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Gradings on simple algebras of finitary matrices

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We describe gradings by finite abelian groups on the associative algebras of infinite matrices with finitely many nonzero entries, over an algebraically closed field of characteristic zero.

1. Introduction

This paper is devoted to the extension of the results of [1–3] about the group gradings on finite-dimensional matrix algebras to the case of infinite-dimensional simple algebras of finitary linear transformations. After reminding the main results in the case of finite dimensions, we describe the G-graded embeddings of one finite-dimensional graded matrix algebra into another (Theorem 3), with G a finite abelian group. Our next result says that if a simple locally finite algebra with minimal one-sided ideals is graded by G as above then it can be presented as the direct limit of finite-dimensional G-graded matrix algebras (Theorem 4). This allows us to describe in Theorem 5 the gradings on the simple algebra of finitary matrices that is, the algebra of infinite matrices such that

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each matrix has only finitely many nonzero entries. Finally, in Theorem 6, we give a necessary and sufficient condition for the equivalence of elementary gradings on the above algebra of infinite matrices.

2. Some notation and simple facts

Let F be an arbitrary field, R a not necessarily associative algebra over an F and G a group. We say that R is a G-graded algebra, if there is a vector space sum decomposition

$$R = \bigoplus_{g \in G} R^{(g)}, \tag{1}$$

such that

$$R^{(g)} R^{(h)} \subset R^{(gh)} \quad \text{for all } g, h \in G. \tag{2}$$

Two G-gradings

$$R = \bigoplus_{g \in G} R^{(g)} \quad \text{and} \quad R = \bigoplus_{g \in G} \left(R'\right)^{(g)} \tag{3}$$

are called *isomorphic* if there is an automorphism φ of R such that $\varphi(R^{(g)}) = (R')^{(g)}$, for all $g \in G$.

A subspace $V \subset R$ is called *graded* (or *homogeneous*) if $V = \bigoplus_{g \in G}(V \cap R^{(g)})$. An element $a \in R$ is called *homogeneous of degree* g if $a \in R^{(g)}$. We also write $\deg a = g$. The *support* of the G-grading is a subset

$$\operatorname{Supp} R = \left\{ g \in G \mid R^{(g)} \neq 0 \right\}.$$

3. Reminder: Group gradings on matrix algebras

Below we briefly recall the results of [1–3], where the full description of a finite group gradings on the full matrix algebra has been given.

A grading $R = \bigoplus_{g \in G} R^{(g)}$ on the matrix algebra $R = M_n(F)$ is called *elementary* if there exists an n-tuple $(g_1, \dots, g_n) \in G^n$, such that the matrix units E_{ij}, $1 \leqslant i, j \leqslant n$ are homogeneous and $E_{ij} \in R^{(g)} \Leftrightarrow g = g_i^{-1} g_j$. If R is a matrix algebra with an elementary G-grading defined by a tuple $(g_1, \dots, g_n) \in G^n$ and B an algebra with a G-grading then the tensor product $R = A \otimes B$ will be given a grading if, given a homogeneous element x of degree h, we set $E_{ij} \otimes x \in R^{(g)}$ provided that $g = g_i^{-1} h g_j$, for any $1 \leqslant i, j \leqslant n$. This grading of the tensor product is called *induced*.

A grading is called *fine* if $\dim R^{(g)} = 1$ for any $g \in \operatorname{Supp} R$. In this case $T = \operatorname{Supp} R$ is always a subgroup of G [3]. In this case if V is a natural R-module then V is the space of a faithful irreducible representation of T (see [2]). If we denote by X_t the image of $t \in T$ in R corresponding to this representation then X_t is a basis of R_t and there is a 2-cocycle $\alpha : G \times G \to F^*$ such that $X_t X_s = \alpha(t, s) X_{ts}$, for any $t, s \in T$. This makes R isomorphic to a twisted group algebra $F^\alpha G$.

The main result of [1, Theorem 6] can be formulated as follows.

Theorem 1. *Let G be a group of order d, F an algebraically closed field and $R = M_n(F)$. Then, as a G-graded algebra, R is isomorphic to the tensor product with induced grading $R \cong A \otimes B$ where $A = M_k(F)$ has an elementary G-grading, with support S, $B = M_l(F)$ has a fine grading, with support T, and $S \cap T = \{e\}$.*

A particular case of the fine gradings is a so-called ε-grading where ε is an nth primitive root of 1. Let $G = \langle a\rangle_n \times \langle b\rangle_n$ be the direct product of two cyclic groups of order n and

$$X_a = \begin{pmatrix} \varepsilon^{n-1} & 0 & \dots & 0 \\ 0 & \varepsilon^{n-2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}, \qquad X_b = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}. \tag{4}$$

Then

$$X_a X_b X_a^{-1} = \varepsilon X_b, \qquad X_a^n = X_b^n = I \tag{5}$$

and all $X_a^i X_b^j$, $1 \leqslant i, j \leqslant n$, are linearly independent. Clearly, the elements $X_a^i X_b^j$, $i, j = 1, \dots, n$, form a basis of R and all the products of these basis elements are uniquely defined by (5).

Now for any $g \in G$, $g = a^i b^j$, we set $X_g = X_a^i X_b^j$ and denote by $R^{(g)}$ a one-dimensional subspace

$$R^{(g)} = \langle X_a^i X_b^j \rangle. \tag{6}$$

Then from (5) it follows that $R = \bigoplus_{g\in G} R^{(g)}$ is a G-grading on $M_n(F)$ which is called an ε-grading.

Now let $R = M_n(F)$ be the full matrix algebra over F graded by an abelian group G. The following result has been proved in [3, Section 4, Theorems 5, 6] and [1, Section 2.2, Theorem 6, Section 2.3, Theorem 8].

Theorem 2. *Let F be an algebraically closed field of characteristic zero. Then as a G-graded algebra R is isomorphic to the tensor product*

$$R_0 \otimes R_1 \otimes \cdots \otimes R_k$$

where $R_0 = M_{n_0}(F)$ has an elementary G-grading, $\operatorname{Supp} R_0 = S$ *is a finite subset of G, $R_i = M_{n_i}(F)$ has the ε_i grading, ε_i being a primitive n_ith root of* 1, $\operatorname{Supp} R_i = H_i \cong \mathbb{Z}_{n_i} \times \mathbb{Z}_{n_i}$, *$i = 1, \dots, k$. Also $H = H_1 \cdots H_k \cong H_1 \times \cdots \times H_k$ and $S \cap H = \{e\}$ in G.*

4. Embeddings of graded matrix algebras

To describe the gradings on the algebra of finitary matrices we will need to consider the embedding of one G-graded finite-dimensional simple algebra into another. It follows from Theorem 1, if $R \cong M_n(F)$ is G-graded then, as a graded algebra, R is isomorphic to a tensor product $C \otimes D$ where $C = M_p(F)$, $D = M_q(F)$, $n = pq$, $M_p(F)$ is a matrix algebra with elementary G-grading, $M_q(F)$ a matrix algebra with a fine T-grading where $T = \operatorname{Supp} D$ is a subgroup in G such that $T \cap \operatorname{Supp} C = \{1\}$ where 1 is the identity element of G. Thus we may think that $R = CD \cong C \otimes D$. Let us notice that the subalgebra D is not defined uniquely and once D has been fixed, C is uniquely defined as the centralizer of D in R.

Theorem 3. *Let F be an algebraically closed field and G a finite abelian group. Let $R_1 \cong M_k(F)$ and $R_2 \cong M_n(F)$ be two G-graded matrix algebras with identity elements e_1 and e_2, respectively, $R_1 = C_1 D_1$, $R_2 = C_2 D_2$ their decompositions in which C_1, C_2 have elementary grading while D_1, D_2 have fine grading. Let also $D_1 \cong D_2$ as G-graded algebras and $\varphi : R_1 \to R_2$ be an injective homomorphism of graded algebras. Then there exists a decomposition $R_2 = \widetilde{C}_2 \widetilde{D}_2 \cong \widetilde{C}_2 \otimes \widetilde{D}_2$ such that $\widetilde{C}_2$ is a matrix algebra with elementary grading, $\widetilde{D}_2$ is a matrix algebra with fine grading, $\widetilde{D}_2 \cong D_2$ as graded algebras and $\varphi(C_1) \subseteq \widetilde{C}_2$. If $\varphi(e_1) R_2 \varphi(e_1) = \varphi(R_1)$ then also $\varphi(e_1)\widetilde{C}_2\varphi(e_1) = \varphi(C_1)$. Besides, there is an isomorphism $\psi : D_1 \to \widetilde{D}_2$ such that $\varphi(a)\varphi(d) = \varphi(a)\psi(d)$, for any $a \in R_1$, $d \in D_1$.*

Proof. Since $D_1 \cong D_2$, in particular, $\operatorname{Supp} D_1 \cong \operatorname{Supp} D_2$. We denote $T = \operatorname{Supp} D_1$. Then for any $t \in T$ there exist invertible matrices $X_t \in R_1$ and $X'_t \in R_2$ such that

$$D_1 = \operatorname{Span}\{X_t \mid t \in T\}, \qquad D_2 = \operatorname{Span}\{X'_t \mid t \in T\}.$$

In addition, for any $t, s \in T$ there is $\alpha(t,s) \in F$ such that

$$X_t X_s = \alpha(t,s) X_{ts}, \qquad X'_t X'_s = \alpha(t,s) X'_{ts}, \tag{7}$$

following because D_1 and D_2 are isomorphic. One may also assume that X_1 and X'_1 are the identity elements of R_1 and R_2, respectively.

Since $T \cap \operatorname{Supp} C_2 = \{1\}$ it follows that $\varphi(X_t) = A_t X'_t$ for some matrix $A_t \in C_2$, $\deg A_t = 1$ in the G-grading. We then set

$$A'_t = A_t \varphi(e_1) + e_2 - \varphi(e_1) \quad \text{and} \quad X''_t = A'_t X'_t,$$

where e_2 is the identity of R_2.

We will first show that

$$\widetilde{D}_2 = \operatorname{Span}\{X''_t \mid t \in T\}$$

is a graded subalgebra in R_2 isomorphic to D_1 (or D_2). Now since

$$\varphi(e_1) A_t X'_t = \varphi(e_1)\varphi(X_t) = \varphi(X_t) = A_t X'_t$$

and X'_t is nondegenerate, it follows that $\varphi(e_1) A_t = A_t$. Since A_t and X'_t commute in R_2, it follows that

$$X'_t A_t \varphi(e_1) = \varphi(X_t)\varphi(e_1) = \varphi(X_t) = X'_t A_t,$$

and so

$$\varphi(e_1) A_t = A_t \varphi(e_1) = A_t. \tag{8}$$

In particular, $(e_2 - \varphi(e_1)) A_t = 0$.

Now let us recall that $\varphi(e_1) = \varphi(X_1) = A_1 X'_1 = A_1 \in C_{R_2}(D_2)$ and so

$$\varphi(e_1) X'_t = X'_t \varphi(e_1) \quad \text{for any } t \in T. \tag{9}$$

If we use (7), (8), and (9) we will obtain the following

$$\begin{aligned}
X''_t X''_s &= \big(A_t + e_2 - \varphi(e_1)\big) X'_t \big(A_s + e_2 - \varphi(e_1)\big) X'_s \\
&= A_t X'_t A_s X'_s + \big(e_2 - \varphi(e_1)\big) X'_t X'_s \\
&= \varphi(X_t)\varphi(X_s) + \big(e_2 - \varphi(e_1)\big) X'_t X'_s \\
&= \varphi(X_t X_s) + \big(e_2 - \varphi(e_1)\big) X'_t X'_s \\
&= \varphi\big(\alpha(t,s) X_{ts}\big) + \big(e_2 - \varphi(e_1)\big) \alpha(t,s) X'_{ts} \\
&= \alpha(t,s) A_{ts} X'_{ts} + \alpha(t,s)\big(e_2 - \varphi(e_1)\big) X'_{ts} \\
&= \alpha(t,s) X''_{ts}.
\end{aligned}$$

Since the elements X_t'', $t \in T$ are linearly independent, it follows that the mapping $X_t \mapsto X_t''$ is a (graded) isomorphism of algebras D_1 and $\widetilde{D}_2$.

Now we denote by $\widetilde{C}_2$ the centralizer $C_{R_2}(\widetilde{D}_2)$ of $\widetilde{D}_2$ in R_2. Then $\widetilde{C}_2$ is a graded subalgebra of R_2. The identity element e_2 of R_2 is in $\widetilde{D}_2$ since $X_1 = e_1$ and

$$X_1'' = \varphi(X_1) + e_2 - \varphi(e_1) = \varphi(e_1) + e_2 - \varphi(e_1) = e_2.$$

In this case $R_2 = \widetilde{C}_2\widetilde{D}_2 \cong \widetilde{C}_2 \otimes \widetilde{D}_2$ (see, for instance, [3]).

Now we would like to show that $\widetilde{C}_2$ is an algebra with elementary G-grading. Since $\widetilde{C}_2$ is a central simple F-algebra, by the main result of [3] $\widetilde{C}_2 = \widetilde{C}_0\widetilde{D}_0 \cong \widetilde{C}_0 \otimes \widetilde{D}_0$ where the grading on $\widetilde{C}_0$ is elementary while on $\widetilde{D}_0$ fine. We set $T_0 = \operatorname{Supp}\widetilde{D}_0$. Then T_0 is a subgroup in G such that $T_0 \cap T = \{1\}$. It follows that $\widetilde{D}_0\widetilde{D}_2 \cong \widetilde{D}_0 \otimes \widetilde{D}_2$ is a graded subalgebra in R_2 with fine grading such that $\operatorname{Supp}\widetilde{D}_0\widetilde{D}_2 = T_0T \cong T_0 \times T$. Besides, $R_2 \cong \widetilde{C}_0 \otimes (\widetilde{D}_0 \otimes \widetilde{D}_2)$ is another decomposition of R_2 as the tensor product of algebras with elementary and fine grading. From the above mentioned property of the supports, the identity component R_e of $R = C \otimes D$, C elementary, D fine, is $C_e \otimes I$. The centralizer of R_e is a graded subalgebra $C_e \otimes D$ which has the same support as D. This uniquely defines the support of the fine component. As a result, we have $T = \operatorname{Supp}\widetilde{D}_2 = \operatorname{Supp}\widetilde{D}_0\widetilde{D}_2 = T_0T$. Then $T_0 = \{1\}$ implying that $\widetilde{C}_2 = \widetilde{C}_0$ is an algebra with elementary grading.

Now we need to show that $\varphi(C_1) \subset \widetilde{C}_2 = C_{R_2}(\widetilde{D}_2)$. Let $a \in C_1 = C_{R_1}(D_1)$. Then $aX_t = X_ta$ for any $t \in T$. Then we have the following

$$\begin{aligned}\varphi(a)X_t'' &= \varphi(a)\big(\varphi(X_t) + e_2 - \varphi(e_1)\big)\\ &= \varphi(a)\varphi(X_t) + \varphi(a)\varphi(e_1)\big(e_2 - \varphi(e_1)\big) = \varphi(aX_t)\\ &= \varphi(X_ta) = \big(\varphi(X_t) + e_2 - \varphi(e_1)\big)\varphi(a)\\ &= X_t''\varphi(a),\end{aligned}$$

proving that $\varphi(a) \in C_{R_2}(\widetilde{D}_2)$, that is, $\varphi(C_1) \subset \widetilde{C}_2$.

Finally, let us assume $\varphi(e_1)R_2\varphi(e_1) = \varphi(R_1)$. Since $\varphi(C_1) \subset \widetilde{C}_2$ the containment $\varphi(C_1) \subset \varphi(e_1)\widetilde{C}_2\varphi(e_1)$ is obvious. To prove the converse, we notice that

$$\varphi(R_1) \cap \widetilde{C}_2 = \varphi(e_1)\widetilde{C}_2\varphi(e_1).$$

Now $C_1 = C_{R_1}(D_1)$ and so $b \in R_1$ satisfies $\varphi(b) \in C_{R_2}(\widetilde{D}_2)$ if and only if $b \in C_1$, that is, $\varphi(R_1) \cap \widetilde{C}_2 = \varphi(C_1)$ which now implies $\varphi(C_1) = \varphi(e_1)\widetilde{C}_2\varphi(e_1)$.

It remains to look at the homomorphism $\psi : D_1 \to \widetilde{D}_2$ given by $\psi(X_t) = X_t''$. We have the following

$$\begin{aligned}\varphi(e_1)X_t'' &= \varphi(e_1)A_tX_t' + \varphi(e_1)\big(e_2 - \varphi(e_1)\big)X_t'\\ &= \varphi(e_1)A_tX_t' = \varphi(e_1)\varphi\big(X_t'\big)\end{aligned}$$

so that $\varphi(e_1)\psi(d) = \varphi(e_1)\varphi(d)$ for any $d \in D_1$ and thus

$$\varphi(a)\psi(d) = \varphi(a)\varphi(e_1)\varphi(d) = \varphi(a)\varphi(d).$$

Now the proof is complete. $\square$

To describe the elementary gradings on infinite-dimensional simple algebras we first consider the case of one finite-dimensional matrix algebra embedded in another, both having elementary gradings. Notice that in the claims to follow the grading group G may be infinite and nonabelian.

To start with we notice that if $R = M_n(F)$ is an algebra with an elementary grading given by a tuple $(h_1, \dots, h_n)$, $n = km + r$ for some $k, m \geqslant 1$, $r \geqslant 0$ and $h_1, \dots, h_n$ satisfy the conditions

$$h_{i+1}^{-1} h_{i+2} = h_{i+k+1}^{-1} h_{i+k+2} = \cdots = h_{i+(m-1)k+1}^{-1} h_{i+(m-1)k+2} \quad \text{for } 1 \leqslant i \leqslant k-2 \tag{10}$$

then the subalgebra C consisting of all block-diagonal matrices of the form $\operatorname{diag}\{X, \dots, X, 0\}$ where X is an arbitrary $(k \times k)$-matrix repeated m times on the diagonal is G-graded and isomorphic to a matrix algebra $M_k(F)$ with an elementary grading given by the tuple $(h_1, \dots, h_k)$. This easily follows because by (10) all matrix units $E_{\alpha+ik,\beta+ik}$, $i = 0, 1, \dots, m-1$, have the same degree for fixed $1 \leqslant \alpha, \beta \leqslant k$. We would like now to prove that any embedding of simple algebras with elementary gradings amounts to this construction.

Let us recall that if $V = \bigoplus_{g \in G} V_g$ is a G-graded space then $R = \operatorname{End} V$ canonically becomes G-graded if, given a G-graded basis $\{v_1, \dots, v_n\}$ of V with $\deg v_i = g_i^{-1}$, $1 \leqslant i \leqslant n$, one gives the matrix unit E_{ij} the degree equal $g_i^{-1} g_j$. Thus the grading of $M_n(F)$ induced from V is elementary.

Lemma 1. *Let V be an n-dimensional G-graded space over a field F and* $\operatorname{End} V = R = \bigoplus_{g \in G} R^{(g)}$ *the algebra of all linear transformations of V with induced elementary grading. Let C be a graded subalgebra in R which is isomorphic to the matrix algebra $M_k(F)$ with an elementary grading given by the tuple $(g_1, \dots, g_k)$. Then V splits as the sum of C-invariant subspaces*

$$V = V_1 \oplus \cdots \oplus V_m \oplus V_0 \tag{11}$$

where $\dim V_1 = \cdots = \dim V_m = k$, *$V_1, \dots, V_m$ are faithful irreducible C-modules while $CV_0 = \{0\}$. Besides, there is a homogeneous basis of V in which all matrices of the transformations in C have the block-diagonal form* $\operatorname{diag}\{X, \dots, X, 0\}$ *where X is a $(k \times k)$-matrix and the tuple $(h_1, \dots, h_n)$ giving the induced elementary grading on $R = M_n(F)$ satisfies* (10).

Proof. Since the grading on $C \cong M_k(F)$ is elementary, any subspace spanned by a set of matrix units is graded. In particular, this is true for any minimal left ideal spanned by all matrix units in a fixed column. Let L be one of such minimal ideals, corresponding to the last, kth column of C. If we fix any $v \in V$ then the left C-module Lv is either irreducible or equal zero. Moreover, if v is homogeneous then the C-submodule Lv is also G-graded. These remarks are sufficient to prove the existence of the decomposition (11).

Now let E_{ij}, $1 \leqslant i, j \leqslant k$ be the set of all matrix units of C. Since V_1 in (11) is a faithful C-module, there exists a homogeneous element $v \in V$ such that $E_{1k} v \neq 0$. In this case the vectors $v_i = E_{ik} v = E_{i,i+1} \cdots E_{k-1,k} v$, $i = 1, \dots, k-1$, $v_k = v$, form a homogeneous basis of V_1 and the elementary grading on C induced from this grading is given by the tuple $(g_1, \dots, g_k)$. Indeed, if $\deg v = h$ then $\deg v_i = g_i^{-1} g_k h$, $\deg v_j = g_j^{-1} g_k h$ and so $\deg E_{ij} = g_i^{-1} g_j$ and still $E_{ij} v_j = v_i$. If we choose the bases in other $V_2, \dots, V_m$ and an arbitrary homogeneous basis in V_0 then we obtain a realization of C by the block-diagonal matrices of the form $\operatorname{diag}\{X, \dots, X, 0\}$.

It remains to consider the tuple $(h_1, \dots, h_n)$ which defines the elementary grading on $R = M_n(F)$ induced from the graded basis of V just constructed. If we denote by $\widetilde{E}_{st}$, $1 \leqslant s, t \leqslant n$, the matrix units of R corresponding to this basis, then, as usual, $\deg \widetilde{E}_{st} = g_s^{-1} g_t$. Also, for any $1 \leqslant i, j \leqslant k$ we will have

$$E_{ij} = \widetilde{E}_{ij} + \widetilde{E}_{i+k,j+k} + \cdots + \widetilde{E}_{i+(m-1)k,j+(m-1)k}$$

in R and $\deg E_{ij} = g_i^{-1} g_j$ in C, hence in R, since the embedding of C in R is graded. Now all $\widetilde{E}_{st}$ are homogeneous and so the conditions (10) must be satisfied. Now the proof is complete. $\square$

Example 1. The condition of C having an elementary grading is essential. For example, suppose $C \cong M_n(F)$ with a fine grading. Let V be C itself as a graded vector space and let us assume that C acts on itself by multiplication on the left. Then $R = \operatorname{End} V$ is an algebra with elementary grading induced from V and C a graded subalgebra. So, C is a graded matrix subalgebra of a matrix algebra with an elementary grading but the grading of C is not elementary. So the conclusion of the previous lemma cannot hold for C.

Lemma 1 enables one to describe the gradings on all possible direct limits of matrix algebras with elementary gradings. Here we will need a special case where $C = \bigcup_{i \geqslant 1} C_i$ where $C_1 \subset C_2 \subset \cdots$ is an ascending chain of matrix algebras and $C_i = e_i C_j e_i$, for any $1 \leqslant i \leqslant j$, where e_i is the identity element of C_i. To start with we generalize the notion of the elementary grading to the case of finitary matrices.

Definition 1. Let R be the algebra of finitary matrices and $\mathbf{g} = (g_1, g_2, \ldots)$ a sequence of elements in a group G. Then a grading $R = \bigoplus_{g \in G} R^{(g)}$ is called elementary defined by $\mathbf{g}$ if $R^{(g)} = \operatorname{Span}\{E_{ij} \mid g_i^{-1} g_j = g\}$.

Lemma 2. *Let $C = \bigoplus_{g \in G} C^{(g)}$ be a G-graded algebra over a field F which is the union $C = \bigcup_{i \geqslant 1} C_i$ of an ascending chain of graded matrix subalgebras of orders $n_1, n_2, \ldots,$ with identity elements $e_1, e_2, \ldots.$ Suppose all the gradings on the subalgebras $C_1, C_2, \ldots$ are elementary and $C_i = e_i C_j e_i$ for all i, j with $1 \leqslant i \leqslant j$. Then C is isomorphic to the algebra R of finitary matrices with elementary grading given by a sequence $\mathbf{g} = (g_1, g_2, \ldots)$ of elements of G in which every C_i is embedded as a graded subalgebra of all matrices with zero entries in all rows and columns whose numbers are greater than n_i, $i = 1, 2, \ldots.$ The G-grading on C_i is elementary given by an n-tuple $(g_1, \ldots, g_{n_i})$.*

Proof. By Lemma 1, we may adjust our graded embeddings in the sequence $C_1 \subset C_2 \subset \cdots$ in such a way that each C_i can be viewed as a graded subalgebra of C_{i+1} consisting of all $n_i \times n_i$ matrices in the left upper corner. These adjustments do not change the isomorphism class of the limit since this depend only on the module structure of C_{i+1} over C_i, for each i (see [4]). But then the set of all matrices L_i in C_i with zeros outside the first column is a graded subspace of the similar subspace L_{i+1} in C_{i+1}. If $\{e = g_1^{-1}, g_2^{-1}, \ldots, g_{n_i}^{-1}\}$ is the set of degrees of the matrix units spanning L_i then the elementary grading of C_i is defined by the tuple $(g_1, \ldots, g_{n_i})$. Then the set of degrees of the matrix units in $L = \bigcup_{i=1}^{\infty}$ is the desired sequence of elements of G defining the elementary grading on the algebra of finitary matrices C. $\square$

5. Gradings on simple algebras with minimal one sided ideals

In this section we consider the gradings by finite abelian groups on simple locally finite algebras with minimal one sided ideals. Suppose that R is such an algebra. Using the Structure Theorem in [7, Chapter 4, Section 9], we find a pair of mutually dual spaces V and $\Pi \subset V^*$ such that $R \cong V \otimes \Pi$ with the product given by

$$(v_1 \otimes \pi_1)(v_2 \otimes \pi_2) = \pi_1(v_2)(v_1 \otimes \pi_2)$$

where $v_1, v_2 \in V$, $\pi_1, \pi_2 \in \Pi$ and the kernel of the bilinear mapping $(v, \pi) \mapsto \pi(v)$ is trivial. If $\dim V = \dim \Pi = n < \infty$ we have $R \cong M_n(F)$, the matrix algebra of order n over F.

The linear mapping $S : V \to V$ and $T : \Pi \to \Pi$ are called *adjoint* if $(T(\pi))(v) = \pi(S(v))$. Actually, T is completely defined by S and we write $T = S^*$. The Isomorphism Theorem [7, Chapter 4, Section 11] describes the automorphisms of $V \otimes \Pi$ with the help of the automorphisms of V in the following way. If $\varphi \in \operatorname{Aut}(V \otimes \Pi)$ then there exists a linear automorphism $S : V \to V$, for which there exists the adjoint automorphism $S^* : \Pi \to \Pi$, such that

$$\varphi(v \otimes \pi) = S^{-1}(v) \otimes S^*(\pi) \quad \text{for any } v \in V,\ \pi \in \Pi.$$

The automorphism S is defined by φ uniquely up to a nonzero scalar multiple.

The finite-dimensional subspaces $V' \subset V$ and $\Pi' \subset \Pi$ are called *compatible* if they are of the same dimension n and the annihilator of V' in Π' is zero. As mentioned above, in this case $V' \otimes \Pi' \cong M_n(F)$. A simple remark is that $V' \otimes \Pi' \subset V'' \otimes \Pi''$ if and only if $V' \subset V''$ and $\Pi' \subset \Pi''$. It is shown in [7, Chapter 4, Section 16] that $R = V \otimes \Pi$ has a local system of matrix subalgebras of such form. It will be convenient to label the subalgebras in this local system by the elements of a directed set I, that is, an ordered set such that for any $\alpha, \beta \in I$ there is $\gamma \in I$ with $\alpha \prec \gamma$ and $\beta \prec \gamma$. We will have $V_\alpha \otimes \Pi_\alpha \subset V_\beta \otimes \Pi_\beta$ if and only if $V_\alpha \subset V_\beta$ and $\Pi_\alpha \subset \Pi_\beta$. The latter holds if and only if $\alpha \prec \beta$.

Our aim is to prove the following.

Theorem 4. *Let a simple locally finite algebra R with minimal one sided ideals over an algebraically closed field of characteristic zero be given a grading by a finite abelian group G. Then R has a local system of graded finite-dimensional matrix algebras.*

Proof. Using Litoff's Theorem [7, Chapter 4, Section 15], we find that R is locally matrix, that is, there a local system $\{V_\alpha \otimes \Pi_\alpha \mid \alpha \in I\}$ of matrix subalgebras in a G-graded algebra $R = V \otimes \Pi$. We need to prove that there is another local system whose terms are G-graded matrix subalgebras of the form $\{\overline{V}_\alpha \otimes \overline{\Pi}_\alpha \mid \alpha \in I\}$.

Now the conditions imposed on the field allow one to replace the graded subspaces by the invariant subspaces with respect to the automorphisms corresponding to the multiplicative characters $\chi \in \widehat{G}$, given by $\chi * r = \chi(g)r$, for any $r \in R^{(g)}$. As mentioned before, to each such χ one can associate an automorphism $S_\chi : V \to V$ and its adjoint $S_\chi^* : \Pi \to \Pi$ so that $\chi * (v \otimes \pi) = S_\chi^{-1}(v) \otimes S_\chi^*(\pi)$, for any $v \in V$ and $\pi \in \Pi$. Since S_χ is defined up to scalar, the mappings $\chi \mapsto S_\chi$ and $\chi \mapsto S_\chi^*$ are projective representations of $\widehat{G}$ by linear transformations of V and Π. It is obvious that given a $\widehat{G}$-invariant subspace U in V, the annihilator $U^\perp$ in Π is also $\widehat{G}$-invariant. With these facts in mind, we first pick, for each $\alpha \in I$ a subspace of finite codimension $\Pi_\alpha^\perp$. Set

$$U_\alpha = \bigcap_{\chi \in \widehat{G}} S_\chi\left(\Pi_\alpha^\perp\right) \subset \Pi_\alpha^\perp.$$

This is a $\widehat{G}$-invariant subspace in V of finite codimension. Since $\bigcup_{\alpha \in I} \Pi_\alpha = \Pi$ we must have $\bigcap_{\alpha \in I} \Pi_\alpha^\perp = 0$, hence $\bigcap_{\alpha \in I} U_\alpha = 0$. Note that $U_\gamma \subset U_\beta$ as soon as $\beta \prec \gamma$. Let us now consider a finite-dimensional $\widehat{G}$-invariant subspace $\widehat{G}(V_\alpha) = \sum_{\chi \in \widehat{G}} \chi(V_\alpha)$. Then there exists U_β such that $\widehat{G}(V_\alpha) \cap U_\beta = 0$. Since I is a directed set, there is $\gamma \in I$ such that $\alpha, \beta \prec \gamma$ hence $\widehat{G}(V_\alpha) \cap U_\gamma = 0$. Since the projective representation of a finite group is fully reducible there is a $\widehat{G}$-invariant subspace L in V such that $V = L \oplus (\widehat{G}(V_\alpha) \oplus U_\gamma) = 0$. We then set $\overline{V}_\alpha = L \oplus \widehat{G}(V_\alpha)$. Also, we set $\overline{\Pi}_\alpha = U_\gamma^\perp$. Since $U_\gamma \subset U_\alpha \subset \Pi_\alpha^\perp$, we have that $\Pi_\alpha \subset \overline{\Pi}_\alpha$. Being an orthogonal complement to a $\widehat{G}$-invariant space, $\overline{\Pi}_\alpha$ is $\widehat{G}$-invariant. By construction, $\dim \overline{\Pi}_\alpha = \dim \overline{V}_\alpha$ and also $\overline{\Pi}_\alpha^\perp = U_\gamma$ has trivial intersection with $\overline{V}_\alpha$. This proves that $\overline{V}_\alpha$ and $\overline{\Pi}_\alpha$ are compatible invariant spaces so that $\overline{V}_\alpha \otimes \overline{\Pi}_\alpha$ is a $\widehat{G}$-invariant matrix subalgebra. Since $V \otimes \Pi_\alpha \subset \overline{V}_\alpha \otimes \overline{\Pi}_\alpha$, we obtain a $\widehat{G}$-invariant local system, hence a local system of graded matrix subalgebras. □

6. Gradings on simple algebras of finitary matrices

Now we are ready to prove our main result.

Theorem 5. *Let G be a finite abelian group, $R = \bigoplus_{g \in G} R^{(g)}$ be a G-graded algebra of infinite matrices each having only finitely many nonzero entries over an algebraically closed field F of characteristic zero. Then R is isomorphic to a graded tensor product $C \otimes D$ where C is such with an elementary grading and $D = M_n(F)$ is a matrix algebra of order n with a fine grading. Additionally, we have* $\operatorname{Supp} C \cap \operatorname{Supp} D = \{1\}$.

Proof. According to [7, Chapter 4, Section 15] R is the same as the simple algebra with minimal one sided ideals since in our case $\dim R$ is countable. Clearly, in this case we can remove unnecessary terms from the local system provided by Theorem 4 and conclude that R is the union of the ascending chain $R_1 \subset R_2 \subset \cdots$ of graded simple finite-dimensional subalgebras. Each R_i decomposes as the tensor product $R_i = C_i D_i \cong C_i \otimes D_i$ of a simple subalgebra C_i with an elementary grading and a simple subalgebra D_i with a fine grading. The support $T_i = \operatorname{Supp} D_i$ is a subgroup in G. Since the number of subgroups in G is finite, by excluding unnecessary subalgebras R_i we may assume that $T_1 = T_2 = \cdots$ is the same subgroup T of G. In particular, $\dim D_i = |T|$ for all i and that the D_i as ungraded algebras all isomorphic to the same $M_n(F)$. Since the number of different fine gradings is also finite, we may, as before, assume that all D_i are isomorphic as graded algebras.

Let $\varphi_{i+1,i}$ be the graded embedding of R_i in R_{i+1}. We also set $\varphi_{ji} = \varphi_{j,j-1} \cdots \varphi_{i+1,i}$. If we apply Theorem 3 to each embedding $R_i \subset R_{i+1}$ then we may assume that $\varphi_{i+1,i}(C_i) \subset C_{i+1}$ and $\varphi_{i+1,i}(C_i) = \varphi_{i+1,i}(e_i) C_{i+1} \varphi_{i+1,i}(e_i)$ where e_i is the identity element of C_i and that there is an isomorphism $\psi_{i+1,i}(D_i) \to D_{i+1}$ such that

$$\varphi_{i+1,i}(a)\varphi_{i+1,i}(d) = \varphi_{i+1,i}(a)\psi_{i+1,i}(d) \quad \text{for all } a \in C_i,\ d \in D_i. \tag{12}$$

We set $\psi_1 = \mathrm{id}_{D_1}$ and $\psi_i = \psi_{i,i-1} \cdots \psi_{2,1} : D_1 \to D_i$, for all $i \geqslant 2$. Then $\psi_j(d) = \psi_{j,i}(\psi_i(d))$ for any i, j with $1 \leqslant i \leqslant j$, and any $d \in D_1$. Besides, using (12), we may write

$$\varphi_{j,i}(a)\varphi_{j,i}(d) = \varphi_{i,j}(a)\psi_j\bigl(\psi_i^{-1}(d)\bigr) \quad \text{for all } a \in C_i,\ d \in D_i. \tag{13}$$

Let us set $C = \bigcup_{i \geqslant 1} C_i$ and construct an isomorphism $\rho : R \to C \otimes D_1$. If $a \in C_i$, $d \in D_i$ then we set

$$\rho(ad) = a \otimes \psi_i^{-1}(d) \quad \text{for any } a \in C_i,\ d \in D_i. \tag{14}$$

Clearly, (14) defines an injective homomorphism of $R_i = C_i D_i$ into $C \otimes D_1$. Actually, the same formula defines an isomorphism of R to $C \otimes D_1$. To prove this we only need to check that ρ is well defined on R. Indeed, if $a \in C_i$, $d \in D_i$ and $i < j$ then $\varphi_{j,i}(ad) = \varphi_{j,i}(a)\varphi_{j,i}(d)$ in R and $a = \varphi_{j,i}(a)$ in C since we identify $a \in C_i$ with its image $\varphi_{j,i}(a)$ in C_j. But then, according to (13) we should have

$$\begin{aligned}\rho\bigl(\varphi_{j,i}(a)\varphi_{j,i}(d)\bigr) &= \rho\bigl(\varphi_{i,j}(a)\psi_j\bigl(\psi_i^{-1}(d)\bigr)\bigr)\\ &= \varphi_{j,i}(a) \otimes \psi_i^{-1}(d) = a \otimes \psi_i^{-1}(d),\end{aligned}$$

proving that, indeed, ρ is defined correctly.

By Lemma 2 C is isomorphic to the algebra of finitary matrices with an elementary G-grading. Since $\operatorname{Supp} C = \bigcup_{i \geqslant 1} \operatorname{Supp} C_i$ and $T \cap \operatorname{Supp} C_i = \{1\}$, for all $i \geqslant 1$, we have $T \cap \operatorname{Supp} C = \{1\}$, and the proof is complete. □

7. The uniqueness theorem for the elementary gradings of simple algebras of finitary matrices

The defining sequence **g** of an elementary grading is not defined uniquely. In what follows we prove a theorem that gives necessary and sufficient conditions for two sequences to define isomorphic gradings. It will be convenient to denote such sequence as a function $\tau : I \to G$ such that $\tau(i) = g_i$. Here I is the sequence of natural numbers or any initial segment of this. In the latter case we simply deal with $R = M_n(F)$ for a natural number n. With each such function we associate a function $S_\tau : G \to \mathbb{N} \cup \{\infty\}$ given by $S_\tau(g) = \operatorname{Card}(\tau^{-1}(g))$.

Further notice that for each elementary grading defined by a function τ there is a graded vector space V with a basis $\{v_i \mid i \in I\}$ such that $\deg v_i = g_i^{-1}$. We denote the subspace spanned by all v_i with $\tau(i) = g$ by $V_{g^{-1}}$. In this case the algebra of finitary matrices can be identified with the set all linear transformations of V spanned by the linear transformations with matrices E_{ij} with respect to the above basis. The homogeneous component $R^{(g)}$ is then the set of all linear transformations φ such that $\varphi(V_h) \subset V_{gh}$.

Theorem 6. *Let G be a group, R and R' the algebras of finitary matrices endowed by two elementary gradings $R = \bigoplus_{g\in G} R^{(g)}$ and $R' = \bigoplus_{g\in G}(R')^{(g)}$ defined by the tuples τ and τ', respectively. Then R and R' are isomorphic as graded algebras if and only if there is an element $g_0 \in G$ such that $S_\tau(g) = S_{\tau'}(g_0 g)$, for all $g \in G$.*

Proof. First we assume that the gradings defined by τ and τ' are isomorphic. Note that two sequences $\mathbf{g} = (g_1, g_2, \ldots)$ and $\mathbf{h} = (ag_1, ag_2, \ldots)$ define the same gradings on R and $S_\tau(g) = S_\rho(ag)$ for all $g \in G$ where $\rho(i) = ag$. Hence we can suppose that $\rho(1) = e$ that is $g_1 = 1$ in $\mathbf{g}$.

Let $f : R \to R'$ be the graded isomorphism of R and R', that is, $f(R^{(g)}) = (R')^{(g)}$, for all $g \in G$. Let us consider the identity components $R^{(e)}$ and $(R')^{(e)}$. Each of these algebras is the sum of simple ideals $M^{(g)}$ and $(M')^{(g)}$ each defined as the linear span of the set of matrix units E_{ij} or E'_{ij}, respectively, such that $\tau(i) = \tau(j) = \tau'(i) = \tau'(j) = g$.

Since $f(R^{(e)}) = (R')^{(e)}$ we must have $f(M^{(g)}) = (M')^{(\sigma(g))}$ for a bijective map $\sigma : \operatorname{Supp} R \mapsto \operatorname{Supp} R' = \operatorname{Supp} R$ on G. Let us also recall [7, Corollary 2, Section 4.11] that there is a linear bijective map $\alpha : V \to V'$ such that $f(\varphi) = \alpha\varphi\alpha^{-1}$ for any $\varphi \in R$. Let us notice first that such α must satisfy the equation $\alpha(V_{g^{-1}}) = V'_{\sigma(g)^{-1}}$. Indeed, we have

$$M^{(g)} = \{\varphi \in R^{(e)} \mid \varphi(V) \subset V_{g^{-1}}\} \quad \text{and} \quad (M')^{(g)} = \{\varphi \in (R')^{(e)} \mid \varphi(V') \subset V'_{g^{-1}}\}.$$

We have $\alpha M^{(g)}\alpha^{-1} = (M')^{(\sigma(g))}$ and so $\alpha M^{(g)} = (M')^{(\sigma(g))}\alpha$. Applying both sides to V and having in mind the equations

$$\alpha(V) = V', \qquad M^{(g)}(V) = V_{g^{-1}} \quad \text{and} \quad (M')^{(\sigma(g))}(V') = V'_{\sigma(g)^{-1}}$$

we obtain $\alpha(V_{g^{-1}}) = V'_{\sigma(g)^{-1}}$.

Now let us use $\alpha R^{(g)}\alpha^{-1} = (R')^{(g)}$ or $\alpha R^{(g)} = (R')^{(g)}\alpha$, for all $g \in \operatorname{Supp} R \subset G$. Applying both sides of this equation to any V_h, $h \in \operatorname{Supp} V \subset G$, we obtain $\alpha R^{(g)}(V_h) = (R')^{(g)}\alpha(V_h)$ and so $\alpha(V_{gh}) = (R')^{(g)}(V'_{\sigma(h^{-1})^{-1}})$. In other words, $V'_{\sigma(h^{-1}g^{-1})^{-1}} = V'_{g\sigma(h^{-1})^{-1}}$ and

$$\sigma(h^{-1}g^{-1})^{-1} = g\sigma(h^{-1})^{-1} \tag{15}$$

for any $h \in \operatorname{Supp} R$, $g \in \operatorname{Supp} R$. Recall that $g_1 = e$ in $\mathbf{g}$ that is $e^{-1} = e \in \operatorname{Supp} V$. Substituting $h = e$ in (15) and setting $g_0 = \sigma(e)$, we obtain $\sigma(g^{-1}) = g_0 g^{-1}$, for any $g \in \operatorname{Supp} R$. Note that for any elementary grading oh $g^{-1} \in \operatorname{Supp} R$ if and only if $g \in \operatorname{Supp} R$. Hence also $\sigma(g) = g_0 g$ for all $g \in \operatorname{Supp} R$. So we have $\dim V_g = \dim V'_{g_0 g}$. Since $S_\tau(g) = \dim V_g$, we easily obtain the desired condition: there is $g_0 \in G$ such that $S_\tau(g) = S_{\tau'}(g_0 g)$ for all $g \in G$.

To prove the converse, we consider two G-graded finitary matrix algebras $R = \bigoplus_{g\in G} R^{(g)}$, $R' = \bigoplus_{g\in G}(R')^{(g)}$ and assume that there is $g_0 \in G$ such that $S_\tau(g) = S_{\tau'}(g_0 g)$, for any $g \in \operatorname{Supp} R \subset G$.

We define an isomorphism $f : R \to R'$ in the following way. For each $g \in G$, let the ordered subset I_g label the elements v_i of the basis of $V \cap V_{g^{-1}}$. Let I'_g be the same thing for V'. Then there is an ordered map $\beta_g : I_g \to I'_{g_0 g}$. We extend it to a bijection β of I into itself. Then β satisfies the following condition. If $\mathbf{g} = (g_1, g_2, \ldots)$ and $\mathbf{h} = (h_1, h_2, \ldots)$ then

$$h_{\beta(i)} = g_0 g_i. \tag{16}$$

Denote by f the linear map $R \to R'$ such that $f(E_{ij}) = E_{\beta(i)\beta(j)}$. Then f is an isomorphism and $f(R^{(g)}) = (R')^{(g)}$ due to (16). $\square$

Remark 1. The theorem above is no longer true if we replace the algebra of finitary matrices by the other direct limits of matrix algebras. For example, suppose an algebra R is the direct limit of the algebras $R_i = M_{2^i}$, $i = 1, 2, \ldots,$ where the structure mappings $\varphi_i : R_i \to R_{i+1}$ are given by $X \mapsto \operatorname{diag}\{X, X\}$. Then the elementary grading by $G = \langle a \rangle_2$ of R_i given by a tuple τ can be extended to the grading of R_{i+1} defined by τ' to make φ_i graded if we either choose $\tau' = (\tau, \tau)$ or $\tau' = (\tau, a\tau)$. If we start with the grading of R_1 defined by $\tau = (e, a)$ and consider the identity component of the grading in each of the two cases then we will see the limits of semisimple algebras, each of which is the sum of two isomorphic matrix subalgebras. But the Bratteli diagrams [5] of these limits are different and so the limits are not isomorphic. At the same time the "Steinitz numbers" S_τ and S'_τ are the same and both equal to $e^\infty a^\infty$.

Remark 2. A uniqueness theorem for the G-gradings of matrix algebras over algebraically closed field F of characteristic zero has been established by A.A. Chasov [6].

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