp|, \quad \forall F \in G_{n,k}. \tag{4.1}$$

Note that $I_k(K)$ (if it exists) is unique. Also if $t > 0$ and $T \in GL_n$, one has that

$$I_k(tK) = t^{\frac{n-k}{k}} I_k(K) \quad \text{and} \quad I_k(TK) = |\det T| (T^t)^{-1} (I_k(K)). \tag{4.2}$$

The last equality follows from the Fourier characterization of the k -intersection bodies (see [30]).

Let K be a symmetric convex body of volume 1 and $p \geq 1$. We define the L_p -centroid body of K [36,37,43], as the symmetric convex body that has support function

$$h_{Z_p(K)}(\theta) := \left(\int_K |\langle x, \theta \rangle|^p dx \right)^{\frac{1}{p}}. \tag{4.3}$$

We will use the following L_q -version of Rogers–Shephard inequality [43,44]:

Proposition 4.1. *Let K be a symmetric convex body of volume 1 in \mathbb{R}^n and $1 \leq k \leq n - 1$. Then for every $F \in G_{n,k}$,*

$$|K \cap F^\perp|^{\frac{1}{k}} |P_F Z_k(K)|^{\frac{1}{k}} \simeq 1. \tag{4.4}$$

We will also use the Santaló and reverse Santaló inequality [5]: If K is a symmetric convex body in \mathbb{R}^n , then

$$(|K| |K^\circ|)^{\frac{1}{n}} \simeq \frac{1}{n}. \tag{4.5}$$

Proposition 4.2. *Let K be a symmetric convex body in \mathbb{R}^n of volume 1 and $1 \leq k \leq n - 1$. Assume that $I_k(K)$ exists. Then for $F \in G_{n,k}$,*

$$|I_k(K) \cap F|^{\frac{1}{k}} \simeq \left| \left(\frac{Z_k(K)}{k} \right)^\circ \cap F \right|^{\frac{1}{k}}. \tag{4.6}$$

Proof. Using the definition (4.1) and Eqs. (4.4), (4.5), we get that

$$\begin{aligned} |I_k(K) \cap F|^{\frac{1}{k}} &= |K \cap F^\perp|^{\frac{1}{k}} \simeq |P_F Z_k(K)|^{-\frac{1}{k}} \\ &\simeq k |Z_k^\circ(K) \cap F|^{\frac{1}{k}} = \left| \left(\frac{Z_k(K)}{k} \right)^\circ \cap F \right|^{\frac{1}{k}}. \quad \square \end{aligned}$$

The following lemma is a well-known application of the Brunn–Minkowski inequality (see [41]):

Lemma 4.3. *Let K be a symmetric convex body of volume 1 in \mathbb{R}^n and let $k \geq 2$. Then*

$$Z_2(K) \subseteq Z_k(K) \subseteq ckZ_2(K), \tag{4.7}$$

where $c > 0$ is an absolute constant.

Note that $Z_2(K)$ is an ellipsoid. Moreover, since $Z_p(TK) = TZ_p(K)$ for $T \in SL_n$, there exists $T \in SL_n$ such that $Z_2(TK) := L_K B_2^n$. In this case we say that K is isotropic (see [41,14] for more information on isotropicity). Note that in the case where K is convex the definition of the isotropic constant L_K that we give here is equivalent to the definition given in Section 2 (see e.g. [14]).

We have the following

Corollary 4.4. *Let K be an isotropic convex body in \mathbb{R}^n , $1 \leq k \leq n - 1$ and assume that $I_k(K)$ exists. Then for all $F_1, F_2 \in G_{n,k}$,*

$$\frac{1}{(c_1k)^k} \leq \frac{|I_k(K) \cap F_1|}{|I_k(K) \cap F_2|} \leq (c_1k)^k, \tag{4.8}$$

where $c_1 > 1$ is an absolute constant.

Proof. Using Eqs. (4.6) and (4.7) we have that

$$\begin{aligned} \frac{|I_k(K) \cap F_1|}{|I_k(K) \cap F_2|} &\leq c^k \frac{|(\frac{Z_k(K)}{k})^\circ \cap F_1|}{|(\frac{Z_k(K)}{k})^\circ \cap F_2|} \\ &\leq c^k \frac{|(\frac{Z_2(K)}{k})^\circ \cap F_1|}{c_0^k |Z_2^\circ(K) \cap F_2|} \leq \frac{k^k c^k |Z_2^\circ(K) \cap F_1|}{c_0^k |Z_2^\circ(K) \cap F_2|} \leq (c_1k)^k, \end{aligned}$$

since K is isotropic. We work similarly for the left-hand side inequality. \square

We will also need the following

Lemma 4.5. *Let K be a symmetric convex body in \mathbb{R}^{m+1} . Let $r := r_K(e_{m+1})$ and $R := h_K(e_{m+1})$. Then*

$$\frac{2r}{m+1} |K \cap \mathbb{R}^m| \leq |K| \leq 2R |K \cap \mathbb{R}^m|. \tag{4.9}$$

Proof. To estimate the left-hand side observe that $re_{m+1} \in K$, so

$$|K| \geq |\text{co}\{K \cap \mathbb{R}^m, re_{m+1}, -re_{m+1}\}| = 2 \frac{r}{m+1} |K \cap \mathbb{R}^m|.$$

For the right-hand side, observe that the function $f(s) := |K \cap (se_{m+1} + e_{m+1}^\perp)|$ is even and log-concave by the Brunn–Minkowski inequality, therefore attains its maximum at 0. Then by Fubini’s theorem,

$$|K| = 2 \int_0^R f(s) ds \leq 2Rf(0) = 2R|K \cap \mathbb{R}^m|.$$

This finishes the proof. \square

Proposition 4.6. *Let K be a symmetric convex body in \mathbb{R}^n , $n \geq 3$ and $2 \leq k \leq n - 1$. Assume that there exists $\delta \geq 1$ such that for every $F_1, F_2 \in G_{n,k}$,*

$$\frac{|K \cap F_1|}{|K \cap F_2|} \leq \delta^k.$$

Then

$$d_G(K, B_2^n) \leq k\delta^k. \tag{4.10}$$

Proof. Let $R(K)$ and $r(K)$ be the circumradius and the inradius of K , respectively. Then $R(K) = \inf\{a > 0: K \subseteq aB_2^n\}$ and $r(K) = \sup\{b > 0: bB_2^n \subseteq K\}$. Note that there exist $\theta_1, \theta_2 \in S^{n-1}$ such that $R(K) = h_K(\theta_1) = \rho_K(\theta_1)$ and $r(K) = h_K(\theta_2) = \rho_K(\theta_2)$. It is enough to show that $\frac{R}{r} \leq k\delta^k$.

We may assume that $\theta_1 \neq \theta_2$, or else we have nothing to prove. Let $F_0 := \text{span}\{\theta_1, \theta_2\}$. Let $F \in G_{n,k-1}$, $F \perp F_0$, $F_1 := \text{span}\{\theta_1, F\}$ and $F_2 := \text{span}\{\theta_2, F\}$. Let $S := |K \cap F|$. Then by Lemma 4.5 we have that

$$\begin{aligned} \frac{2RS}{k} \leq |K \cap F_1| \leq 2RS \quad \text{and} \quad \frac{2rS}{k} \leq |K \cap F_2| \leq 2rS \quad \text{or} \\ \frac{R}{kr} \leq \frac{|K \cap F_1|}{|K \cap F_2|} \leq \frac{kR}{r}. \end{aligned} \tag{4.11}$$

Then $\frac{R}{kr} \leq \delta^k$. \square

Proof of Theorem 1.2. The case $k = 1$ is covered by the result of Hensley as noted in the Introduction. So we assume that $n \geq 3$ and $2 \leq k \leq n - 1$. Using (4.2) we may assume that K is isotropic. Then we want to show that $d_G(I_k(K), B_2^n) \leq k(ck)^k$, where $c > 0$ absolute constant. This follows from Corollary 4.4 and Proposition 4.6. \square

The dependence on k in Theorem 1.2 is very bad, $c(k) \leq k(ck)^k$. This becomes meaningless for $k \geq \frac{\log n}{\log \log n}$. We can give a (slightly) better bound (but still exponential) using certain tools that were developed in order to attack the Hyperplane conjecture. For a proof of the following lemma see [41].

Lemma 4.7. *Let K be an isotropic convex body in \mathbb{R}^n and let $1 \leq k \leq n - 1$. Then for any $F \in G_{n,k}$ there exists a symmetric convex body B in F such that $B := B(F)$ is also isotropic and*

$$|K \cap F^\perp|^{\frac{1}{k}} \simeq \frac{L_B}{L_K}. \tag{4.12}$$

Second proof of Theorem 1.2. Again we may assume that K is isotropic. We will use the best known bound for the isotropic constant due to B. Klartag [24] (see also [25]): for every B in \mathbb{R}^k ,

$$L_B \leq ck^{\frac{1}{4}}, \tag{4.13}$$

where $c > 0$ is a universal constant. Moreover, it is known (e.g. [41]) that $L_K \geq L_{B_2^n} \simeq 1$. So using (4.12) we get that for every $F_1, F_2 \in G_{n,k}$,

$$\frac{|I_k(K) \cap F_1|}{|I_k(K) \cap F_2|} = \frac{|K \cap F_1^\perp|}{|K \cap F_2^\perp|} \leq \left(c \frac{L_{B(F_1)}}{L_{B(F_2)}} \right)^k \leq (c'k)^{\frac{k}{4}}. \tag{4.14}$$

So, using Proposition 4.6 again we get that $d_G(K, B_2^n) \leq k(ck)^{\frac{k}{4}}$. This finishes the proof. \square

5. Volumetric estimates for k -intersection bodies

Let $p \neq 0$ and K be a symmetric star body in \mathbb{R}^n . We define

$$M_p(L) := \left(\int_{S^{n-1}} \|\theta\|_L^p d\sigma(\theta) \right)^{\frac{1}{p}}. \tag{5.1}$$

Moreover, if L is convex we write

$$W_p(L) := \left(\int_{S^{n-1}} h_L^p(\theta) d\sigma(\theta) \right)^{\frac{1}{p}}. \tag{5.2}$$

Let K be a compact set in \mathbb{R}^n with $|K| = 1$. Let $p > -n$, $p \neq 0$. We define

$$E_p(K) := \left(\int_K \|x\|_2^p dx \right)^{\frac{1}{p}}. \tag{5.3}$$

We have the following identity (see [44] for a generalization to the case of measures):

Lemma 5.1. *Let K be a symmetric star body in \mathbb{R}^n of volume 1 and $1 \leq k \leq n - 1$. Then*

$$E_{-k}(K) \left(\int_{G_{n,k}} |K \cap F^\perp| d\mu_{n,k}(F) \right)^{\frac{1}{k}} = E_{-k}(D_n) \left(\int_{G_{n,k}} |D_n \cap F^\perp| d\mu_{n,k}(F) \right)^{\frac{1}{k}}. \tag{5.4}$$

Proof. Indeed, writing in polar coordinates, we have that

$$E_{-k}^{-k}(K) = \frac{n\omega_n}{n-k} \int_{G_{n,n-k}} \int_{S_H} \frac{d\sigma(\theta)}{\|\theta\|_K^{n-k}} d\mu_{n,n-k}(H)$$

$$= \frac{n\omega_n}{(n-k)\omega_{n-k}} \int_{G_{n,n-k}} \omega_{n-k} \int_{S_H} \frac{d\sigma(\theta)}{\|\theta\|_K^{n-k}} d\mu_{n,n-k}(H),$$

where $S_H = S^{n-1} \cap H$. We now compute the inner integral of the latter and get that

$$\begin{aligned} &= \frac{n\omega_n}{(n-k)\omega_{n-k}} \int_{G_{n,n-k}} |K \cap H| d\mu_{n,n-k}(H) \\ &= \frac{n\omega_n}{(n-k)\omega_{n-k}} \int_{G_{n,k}} |K \cap F^\perp| d\mu_{n,k}(F). \end{aligned}$$

Since the same holds also for D_n , Eq. (5.4) follows. \square

We have the following application of the previous lemma and a definition of k -intersection bodies.

Lemma 5.2. *Let K be a symmetric star body in \mathbb{R}^n of volume 1 and $1 \leq k \leq n - 1$. Assume that $I_k(K)$ exists. Then*

$$M_{-k}(I_k(K)) \left(\int_{G_{n,k}} |K \cap F^\perp| d\mu_{n,k}(F) \right)^{\frac{1}{k}} = \omega_k^{\frac{1}{k}}. \tag{5.5}$$

Moreover,

$$\frac{M_{-k}(I_k(K))}{M_{-k}(I_k(D_n))} = \frac{E_{-k}(K)}{E_{-k}(D_n)}. \tag{5.6}$$

Proof. Integrating (4.1) over $G_{n,k}$ we have that

$$\int_{G_{n,k}} |I_k(K) \cap F| d\mu_{n,k}(F) = \int_{G_{n,k}} |K \cap F^\perp| d\mu_{n,k}(F). \tag{5.7}$$

Moreover,

$$\begin{aligned} \int_{G_{n,k}} |I_k(K) \cap F| d\mu_{n,k}(F) &= \int_{G_{n,k}} \omega_k \int_{S_F} \frac{d\sigma_F(\theta)}{\|\theta\|_{I_k(K)}^k} d\mu_{n,k}(F) \\ &= \omega_k \int_{S^{n-1}} \frac{d\sigma(\theta)}{\|\theta\|_{I_k(K)}^k} = \omega_k M_{-k}^{-k}(I_k(K)). \end{aligned}$$

So, by (5.7) we get (5.5). (5.6) follows from (5.5) and (5.4). \square

The following two propositions deal with the behavior of the ratio $\frac{E_p(K)}{E_p(D_n)}$.

It is not difficult to obtain a lower bound for the quantity $\frac{E_p(K)}{E_p(D_n)}$. The main tool is an argument of Milman and Pajor [41].

Proposition 5.3. *Let K be a compact set in \mathbb{R}^n of volume 1 and let $p > -n$, $p \neq 0$. Then*

$$E_p(K) \geq E_p(D_n), \tag{5.8}$$

with equality if and only if $|K \cap D_n| = 1$.

In particular, if K is a star body, then we have equality if and only if $K = D_n$.

Proof. Note that $|K \setminus D_n| = |D_n \setminus K|$. Also if $x \in K \setminus D_n$ then $\|x\|_2 \geq |B_2^n|^{-\frac{1}{n}}$ while if $x \in D_n \setminus K$, $\|x\|_2 \leq |B_2^n|^{-\frac{1}{n}}$. So, if $p > 0$,

$$\begin{aligned} E_p^p(K) &= \int_K \|x\|_2^p dx = \int_{K \setminus D_n} \|x\|_2^p dx + \int_{K \cap D_n} \|x\|_2^p dx \\ &\geq \int_{D_n \setminus K} \|x\|_2^p dx + \int_{K \cap D_n} \|x\|_2^p dx = E_2^p(D_n). \end{aligned}$$

It is clear that we have equality if and only if $|K \cap D_n| = 1$. We work similarly if $p < 0$. \square

Proposition 5.4. *Let K be a star body in \mathbb{R}^n of volume 1, $-n < p < q \leq \infty$, $p, q \neq 0$. Then*

$$\frac{E_p(K)}{E_q(K)} \leq \frac{E_p(D_n)}{E_q(D_n)}, \tag{5.9}$$

with equality if and only if $K = D_n$.

Proof. We follow an argument from [3]. A simple computation shows that

$$\frac{E_q(D_n)}{E_p(D_n)} = \frac{(\frac{n}{n+q})^{1/q}}{(\frac{n}{n+p})^{1/p}}. \tag{5.10}$$

For every $q > -n$, $q \neq 0$, by integration in polar coordinates we have

$$E_q^q(K) = n\omega_n \int_0^\infty r^{n+q-1} \sigma\left(\frac{1}{r}K\right) dr := \int_0^\infty r^{n+q-1} g(r) dr, \tag{5.11}$$

where $\sigma(\frac{1}{r}K) = \sigma(S^{n-1} \cap \frac{1}{r}K)$. The function $g(r) := \omega_n \sigma(\frac{1}{r}K)$ is non-increasing on its support, $[r(K), R(K)]$. If we assume that K is not the Euclidean ball, then $[r(K), R(K)]$ is an interval and $g(r)$ can be assumed absolutely continuous. In this case we can write

$$g(r) = n \int_r^\infty \frac{\rho(s)}{s^n} ds \quad (r > 0) \tag{5.12}$$

for some non-negative function ρ on $(0, \infty)$. Then, again by integration in polar coordinates,

$$1 = |K| = \int_0^\infty r^{n-1} g(r) dr = n \int_0^\infty \int_{0 < r < s} r^{n-1} \frac{\rho(s)}{s^n} dr ds = \int_0^\infty \rho(s) ds.$$

Hence, ρ represents a probability density of a positive random variable, say, ξ . Then (5.11) becomes

$$E_q^q(K) = \int_0^\infty r^{q+n-1} g(r) dr = \frac{n}{n+q} \int_0^\infty s^q \rho(s) ds = \frac{n}{n+q} \mathbb{E}(\xi^q).$$

Applying Hölder’s inequality for $-n < p < q \leq \infty$, we see that

$$(\mathbb{E}(\xi^q))^{1/q} > (\mathbb{E}(\xi^p))^{1/p}. \tag{5.13}$$

Note that, since ξ is a non-zero random variable with an absolute continuous density, there is no equality case in (5.13) (see [19, Theorem 188]). So,

$$\frac{E_q(K)}{E_p(K)} = \frac{(\frac{n}{n+q} \mathbb{E}(\xi^q))^{1/q}}{(\frac{n}{n+p} \mathbb{E}(\xi^p))^{1/p}} > \frac{(\frac{n}{n+q})^{1/q}}{(\frac{n}{n+p})^{1/p}} = \frac{E_q(D_n)}{E_p(D_n)},$$

as claimed. \square

We will use the following immediate application of Hölder’s inequality.

Lemma 5.5. *Let L be a symmetric star body in \mathbb{R}^n and $p \leq q$, $p, q \neq 0$. Then*

$$M_p(L) \leq M_q(L), \tag{5.14}$$

with equality if and only if $L = aB_2^n$ for some $a > 0$.

Moreover writing the volume of L in polar coordinates we get that

$$M_{-n}(L) = \left(\frac{|B_2^n|}{|L|} \right)^{\frac{1}{n}}. \tag{5.15}$$

Results of Lewis [34], Figiel and Tomczak-Jaegermann [11], Pisier [45] establish the following “reverse Uryson” inequality:

Proposition 5.6. *Let L be a symmetric convex body in \mathbb{R}^n . Then there exists $T \in SL_n$ such that*

$$W(L_1) \leq c\sqrt{n}|L_1|^{\frac{1}{n}} \log(1 + d_{BM}(L, B_2^n)), \tag{5.16}$$

where $c > 0$ is an absolute constant and $L_1 = TL$.

Proof of Theorem 1.3. Using Lemma 5.5 and (5.15) we have that

$$\frac{M_{-k}(I_k(K))}{M_{-k}(I_k(D_n))} \geq \left(\frac{|I_k(D_n)|}{|I_k(K)|} \right)^{\frac{1}{n}}. \tag{5.17}$$

Using (4.2), we may assume that K is isotropic. Then $E_2(K) = \sqrt{n}L_K$. Then, by (5.6), and Proposition 5.4,

$$\frac{M_{-k}(I_k(K))}{M_{-k}(I_k(D_n))} = \frac{E_{-k}(K)}{E_{-k}(D_n)} \leq \frac{E_2(K)}{E_2(D_n)} = \frac{L_K}{L_{D_n}}. \tag{5.18}$$

So, (5.17) and (5.18) imply Eq. (1.3).

We now assume that $I_k(K)$ is convex. We consider $I_k^\circ(K)$ to be in the position described in Proposition 5.6. (Again by using (4.2).) Then by Lemmas 5.5 and 5.6,

$$\begin{aligned} M_{-k}(I_k(K)) &\leq M_1(I_k(K)) = W(I_k^\circ(K)) \\ &\leq c\sqrt{n}|I_k^\circ(K)|^{\frac{1}{n}} \log(1 + d_{BM}(I_k(K), B_2^n)) \leq \frac{c' \log(1 + d_{BM}(I_k(K), B_2^n))}{\sqrt{n}|I_k(K)|^{\frac{1}{n}}}, \end{aligned}$$

using also Santaló inequality. This implies that

$$\frac{M_{-k}(I_k(K))}{M_{-k}(I_k(D_n))} \leq c \left(\frac{|I_k(D_n)|}{|I_k(K)|} \right)^{\frac{1}{n}} \log(1 + d_{BM}(I_k(K), B_2^n)). \tag{5.19}$$

So again by (5.6) and Proposition 5.3, we have that

$$\frac{M_{-k}(I_k(K))}{M_{-k}(I_k(D_n))} = \frac{E_{-k}(K)}{E_{-k}(D_n)} \geq 1. \tag{5.20}$$

So, Eqs. (5.19) and (5.20) imply (1.4). To conclude we use the fact that the Banach–Mazur distance of any symmetric convex body (here $I(K)$) with the Euclidean ball is at most \sqrt{n} and Theorem 1.2. \square

6. Concluding remarks

Theorem 1.1 indicates that the classes \mathcal{BP}_k^n increase in a “canonical” way. It is not clear to us if the reverse inequality (up to the logarithmic term) holds true even for the Banach–Mazur distance. We pose this as

Question 1. Is it true that for every n and every k , $1 \leq k \leq n - 1$, there exists a convex symmetric body K in \mathbb{R}^n such that

$$d_{BM}(K, \mathcal{BP}_k^n) \geq c\sqrt{\frac{n}{k}},$$

where $c > 0$ is a universal constant?

We can show that this is true in the case $k = 1$:

Proposition 6.1. *There exists $c > 0$ such that for every $n \geq 1$,*

$$d_{BM}(B_\infty^n, \mathcal{I}_1^n) \geq c\sqrt{n}.$$

The proof of the latter proposition depends on the following fact: all convex intersection bodies have bounded volume ratio. Let $\mathcal{VR}_n(a)$ be the class of symmetric convex bodies in \mathbb{R}^n with $v.r.(K) := \inf\{(\frac{|TK|}{|B_2^n|})^{\frac{1}{n}} : T \in GL_n, B_2^n \subseteq TK\} \leq a$. In this notation we have the following

Proposition 6.2. *There exists $c > 0$ such that for every $n \geq 1$ every convex intersection body in \mathbb{R}^n belongs to the class $\mathcal{VR}_n(c)$.*

Proof. The proof is simply a combination of certain known results. Let $K \in \mathcal{I}_1^n$ and be convex, and let X be the n -dimensional Banach space with norm $\|\cdot\| := \|\cdot\|_K$. Then it has been shown in [22] that X embeds isomorphically to $L_{1/2}$. It is known (e.g. [21]) that every Banach subspace of $L_{1/2}$ has (Rademacher) cotype 2. Next, by a result of Bourgain and Milman [5], every finite dimensional subspace with bounded cotype 2 constant, has the property that its unit ball has bounded volume ratio. This finishes the proof. \square

Proof of Proposition 6.1. We will show that $d_{BM}(B_\infty^n, \mathcal{VR}_n(a)) \geq c(a)\sqrt{n}$. Then by Proposition 6.2 the proof will be complete. For simplicity we assume that $n = 4k$ for some $k \in \mathbb{N}$. The general case follows easily. Let $L \in \mathcal{VR}_n(a)$ be such that $d_{BM}(B_\infty^n, \mathcal{VR}_n(a)) = d_G(B_\infty^n, L) =: d$. Then we have that $L \subseteq B_\infty^n \subseteq dL$. By a well-known generalization of Kashin’s theorem [50] there exist $F \in G_{n, \frac{n}{2}}$ and an ellipsoid \mathcal{E} such that $c_1(a)\mathcal{E} \subseteq L \cap F \subseteq c_2(a)\mathcal{E}$. Moreover it is known (see e.g. [53]) that for any ellipsoid in \mathbb{R}^m there exist $E \in G_{m, \frac{m}{2}}$ and some $r > 0$ such that $\mathcal{E} \cap E = rB_E$, where B_E is the Euclidean ball of E . So we get that there exist $E \in G_{n, \frac{n}{4}}$ and $r_1 := r_1(a) > 0, r_2 := r_2(a)$, such that

$$r_1 B_E \subseteq B_\infty^n \cap E \subseteq dr_2 B_E.$$

Hence, it is enough to show that $d_G(B_\infty^n \cap E, B_E) \geq c\sqrt{n}$, for every $E \in G_{n, \frac{n}{4}}$. Considering the polar body of $B_\infty^n \cap E$ it is enough to show that the convex hull of at most $8n$ points in \mathbb{R}^n has geometric distance from the Euclidean ball at least $c\sqrt{n}$. Let $N \geq n + 1, v_1, \dots, v_N \in B_2^n$ and $\|v_1\|_2 = 1$. Let $K := \text{co}\{v_1, \dots, v_N\}$. Note that $R(K) = 1$. Now it is enough to show that if $N = 8n$ then $r(K) = \min_{\theta \in S^{n-1}} \rho_K(\theta) \leq \frac{c}{\sqrt{n}}$. But (see e.g. [2,1,10,15]) one has that $|K|^{\frac{1}{n}} \leq c' \sqrt{\frac{\log \frac{eN}{n}}{n}}$. Writing the volume of K in polar coordinates we get that there exists at least one $\theta \in S^{n-1}$ such that $\rho_K(\theta) \leq c'' \sqrt{\frac{\log \frac{eN}{n}}{n}}$. We complete the proof by choosing $N = 8n$. \square

The estimate in Theorem 1.2 is exponential with respect to k . Even if we assume that the Hyperplane conjecture has a positive answer the existing proof would still give an estimate exponential with respect to k . We believe that a better estimate (polynomial) must be true. Having in mind Eq. (4.6), we pose the following question:

Question 2. Is it true that if K is symmetric and convex in \mathbb{R}^n , $1 \leq k \leq n$ and $I_k(K)$ exists and it is convex then $I_k(K)$ is isomorphic to $Z_k^{\circ}(K)$?

Note that a positive answer to the previous question would easily imply a linear in k estimate in Theorem 1.2.

The second conclusion of Theorem 1.3 can be viewed as a generalization of the classical Busemann inequality (see the Introduction). We don't know if the assumption that $I_k(K)$ is convex is necessary in Theorem 1.3 and whether the estimate can be replaced by 1:

Question 3. Is it true that if K is a symmetric star body of volume 1, $1 \leq k \leq n - 1$ and if $I_k(K)$ exists, then

$$|I_k(K)| \leq |I_k(D_n)|?$$

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