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# On the topological entropy of families of braids

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ABSTRACT

A method for computing the topological entropy of each braid in an infinite family, making use of Dynnikov's coordinates on the boundary of Teichmüller space, is described. The method is illustrated on two two-parameter families of braids.



## 1. Introduction

In the dynamical study of iterated surface homeomorphisms, it is common to seek to compute the topological entropy of each member of an infinite family of isotopy classes, perhaps on varying surfaces—the topological entropy of an isotopy class being the minimum topological entropy of a homeomorphism in the class, which is realised by a Nielsen–Thurston canonical representative [13,7,3]. The normal approach to such a problem is to use train-track methods [1,9,11], which not only make it possible to compute topological entropy, but also, in the pseudo-Anosov case, provide a Markov partition for the pseudo-Anosov homeomorphism in the isotopy class, and hence information about the structure of its invariant singular measured foliations.

One drawback of this approach is that even single train tracks are fairly unwieldy objects. It is usually far from straightforward to describe an infinite family of train tracks, to verify that they are indeed invariant under the relevant isotopy classes, and to compute the transition matrices and hence the topological entropy of the induced train track maps: very often, the best that one can reasonably do is to draw pictures of typical train tracks in the family and rely on the reader's ability to observe that they are invariant.

In this paper an alternative approach to the problem is described in the case of families of isotopy classes of orientation-preserving homeomorphisms of punctured disks—such isotopy classes can be described by elements of Artin's braid groups. The method is illustrated by applying it to two families of braids considered by Hironaka and Kin [10], which are of interest in the study of braids with low topological entropy. The results presented here about these families are not new, therefore: the emphasis is on the method used to obtain them, which can be contrasted with the train track methods of Hironaka and Kin.

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The methods developed are a relatively straightforward application of Dynnikov's coordinate system [6] on the boundary of the Teichmüller space of the punctured disk, together with the *update rules* which describe the action of the Artin braid generators on the boundary of Teichmüller space in terms of Dynnikov coordinates. This background material is described in Section 2. The practical application of this theory is very much eased by the results presented in Section 3, which give update rules for braids which can be written as ascending or descending sequences of contiguous Artin generators (or their inverses), such as $\sigma_3\sigma_4\sigma_5\sigma_6$. Examples of the application of the method to two two-parameter families of braids are given in Section 4: the examples include showing that an infinite family of braids is of reducible type, as well as computing topological entropies in the pseudo-Anosov case.

## 2. Dynnikov coordinates of measured foliations

This section is essentially an expansion of parts of Dynnikov's very terse paper [6]: see also [5,4], and [12,8] for dynamical applications. One difference is that in the papers cited above the action of the $n$-braid group $B_n$ on an $(n+2)$-punctured disk is considered, whereas here, as is more appropriate in a dynamical setting, $B_n$ acts on an $n$-punctured disk. This modification requires separate consideration of the action of the "end" Artin generators $\sigma_1$ and $\sigma_{n-1}$. In addition, the useful Lemma 1 does not seem to have appeared explicitly in the literature.

### *2.1. The Dynnikov coordinates of a measured foliation*

Let $D_n$ be a standard model of the $n$-punctured disk ($n \geqslant 3$). Write $\mathcal{F}_n$ for the set of singular measured foliations $(\mathcal{F},\mu)$ on $D_n$, and $\mathfrak{F}_n$ for $\mathcal{F}_n$ up to isotopy and Whitehead equivalence (see for example [7]): the element of $\mathfrak{F}_n$ containing $(\mathcal{F},\mu)\in\mathcal{F}_n$ is denoted $[\mathcal{F},\mu]$. Dynnikov's coordinate system provides an explicit bijection $\rho:\mathfrak{F}_n\to\mathbb{R}^{2n-4}\setminus\{0\}$.

Let $(\mathcal{F},\mu)\in\mathcal{F}_n$. Write $\mathcal{A}_n$ for the set of arcs in $D_n$ which have each endpoint either on the boundary or at a puncture. Recall that if $\alpha\in\mathcal{A}_n$, then its measure $\mu(\alpha)$ is defined to be

$$\mu(\alpha)=\sup\sum_{i=1}^{k}\mu(\alpha_i),$$

where the supremum is taken over all finite collections $\alpha_1,\dots,\alpha_k$ of mutually disjoint subarcs of $\alpha$ which are transverse to $\mathcal{F}$. Denoting by $[\alpha]$ the isotopy class of $\alpha$ (under isotopies through $\mathcal{A}_n$), one can then define

$$\mu\big([\alpha]\big)=\inf_{\beta\in[\alpha]}\mu(\beta),$$

which is well defined on $\mathfrak{F}_n$.

Consider the arcs $\alpha_i$ $(1\leqslant i\leqslant 2n-4)$ and $\beta_i$ $(1\leqslant i\leqslant n-1)$ depicted in Fig. 1: the arcs $\alpha_{2j-3}$ and $\alpha_{2j-2}$ (for $2\leqslant j\leqslant n-1$) join the $j$th puncture to the boundary, while the arc $\beta_i$ has both endpoints on the boundary and passes between the $i$th and $i+1$th punctures.

Let $\tau:\mathfrak{F}_n\to\mathbb{R}^{3n-5}_{\geqslant 0}$ be the *triangle coordinate function* defined by

$$\tau\big([\mathcal{F},\mu]\big)=\big(\mu\big([\alpha_1]\big),\dots,\mu\big([\alpha_{2n-4}]\big),\mu\big([\beta_1]\big),\dots,\mu\big([\beta_{n-1}]\big)\big).$$

The function $\tau$ is injective: if $\tau([\mathcal{F},\mu])$ is given, then a representative measured foliation in $[\mathcal{F},\mu]$ can be constructed by gluing together pieces of measured foliation in each of the strips of Fig. 1. However, it is clearly not surjective: $\tau([\mathcal{F},\mu])$ must satisfy the triangle inequality in each of the strips of Fig. 1, as well as additional conditions to ensure that $(\mathcal{F},\mu)$ has no singularities which are centers.

Let $\rho:\mathfrak{F}_n\to\mathbb{R}^{2n-4}\setminus\{0\}$ be the *Dynnikov coordinate function* defined by

$$\rho\big([\mathcal{F},\mu]\big)=(a,b)=(a_1,\dots,a_{n-2},\,b_1,\dots,b_{n-2}),$$

where for $1\leqslant i\leqslant n-2$,

$$a_i=\frac{\mu([\alpha_{2i}])-\mu([\alpha_{2i-1}])}{2}\quad\text{and}\quad b_i=\frac{\mu([\beta_i])-\mu([\beta_{i+1}])}{2}.$$

Let $\mathcal{C}_n=\mathbb{R}^{2n-4}\setminus\{0\}$ denote the space of Dynnikov coordinates.

The Dynnikov coordinate function is a bijection (in fact it is a homeomorphism when $\mathfrak{F}_n$ is endowed with its usual topology). To describe its inverse, it is sufficient to describe a function $\mathcal{C}_n\to\mathbb{R}^{3n-5}_{\geqslant 0}$ which sends each $(a,b)\in\mathcal{C}_n$ to the triangle coordinates of a measured foliation $[\mathcal{F},\mu]$ which has Dynnikov coordinates $(a,b)$.

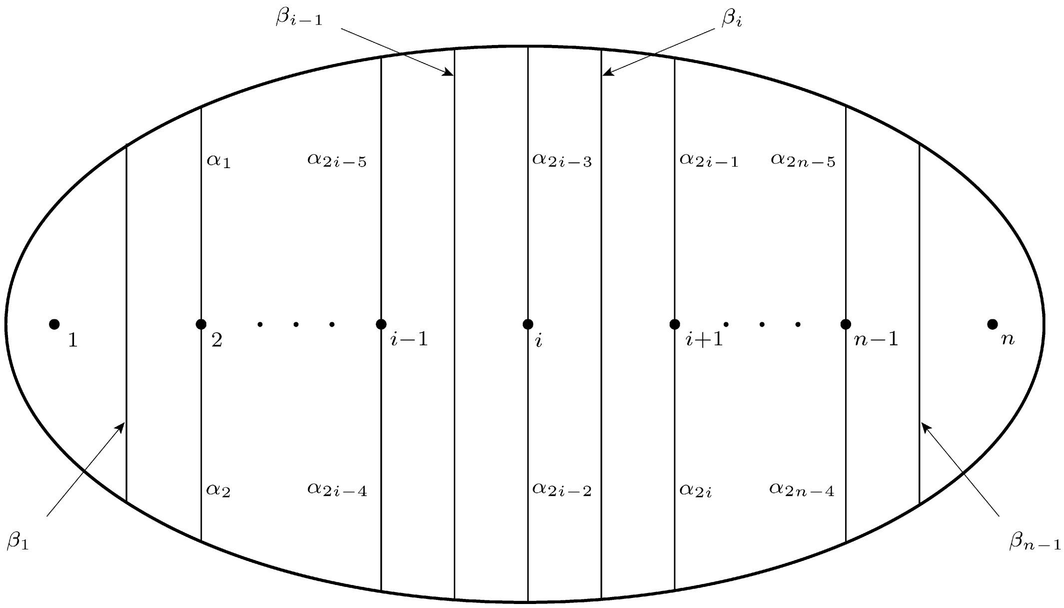


**Fig. 1.** The arcs $\alpha_i$ and $\beta_i$.

**Lemma 1** *(Inversion of Dynnikov coordinates). Let $(a,b) \in \mathcal{C}_n$. Then $(a,b)$ is the Dynnikov coordinate of exactly one element $[\mathcal{F},\mu]$ of $\mathfrak{F}_n$, which has*

$$\mu([\beta_i]) = 2 \max_{1\leqslant k\leqslant n-2}\left(|a_k| + \max(b_k,0) + \sum_{j=1}^{k-1} b_j\right) - 2\sum_{j=1}^{i-1} b_j \quad \textit{and}$$

$$\mu([\alpha_i]) = \begin{cases} (-1)^i a_{\lceil i/2\rceil} + \frac{\mu([\beta_{\lceil i/2\rceil}])}{2} & \textit{if } b_{\lceil i/2\rceil} \geqslant 0, \\ (-1)^i a_{\lceil i/2\rceil} + \frac{\mu([\beta_{1+\lceil i/2\rceil}])}{2} & \textit{if } b_{\lceil i/2\rceil} \leqslant 0. \end{cases}$$

*Here $\lceil x\rceil$ denotes the smallest integer which is not less than x.*

The proof of this lemma is straightforward. Observe that if $\mu([\beta_1])$ is known, then all of the $\mu([\beta_i])$ can be calculated immediately from the coordinates $b_j$, and the $\mu([\alpha_i])$ can then be deduced using the coordinates $a_j$. Finally, $\mu([\beta_1])$ can be determined by using the conditions: that $\mu([\beta_i]) \geqslant 0$ for $1 \leqslant i \leqslant n-1$; that $\mu([\alpha_i]) \geqslant |b_{\lceil i/2\rceil}|$ for $1 \leqslant i \leqslant 2n-4$; and that at least one of these inequalities is an equality (otherwise the foliation would have a leaf parallel to the boundary of $D_n$). These conditions give

$$\mu([\beta_1]) = 2 \max_{1\leqslant k\leqslant n-2}\left(|a_k| + \max(b_k,0) + \sum_{j=1}^{k-1} b_j\right)$$

as in the statement of the lemma.

Projectivizing the Dynnikov coordinates yields an explicit homeomorphism between $S^{2n-5} = \mathcal{C}_n/\mathbb{R}^+$ and the boundary of the Teichmüller space of $D_n$ (that is, the space of projective measured foliations on $D_n$ up to isotopy and Whitehead equivalence).

**Remark 2.** Let $\mathcal{S}_n$ be the set of non-empty finite unions of pairwise disjoint (but not necessarily pairwise non-homotopic) essential simple closed curves on $D_n$, up to isotopy. Denote by $S([\alpha])$ the minimum intersection number of $S \in \mathcal{S}_n$ with an arc $\alpha \in \mathcal{A}_n$. Then there is a bijection $\rho : \mathcal{S}_n \to \mathbb{Z}^{2n-4} \setminus \{0\}$ defined by

$$\rho(S) = (a,b) = (a_1,\ldots,a_{n-2},b_1,\ldots,b_{n-2}),$$

where for $1 \leqslant i \leqslant n-2$,

$$a_i = \frac{S([\alpha_{2i}]) - S([\alpha_{2i-1}])}{2} \quad \text{and} \quad b_i = \frac{S([\beta_i]) - S([\beta_{i+1}])}{2}.$$

This bijection is just the restriction of the Dynnikov coordinate function to the rational measured foliations represented by elements of $\mathcal{S}_n$.

### 2.2. Update rules

The Mapping Class Group of $D_n$ is canonically isomorphic to Artin's braid group $B_n$ modulo its center. $B_n$ thus acts on $\mathfrak{F}_n$, and hence on the space of Dynnikov coordinates. Given $\beta \in B_n$, define $\beta : \mathcal{C}_n \to \mathcal{C}_n$ by $\beta(a,b) = \rho \circ \beta \circ \rho^{-1}(a,b)$.

**Remark 3.** The convention used here for the Artin generators is the normal one in dynamics, i.e. that used in Birman's book [2], where $\sigma_i$ denotes the counter-clockwise interchange of the $i$th and $i+1$th punctures. Note also the unfortunate convention that composition is from left to right when composing braid actions: that is, if $(a,b) \in \mathcal{C}_n$ and $\beta_1, \beta_2 \in B_n$, then $(\beta_1\beta_2)(a,b) = \beta_2(\beta_1(a,b))$.

The *update rules* describe the action of the Artin generators (and their inverses) on $\mathcal{C}_n$. For computational and notational convenience, it is helpful to work in the *max-plus semiring* $(\mathbb{R}, \max, +)$, in which the additive and multiplicative operations are given by $a \oplus b = \max(a,b)$ and $a \otimes b = a + b$. To simplify the notation further, formulae in this semiring will use the normal notation of addition, multiplication, and division, and the fact that these operations are to be interpreted in their max-plus sense will be indicated by enclosing the formulae in square brackets. That is, $[a+b] = \max(a,b)$, $[ab] = a+b$, $[a/b] = a-b$, and $[1] = 0$, the multiplicative identity. For example, the formula

$$a'_i = \left[\frac{a_{i-1}a_ib_i}{a_{i-1}(1+b_i)+a_i}\right]$$

given below is just another way of writing

$$a'_i = a_{i-1} + a_i + b_i - \max\bigl(a_{i-1} + \max(0,b_i), a_i\bigr).$$

**Lemma 4** *(Update rules for Artin generators). Let $(a,b) \in \mathcal{C}_n$ and $1 \leqslant i \leqslant n-1$, and write $\sigma_i(a,b) = (a',b')$. Then $a'_j = a_j$ and $b'_j = b_j$ except when $j = i-1$ or $j = i$, and:*

*if $i = 1$ then*

$$a'_1 = \left[\frac{a_1b_1}{a_1+1+b_1}\right], \qquad b'_1 = \left[\frac{1+b_1}{a_1}\right];$$

*if $2 \leqslant i \leqslant n-2$ then*

$$a'_{i-1} = \bigl[a_{i-1}(1+b_{i-1}) + a_ib_{i-1}\bigr], \qquad b'_{i-1} = \left[\frac{a_ib_{i-1}b_i}{a_{i-1}(1+b_{i-1})(1+b_i)+a_ib_{i-1}}\right],$$

$$a'_i = \left[\frac{a_{i-1}a_ib_i}{a_{i-1}(1+b_i)+a_i}\right], \qquad b'_i = \left[\frac{a_{i-1}(1+b_{i-1})(1+b_i)+a_ib_{i-1}}{a_i}\right];$$

*if $i = n-1$ then*

$$a'_{n-2} = \bigl[a_{n-2}(1+b_{n-2}) + b_{n-2}\bigr], \qquad b'_{n-2} = \left[\frac{b_{n-2}}{a_{n-2}(1+b_{n-2})}\right].$$

The update rules for the inverse generators $\sigma_i^{-1}$ can be obtained from these on conjugating by the involution

$$(a_1, \ldots, a_{n-2}, b_1, \ldots, b_{n-2}) \mapsto \bigl[(1/a_1, \ldots, 1/a_{n-2}, b_1, \ldots, b_{n-2})\bigr]$$

as explained in Section 3 below. These rules are given in the next lemma.

**Lemma 5** *(Update rules for inverse Artin generators). Let $(a,b) \in \mathcal{C}_n$ and $1 \leqslant i \leqslant n-1$, and write $\sigma_i^{-1}(a,b) = (a'',b'')$. Then $a''_j = a_j$ and $b''_j = b_j$ except when $j = i-1$ or $j = i$, and:*

*if $i = 1$ then*

$$a''_1 = \left[\frac{1+a_1(1+b_1)}{b_1}\right], \qquad b''_1 = \bigl[a_1(1+b_1)\bigr];$$

*if $2 \leqslant i \leqslant n-2$ then*

$$a''_{i-1} = \left[\frac{a_{i-1}a_i}{a_{i-1}b_{i-1}+a_i(1+b_{i-1})}\right], \qquad b''_{i-1} = \left[\frac{a_{i-1}b_{i-1}b_i}{a_{i-1}b_{i-1}+a_i(1+b_{i-1})(1+b_i)}\right],$$

$$a''_i = \left[\frac{a_{i-1}+a_i(1+b_i)}{b_i}\right], \qquad b''_i = \left[\frac{a_{i-1}b_{i-1}+a_i(1+b_{i-1})(1+b_i)}{a_{i-1}}\right];$$

*if* $i = n-1$ *then*

$$a''_{n-2} = \left[\frac{a_{n-2}}{a_{n-2}b_{n-2}+1+b_{n-2}}\right], \qquad b''_{n-2} = \left[\frac{a_{n-2}b_{n-2}}{1+b_{n-2}}\right].$$

Using the max-plus notation, the action of any braid $\beta \in B_n$ on $\mathcal{C}_n$ can be computed by composing the functions of Lemmas 4 and 5 in the normal way. For a general braid, of course, the resulting rational functions can be extremely complicated. However, useful results can be obtained for braids which are ascending or descending sequences of contiguous Artin generators (or their inverses): these results are described in the next section.

## 3. Update rules for sequences of contiguous generators

The update rules for the $n$-braids

$$\begin{aligned}
\gamma_n^{k,l} &= \sigma_k\sigma_{k+1}\dots\sigma_{l-1}\sigma_l,\\
\delta_n^{k,l} &= \sigma_l\sigma_{l-1}\dots\sigma_{k+1}\sigma_k,\\
\epsilon_n^{k,l} &= \left(\delta_n^{k,l}\right)^{-1} = \sigma_k^{-1}\sigma_{k+1}^{-1}\dots\sigma_{l-1}^{-1}\sigma_l^{-1}, \quad \text{and}\\
\zeta_n^{k,l} &= \left(\gamma_n^{k,l}\right)^{-1} = \sigma_l^{-1}\sigma_{l-1}^{-1}\dots\sigma_{k+1}^{-1}\sigma_k^{-1},
\end{aligned}$$

where $1 \leqslant k \leqslant l \leqslant n-1$, have a relatively simple form. Their description is, however, complicated by the need to consider separately the "end" cases $k=1$ and $l=n-1$.

**Lemma 6** *(Update rules for $\gamma_n^{k,l}$). Let $n \geqslant 3$, and for $1 \leqslant k \leqslant l \leqslant n-1$ let $\gamma_n^{k,l}$ denote the braid $\sigma_k\sigma_{k+1}\dots\sigma_{l-1}\sigma_l \in B_n$.*

*Given $(a,b) \in \mathcal{C}_n$ and an integer $j$ with $k-1 \leqslant j \leqslant n-2$, write*

$$P_j = P_j(b,k) = \left[(1+b_{k-1})\prod_{i=k}^{j} b_i\right].$$

*(Note the interpretation of this formula in special cases: $P_j(b,k) = [\prod_{i=k}^{j} b_i]$ if $k=1$, $P_j(b,k) = [(1+b_{k-1})]$ if $j=k-1$, and $P_j(b,k) = [1]$ if $k=1$ and $j=0$.) Similarly, for $k \leqslant j \leqslant n-2$, write*

$$S_j = S_j(a,b,k) = \left[\sum_{i=k}^{j} \frac{(1+b_i)P_{i-1}}{a_i}\right].$$

*Let $(a',b') = \gamma_n^{k,l}(a,b)$. Then $a'_j = a_j$ and $b'_j = b_j$ for $j < k-1$ and for $j > l$. Moreover,*

1. *If $k > 1$ and $l < n-1$ then*

$$a'_{k-1} = \left[a_{k-1}(1+b_{k-1}) + a_k b_{k-1}\right], \qquad b'_{k-1} = \left[\frac{a_k b_{k-1} b_k}{a_{k-1}(1+b_{k-1})(1+b_k) + a_k b_{k-1}}\right],$$

$$a'_j = \left[a_{j+1}b_{k-1} + a_{k-1}(a_{j+1}S_j + P_j)\right], \qquad b'_j = \left[b_{j+1}\left(\frac{b_{k-1}+a_{k-1}S_j}{b_{k-1}+a_{k-1}S_{j+1}}\right)\right] \quad (k \leqslant j < l),$$

$$a'_l = \left[\frac{a_{k-1}P_l}{1+b_{k-1}+a_{k-1}S_l}\right], \qquad b'_l = [b_{k-1}+a_{k-1}S_l].$$

2. *If $k > 1$ and $l = n-1$ then the formulae in case 1 hold for $k-1 \leqslant j < n-2$, while*

$$a'_{n-2} = \left[b_{k-1} + a_{k-1}(S_{n-2}+P_{n-2})\right], \qquad b'_{n-2} = \left[\frac{1}{P_{n-2}}\left(\frac{b_{k-1}}{a_{k-1}} + S_{n-2}\right)\right].$$

3. *If $k = 1$ and $l < n-1$ then*

$$a'_j = [P_j + a_{j+1}S_j], \qquad b'_j = [b_{j+1}S_j/S_{j+1}] \quad (1 \leqslant j < l),$$

$$a'_l = \left[P_l/(1+S_l)\right], \qquad b'_l = [S_l].$$

4. *If $k = 1$ and $l = n-1$ then the formulae in case 3 hold for $1 \leqslant j < n-2$, while*

$$a'_{n-2} = [P_{n-2} + S_{n-2}], \qquad b'_{n-2} = [S_{n-2}/P_{n-2}].$$

**Proof.** The proof is a straightforward induction on $l \geqslant k$ for each $k$, with the base case $l = k$ given by the update rules for single braid generators (Lemma 4).

Take, for example, $1 < k < n-1$ (cases 1 and 2). Putting $l = k$ gives $P_l = [(1+b_{k-1})b_k]$ and $S_l = [(1+b_{k-1})(1+b_k)/a_k]$. The rules for $a'_{k-1}$ and $b'_{k-1}$ given in case 1 of the lemma are identical to those of Lemma 4, while

$$a'_k = a'_l = \left[\frac{a_{k-1}P_l}{1+b_{k-1}+a_{k-1}S_l}\right] = \left[\frac{a_{k-1}(1+b_{k-1})b_k}{(1+b_{k-1})+a_{k-1}(1+b_{k-1})(1+b_k)/a_k}\right] = \left[\frac{a_{k-1}a_k b_k}{a_k + a_{k-1}(1+b_k)}\right] \quad \text{and}$$

$$b'_k = b'_l = \left[b_{k-1} + a_{k-1}(1+b_{k-1})(1+b_k)/a_k\right] = \left[\frac{a_k b_{k-1} + a_{k-1}(1+b_{k-1})(1+b_k)}{a_k}\right],$$

in agreement with Lemma 4.

Now assume the result is true for some $l$ with $k \leqslant l < n-1$, so that $\gamma_n^{k,l}(a,b) = (a',b')$ as given by case 1 of the lemma. Let $(a'',b'') = \gamma_n^{k,l+1}(a,b)$, so that $(a'',b'') = \sigma_{l+1}(a',b')$. In particular, $a''_j = a'_j$ and $b''_j = b'_j$ for all $j$ except $l$ and $l+1$. Consider $a''_{l+1}$ for $l+1 < n-1$ and $a''_l$ for $l+1 = n-1$: the other coordinates work similarly.

If $l+1 < n-1$, then Lemma 4 gives

$$\begin{aligned} a''_{l+1} &= \left[\frac{a'_l a'_{l+1} b'_{l+1}}{a'_l(1+b'_{l+1}) + a'_{l+1}}\right] = \left[\frac{a_{l+1}b_{l+1}a_{k-1}P_l/(1+b_{k-1}+a_{k-1}S_l)}{a_{l+1} + (1+b_{l+1})a_{k-1}P_l/(1+b_{k-1}+a_{k-1}S_l)}\right] \\ &= \left[\frac{a_{k-1}P_{l+1}}{1+b_{k-1}+a_{k-1}(S_l + (1+b_{l+1})P_l/a_{l+1})}\right] = \left[\frac{a_{k-1}P_{l+1}}{1+b_{k-1}+a_{k-1}S_{l+1}}\right] \end{aligned}$$

as required. Similarly if $l+1 = n-1$, then Lemma 4 gives

$$\begin{aligned} a''_l &= \left[a'_l(1+b'_l) + b'_l\right] = \left[\frac{a_{k-1}P_l}{1+b_{k-1}+a_{k-1}S_l}(1+b_{k-1}+a_{k-1}S_l) + b_{k-1} + a_{k-1}S_l\right] \\ &= \left[b_{k-1} + a_{k-1}(S_{n-2} + P_{n-2})\right] \quad \text{as required.} \qquad \square \end{aligned}$$

The update rules for $\delta_n^{k,l}$, $\epsilon_n^{k,l}$, and $\zeta_n^{k,l}$, can be derived from Lemma 6 by symmetry, conjugating by an appropriate transformation as described below:

*Reflection in the horizontal diameter of the disk*: sends each braid generator $\sigma_i$ to $\sigma_i^{-1}$. The corresponding transformation of Dynnikov coordinates is given by

$$(a_1,\dots,a_{n-2},b_1,\dots,b_{n-2}) \mapsto (-a_1,\dots,-a_{n-2},b_1,\dots,b_{n-2}),$$

or, in max-plus notation,

$$(a_1,\dots,a_{n-2},b_1,\dots,b_{n-2}) \mapsto \left[(1/a_1,\dots,1/a_{n-2},b_1,\dots,b_{n-2})\right].$$

Thus the update rules for $\epsilon_n^{k,l}$ can be obtained by conjugating the rules of Lemma 6 by this involution.

*Reflection in the vertical diameter of the disk*: sends each braid generator $\sigma_i$ to $\sigma_{n-i}^{-1}$. The corresponding transformation of Dynnikov coordinates is given by

$$(a_1,\dots,a_{n-2},b_1,\dots,b_{n-2}) \mapsto (a_{n-2},\dots,a_1,-b_{n-2},\dots,-b_1),$$

or, in max-plus notation,

$$(a_1,\dots,a_{n-2},b_1,\dots,b_{n-2}) \mapsto \left[(a_{n-2},\dots,a_1,1/b_{n-2},\dots,1/b_1)\right].$$

Thus the update rules for $\zeta_n^{k,l}$ can be obtained by conjugating the rules of Lemma 6 by this involution.

*Rotation through $\pi$ about the center of the disk*: sends each braid generator $\sigma_i$ to $\sigma_{n-i}$. The corresponding transformation of Dynnikov coordinates is given by

$$(a_1,\dots,a_{n-2},b_1,\dots,b_{n-2}) \mapsto (-a_{n-2},\dots,-a_1,-b_{n-2},\dots,-b_1),$$

or, in max-plus notation,

$$(a_1,\dots,a_{n-2},b_1,\dots,b_{n-2}) \mapsto \left[(1/a_{n-2},\dots,1/a_1,1/b_{n-2},\dots,1/b_1)\right].$$

Thus the update rules for $\delta_n^{k,l}$ can be obtained by conjugating the rules of Lemma 6 by this involution.

An example which will be used later is given: here the update rules for $\delta_n^{k,l}$ are derived from those of Lemma 6 for $\gamma_n^{n-l,n-k}$ by conjugating by a rotation through $\pi$ about the center of the disk.

**Lemma 7** *(Update rules for $\delta_n^{k,l}$). Let $n \geqslant 3$, and for $1 \leqslant k \leqslant l \leqslant n-1$ let $\delta_n^{k,l}$ denote the braid $\sigma_l\sigma_{l-1}\dots\sigma_{k+1}\sigma_k \in B_n$. Given $(a,b) \in \mathcal{C}_n$ and an integer $j$ with $\max(k-1,1) \leqslant j \leqslant l$ write*

$$\widetilde{P}_j = \widetilde{P}_j(b,l) = \left[(1+b_l)\prod_{i=j}^{l}\frac{1}{b_i}\right].$$

*(In the special case $l = n-1$, $\widetilde{P}_j(b,n-1) = [\prod_{i=j}^{n-2}\frac{1}{b_i}]$ for $j < l$, while $\widetilde{P}_{n-1}(b,n-1) = [1]$.) Similarly, for $\max(k-1,1) \leqslant j \leqslant l-1$ write*

$$\widetilde{S}_j = \widetilde{S}_j(a,b,l) = \left[\sum_{i=j}^{l-1}\frac{a_i(1+b_i)\widetilde{P}_{i+1}}{b_i}\right].$$

*Let $(a',b') = \delta_n^{k,l}(a,b)$. Then $a'_j = a_j$ and $b'_j = b_j$ for $j < k-1$ and for $j > l$. Moreover,*

1. *If $k > 1$ and $l < n-1$ then*

$$a'_{k-1} = \left[\frac{a_l(1+b_l)+b_l\widetilde{S}_{k-1}}{b_l\widetilde{P}_{k-1}}\right], \qquad b'_{k-1} = \left[\frac{a_lb_l}{a_l+b_l\widetilde{S}_{k-1}}\right],$$

$$a'_j = \left[\frac{a_{j-1}a_lb_l}{a_l+b_l(\widetilde{S}_j+a_{j-1}\widetilde{P}_j)}\right], \qquad b'_j = \left[b_{j-1}\left(\frac{a_l+b_l\widetilde{S}_{j-1}}{a_l+b_l\widetilde{S}_j}\right)\right] \quad (k \leqslant j < l),$$

$$a'_l = \left[\frac{a_{l-1}a_lb_l}{a_{l-1}(1+b_l)+a_l}\right], \qquad b'_l = \left[\frac{a_{l-1}(1+b_{l-1})(1+b_l)+a_lb_{l-1}}{a_l}\right].$$

2. *If $k = 1$ and $l < n-1$ then the formulae in case 1 hold for $2 \leqslant j \leqslant l$, while*

$$a'_1 = \left[\frac{a_lb_l}{a_l+b_l(\widetilde{S}_1+\widetilde{P}_1)}\right], \qquad b'_1 = \left[\frac{b_l\widetilde{P}_1}{a_l+b_l\widetilde{S}_1}\right].$$

3. *If $k > 1$ and $l = n-1$ then*

$$a'_j = \left[\frac{a_{j-1}}{a_{j-1}\widetilde{P}_j+\widetilde{S}_j}\right], \qquad b'_j = [b_{j-1}\widetilde{S}_{j-1}/\widetilde{S}_j] \quad (k \leqslant j \leqslant n-2),$$

$$a'_{k-1} = [(1+\widetilde{S}_{k-1})/\widetilde{P}_{k-1}], \qquad b'_{k-1} = [1/\widetilde{S}_{k-1}].$$

4. *If $k = 1$ and $l = n-1$ then the formulae in case 3 hold for $2 \leqslant j \leqslant n-2$, while*

$$a'_1 = [1/(\widetilde{P}_1+\widetilde{S}_1)], \qquad b'_1 = [\widetilde{P}_1/\widetilde{S}_1].$$

## 4. Computing topological entropy in families of braids

A braid $\beta \in B_n$ is pseudo-Anosov if and only if there is some $(a^u,b^u) \in \mathcal{C}_n$ (corresponding to the unstable foliation of $\beta$) and a number $r > 1$ (the dilatation of $\beta$) such that $\beta(a^u,b^u) = r(a^u,b^u)$. In this case $\beta$ has topological entropy $h(\beta) = \log r$; there is an element $(a^s,b^s)$ of $\mathcal{C}_n$ (corresponding to the stable foliation of $\beta$) with $\beta(a^s,b^s) = \frac{1}{r}(a^s,b^s)$; and any $(a,b) \in \mathcal{C}_n$ satisfying $\beta(a,b) = k(a,b)$ for some $k > 0$ is a multiple either of $(a^u,b^u)$ or of $(a^s,b^s)$.

$\beta$ is a reducible braid if and only if there is some $(a,b) \in \mathbb{Z}^{2n-4} \setminus \{0\}$ (corresponding to a system of reducing curves, see Remark 2) with $\beta(a,b) = (a,b)$.

If there is no $(a,b) \in \mathcal{C}_n$ and $k > 0$ with $\beta(a,b) = k(a,b)$, then $\beta$ is a finite order braid, and hence there is some $N > 0$ such that $\beta^N(a,b) = (a,b)$ for all $(a,b) \in \mathcal{C}_n$.

In many cases it is possible to do a simultaneous analysis of this type of every braid in a family. This provides a method of computing the topological entropy of braids in such families which is more direct and tractable than the train track approach. In this section, this method is illustrated with two families of braids considered in [10], which are of interest in the study of braids of low topological entropy. These families are $\{\beta_{m,n}\colon\ m,n \geqslant 1\}$, and $\{\sigma_{m,n}\colon\ 1 \leqslant m \leqslant n\}$, where

$$\beta_{m,n} = \sigma_1\dots\sigma_m\sigma_{m+1}^{-1}\dots\sigma_{m+n}^{-1} = \gamma_{m+n+1}^{1,m}\epsilon_{m+n+1}^{m+1,m+n} \in B_{m+n+1} \quad \text{and}$$

$$\sigma_{m,n} = \sigma_1\dots\sigma_m\,\sigma_m\dots\sigma_1\,\sigma_1\dots\sigma_{m+n} = \gamma_{m+n+1}^{1,m}\delta_{m+n+1}^{1,m}\gamma_{m+n+1}^{1,m+n} \in B_{m+n+1}.$$

The approach taken here can be contrasted with the method of proof of the same results in [10].

*4.1. A family of pseudo-Anosov braids*

The following result establishes that $\beta_{m,n}$ is a pseudo-Anosov braid for all $m, n \geqslant 1$, and gives the topological entropy $h(\beta_{m,n})$.

**Theorem 8** *(The braids $\beta_{m,n}$). Let $m, n \geqslant 1$. Then $\beta_{m,n} \in B_{m+n+1}$ is a pseudo-Anosov braid, whose dilatation $r$ is the unique root in $(1, \infty)$ of the polynomial*

$$f_{m,n}(r) = (r-1)\big(r^{m+n+1}-1\big) - 2r\big(r^m + r^n\big).$$

*The Dynnikov coordinates $(a, b) \in \mathcal{C}_{m+n+1}$ of the unstable invariant measured foliation of $\beta_{m,n}$ are given by*

$$a_i = \begin{cases} -r(r^n+1)(r^i-1) & \text{if } 1 \leqslant i \leqslant m-1, \\ -(r^{m+1}-1)(r^{n+1}-1) & \text{if } i = m, \\ -(r^{m+1}-1)(r^{m+n+1-i}-1)r^{i-m} & \text{if } m+1 \leqslant i \leqslant m+n-1, \end{cases}$$

$$b_i = \begin{cases} -(r-1)(r^n+1)r^{i+1} & \text{if } 1 \leqslant i \leqslant m-1, \\ -(r+1)(r^{m+1}-1) & \text{if } i = m, \\ -(r-1)(r^{m+1}-1)r^{i-m} & \text{if } m+1 \leqslant i \leqslant m+n-1. \end{cases}$$

**Proof.** $f_{m,n}$ has a root $r > 1$ since $f_{m,n}(1) = -4$. It will be shown that $\beta_{m,n}(a,b) = r(a,b)$, from which the result (and the uniqueness of $r$) follows.

Write $N = m+n+1$ and recall that $\beta_{m,n} = \gamma_N^{1,m}\epsilon_N^{m+1,N-1}$. Thus to show that $\beta_{m,n}(a,b) = r(a,b)$ it suffices to show that $\gamma_N^{1,m}(a,b) = r\delta_N^{m+1,N-1}(a,b)$. It will be shown that each side of this equation is equal to $(a', b')$, where

$$(a'_j, b'_j) = \begin{cases} (ra_j, rb_j), & 1 \leqslant j < m, \\ (a_m + b_m, r(r^n+1)(r+1)), & j = m, \\ (a_j, b_j), & m < j \leqslant m+n-1. \end{cases}$$

Observe that

$$ra_{m-1} - a_m + a_1 = f_{m,n}(r) + 2r\big(1+r^n\big) = 2r\big(1+r^n\big) > 0. \tag{1}$$

Consider first $(a', b') = \gamma_N^{1,m}(a,b)$, which is given by Lemma 6. The first step is to calculate the quantities $P_j$ and $S_j$ from the statement of Lemma 6 for $1 \leqslant j \leqslant m$.

Now $P_j = \sum_{i=1}^{j} b_i$, giving $P_j = -r^2(r^n+1)(r^j-1) = ra_j$ for $1 \leqslant j < m$; and hence $P_m = P_{m-1} + b_m = ra_{m-1} + b_m$. On the other hand,

$$S_j = \max_{1\leqslant i\leqslant j}\big(\max(0, b_i) + P_{i-1} - a_i\big) = \max_{1\leqslant i\leqslant j}(ra_{i-1} - a_i)$$

(setting $a_0 = 0$), since $b_i < 0$ for all $i$. Now $ra_{i-1} - a_i = -a_1$ for all $i < m$, so $S_j = -a_1$ for $1 \leqslant j < m$. Finally $S_m = \max(-a_1, ra_{m-1} - a_m) = ra_{m-1} - a_m$ by (1).

Let $1 \leqslant j \leqslant m-2$. Then (using case 3 of Lemma 6)

$$a'_j = \max(P_j, a_{j+1} + S_j) = \max(ra_j, a_{j+1} - a_1) = \max(ra_j, ra_j) = ra_j \quad \text{and}$$

$$b'_j = b_{j+1} + S_j - S_{j+1} = b_{j+1} = rb_j \quad \text{as required.}$$

Let $j = m-1$. Then $a'_{m-1} = \max(P_{m-1}, a_m + S_{m-1}) = \max(ra_{m-1}, a_m - a_1) = ra_{m-1}$ by (1), and $b'_{m-1} = b_m + S_{m-1} - S_m = b_m - a_1 - (ra_{m-1} - a_m) = rb_{m-1}$ as required.

Let $j = m$. Then $a'_m = P_m - \max(0, S_m) = ra_{m-1} + b_m - (ra_{m-1} - a_m) = a_m + b_m$ as required, while $b'_m = S_m = ra_{m-1} - a_m = 2r(1+r^n) - a_1$ by (1), giving $b'_m = r(r^n+1)(r+1)$ as required.

Now let $(a'', b'') = \delta_N^{m+1,N-1}(a,b)$. Showing that $(a'', b'') = (a', b')/r$, will complete the proof. The argument, using Lemma 7, is similar to the first part of the proof. Calculating the quantities $\widetilde{P}_j$ and $\widetilde{S}_j$ from the statement of Lemma 7 gives

$$\widetilde{P}_j = r^{j-m}\big(r^{m+1}-1\big)\big(r^{m+n-j}-1\big), \qquad \widetilde{S}_j = -r^n(r-1)\big(r^{m+1}-1\big) \quad (j > m),$$

$$\widetilde{P}_m = \big(r^{m+1}-1\big)\big(r^n+1\big), \qquad \widetilde{S}_m = -(r+1)\big(r^n+1\big).$$

Then, by case 3 of Lemma 7,

$$a''_m = \max(0, \widetilde{S}_m) - \widetilde{P}_m = -\widetilde{P}_m = (a_m + b_m)/r,$$
$$b''_m = -\widetilde{S}_m = \big(r^n + 1\big)(r+1),$$
$$a''_{m+1} = a_m - \max(a_m + \widetilde{P}_{m+1}, \widetilde{S}_{m+1}) = a_m - \widetilde{S}_{m+1} = a_{m+1}/r,$$
$$b''_{m+1} = b_m + \widetilde{S}_m - \widetilde{S}_{m+1} = b_{m+1}/r + f_{m,n}(r) = b_{m+1}/r,$$
$$a''_j = a_{j-1} - \max(a_{j-1} + \widetilde{P}_j, \widetilde{S}_j) = a_{j-1} - \widetilde{S}_j = a_j/r \quad (j > m+1), \quad \text{and}$$
$$b''_j = b_{j-1} + \widetilde{S}_{j-1} - \widetilde{S}_j = b_{j-1} = b_j/r \quad (j > m+1)$$

as required. □

**Remark 9.** The proof of Theorem 8 is self-contained. However, one might ask how the polynomial $f_{m,n}$ and the Dynnikov coordinates of the unstable measured foliation $[\mathcal{F}_{m,n}, \mu_{m,n}]$ of $\beta_{m,n}$ were found.

To find the train tracks for an infinite family of braids, the usual method would be to compute train tracks (using, for example, the Bestvina–Handel algorithm [1]) for enough examples to spot a general pattern, and then to prove that the conjectured pattern does indeed hold for all braids in the family. The method here is similar. Since $[\mathcal{F}_{m,n}, \mu_{m,n}]$ is an attracting fixed point for the action of $\beta_{m,n}$ on the boundary of Teichmüller space, it is easy to find its Dynnikov coordinates numerically. Having done this for several cases of $m$ and $n$, one can guess how the various maxima in the statements of Lemmas 6 and 7 are resolved. This yields the following statement (provided $m, n \geqslant 2$):

Assume that $a_i \leqslant 0$; $b_i \leqslant 0$; $a_{i+1} = a_i + b_i$ for $1 \leqslant i \leqslant m-2$; $a_m \leqslant a_{m-1} + b_{m-1}$; $a_{m+1} \geqslant a_m + b_m$; $a_{i+1} = a_i - b_i$ for $m+1 \leqslant i \leqslant m+n-2$; and $a_{m+n-1} \leqslant b_{m+n-1}$.

Let $\xi = -a_1 + (a_{m-1} + b_{m-1} - a_m) + (a_{m+1} - a_m - b_m) \geqslant 0$. Then

$$\beta_{m,n}(a, b) = (a', b'),$$

where

$$a'_i = \begin{cases} b_1, & i = 1, \\ a_{i+1} - a_1, & 2 \leqslant i \leqslant m-2, \\ a_{m-1} + b_{m-1} - a_1, & i = m-1, \\ a_{i+1} - \xi, & m \leqslant i \leqslant m+n-2, \\ a_{m+n-1} - b_{m+n-1} - \xi, & i = m+n-1, \end{cases}$$

$$b'_i = \begin{cases} b_{i+1}, & 1 \leqslant i \leqslant m-2, \\ a_m - a_{m-1} + b_m - b_{m-1}, & i = m-1, \\ a_m - a_{m+1} + b_m + b_{m+1}, & i = m, \\ b_{i+1}, & m+1 \leqslant i \leqslant m+n-2, \\ a_{m+n-1} - b_{m+n-1}, & i = m+n-1. \end{cases}$$

Solving for an eigenvalue $r \in (1, \infty)$ and the associated eigenvector $(a, b)$ yields the statement of Theorem 8.

**Remark 10.** The singularity structure of the invariant foliation $[\mathcal{F}_{m,n}, \mu_{m,n}]$ can be seen in its Dynnikov coordinates. The equations

$$a_{i+1} = a_i + b_i \quad \text{if } 1 \leqslant i \leqslant m-2,$$
$$a_{i+1} = a_i - b_i \quad \text{if } m+1 \leqslant i \leqslant m+n-2,$$

of Remark 9 correspond to the existence of an $(m+1)$-pronged singularity and an $(n+1)$-pronged singularity, respectively.

### *4.2. The reducible case*

Consider now the braids $\sigma_{m,n}$ for $1 \leqslant m \leqslant n$. If $n \geqslant m+2$ then $\sigma_{m,n}$ is a pseudo-Anosov braid: the following result can be proved analogously to Theorem 8.

**Theorem 11** *(The braids $\sigma_{m,n}$ for $n \geqslant m+2$). Let $1 \leqslant m \leqslant n-2$. Then $\sigma_{m,n} \in B_{m+n+1}$ is a pseudo-Anosov braid, whose dilatation $r$ is the unique root in $(1, \infty)$ of the polynomial*

$$g_{m,n}(r) = (r-1)\big(r^{m+n+1} + 1\big) + 2r\big(r^m - r^n\big).$$

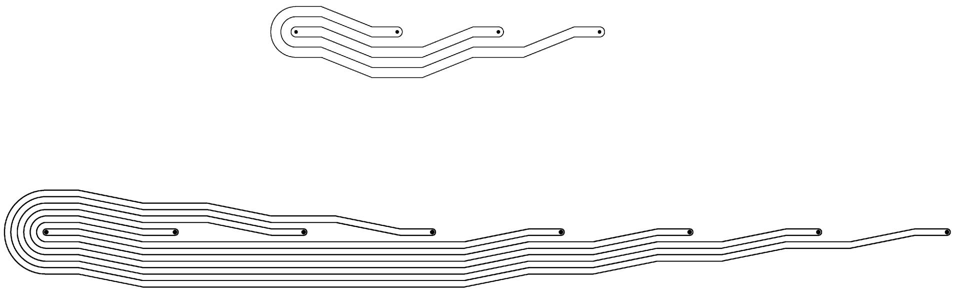

**Fig. 2.** The reducing systems $S_1 \in \mathcal{S}_4$ and $S_3 \in \mathcal{S}_8$.

*The Dynnikov coordinates* $(a,b) \in \mathcal{C}_{m+n+1}$ *of the unstable invariant measured foliation of* $\sigma_{m,n}$ *are given by*

$$a_i = \begin{cases} r(r^n-1)(r^{i+1}-1) & \text{if } 1 \leqslant i \leqslant m-1, \\ (r^{m+1}-1)(r^{m+n-i}-1)r^{i+1-m} & \text{if } m \leqslant i \leqslant m+n-1, \end{cases}$$

$$b_i = \begin{cases} (r-1)(r^n-1)r^{i+1} & \text{if } 1 \leqslant i \leqslant m-1, \\ (r-1)(r^{m+1}-1)r^{i-m} & \text{if } m \leqslant i \leqslant m+n-1. \end{cases}$$

However, the focus in this subsection is on the case $n = m+1$, when $\sigma_{m,n}$ is a reducible braid. Again, the emphasis in the next result is on the transparent computational nature of the proof, when compared with a more direct approach such as conjugating the braids in some suitable way and then appealing to the reader to observe that the resulting braids leave a certain system of curves invariant.

**Theorem 12.** *Let* $m \geqslant 1$. *Then the braid* $\sigma_{m,m+1} \in B_{2m+2}$ *is reducible, having a system of reducing curves* $S_m \in \mathcal{S}_{2m+2}$ *with* $\rho(S_m) = (a,b) \in \mathbb{Z}^{4m} \setminus \{0\}$ *given by*

$$(a_i, b_i) = \begin{cases} (i+1, 1), & 1 \leqslant i \leqslant m, \\ (2m+1-i, 1), & m+1 \leqslant i \leqslant 2m \end{cases}$$

(*see Fig.* 2).

**Proof.** Recall that $\sigma_{m,m+1} = \gamma^{1,m}_{2m+2}\delta^{1,m}_{2m+2}\gamma^{1,2m+1}_{2m+2}$. The method of proof is to compute successively $(a^{(1)}, b^{(1)}) = \gamma^{1,m}_{2m+2}(a,b)$, $(a^{(2)}, b^{(2)}) = \delta^{1,m}_{2m+2}(a^{(1)}, b^{(1)})$, and $(a^{(3)}, b^{(3)}) = \gamma^{1,2m+1}_{2m+2}(a^{(2)}, b^{(2)})$, and then to observe that $(a^{(3)}, b^{(3)}) = (a,b)$. The calculations are straightforward using Lemmas 6 and 7.

1. $(a^{(1)}, b^{(1)})$ is computed using case 3 of Lemma 6. The quantities $P_j$ and $S_j$ are given for $j \leqslant m$ by $P_j = \sum_{i=1}^{j} b_i = j$ and

   $$S_j = \max_{1 \leqslant i \leqslant j}\big(\max(b_i, 0) + P_{i-1} - a_i\big) = \max_{1 \leqslant i \leqslant j}\big(1 + (i-1) - (i+1)\big) = -1.$$

   Then for $1 \leqslant j < m$,

   $$a^{(1)}_j = \max(P_j, a_{j+1} + S_j) = \max(j, j+2-1) = j+1,$$
   $$b^{(1)}_j = b_{j+1} + S_j - S_{j+1} = 1 - 1 + 1 = 1.$$

   Finally $a^{(1)}_m = P_m - \max(S_m, 0) = m - \max(-1, 0) = m$, and $b^{(1)}_m = S_m = -1$. Thus

   $$\big(a^{(1)}_i, b^{(1)}_i\big) = \begin{cases} (i+1, 1), & 1 \leqslant i < m, \\ (m, -1), & i = m, \\ (2m+1-i, 1), & m+1 \leqslant i \leqslant 2m. \end{cases}$$

2. $(a^{(2)}, b^{(2)})$ is computed using case 2 of Lemma 7. The quantities $\widetilde{P}_j$ and $\widetilde{S}_j$ are given for $j \leqslant m$ by

   $$\widetilde{P}_j = \max\big(b^{(1)}_m, 0\big) - \sum_{i=j}^{m} b^{(1)}_i = 1 + j - m$$

   and $\widetilde{S}_j = \max_{j \leqslant i \leqslant m-1}(a^{(1)}_i + \max(b^{(1)}_i, 0) + \widetilde{P}_{i+1} - b^{(1)}_i) = m+1$. Hence

$$a_1^{(2)} = a_m^{(1)} + b_m^{(1)} - \max\bigl(a_m^{(1)}, b_m^{(1)} + \max(\widetilde{S}_1, \widetilde{P}_1)\bigr) = m - 1 - \max\bigl(m, -1 + \max(m+1, 2-m)\bigr) = -1,$$
$$b_1^{(2)} = b_m^{(1)} + \widetilde{P}_1 - \max\bigl(a_m^{(1)}, b_m^{(1)} + \widetilde{S}_1\bigr) = -1 + (2-m) - \max(m, -1 + m + 1) = 1 - 2m,$$
$$a_m^{(2)} = a_{m-1}^{(1)} + a_m^{(1)} + b_m^{(1)} - \max\bigl(a_{m-1}^{(1)} + \max\bigl(b_m^{(1)}, 0\bigr), a_m^{(1)}\bigr) = m + m - 1 - \max(m, m) = m - 1,$$
$$b_m^{(2)} = \max\bigl(a_{m-1}^{(1)} + \max\bigl(b_{m-1}^{(1)}, 0\bigr) + \max\bigl(b_m^{(1)}, 0\bigr), a_m^{(1)} + b_{m-1}^{(1)}\bigr) - a_m^{(1)} = \max(m + 1 + 0, m + 1) - m = 1,$$

and for $2 \leqslant j < m$,

$$\begin{aligned} a_j^{(2)} &= a_{j-1}^{(1)} + a_m^{(1)} + b_m^{(1)} - \max\bigl(a_m^{(1)}, b_m^{(1)} + \max\bigl(\widetilde{S}_j, a_{j-1}^{(1)} + \widetilde{P}_j\bigr)\bigr) \\ &= j + m - 1 - \max\bigl(m, -1 + \max(m+1, 2j + 1 - m)\bigr) = j + m - 1 - m = j - 1, \end{aligned}$$

$$b_j^{(2)} = b_{j-1}^{(1)} + \max\bigl(a_m^{(1)}, b_m^{(1)} + \widetilde{S}_{j-1}\bigr) - \max\bigl(a_m^{(1)}, b_m^{(1)} + \widetilde{S}_j\bigr) = b_{j-1}^{(1)} = 1.$$

Thus

$$\bigl(a_i^{(2)}, b_i^{(2)}\bigr) = \begin{cases} (-1, 1 - 2m), & i = 1, \\ (i - 1, 1), & 2 \leqslant i \leqslant m, \\ (2m + 1 - i, 1), & m + 1 \leqslant i \leqslant 2m. \end{cases}$$

3. $(a^{(3)}, b^{(3)})$ is computed using case 4 of Lemma 6. The quantities $P_j$ and $S_j$ are given by $P_j = \sum_{i=1}^{j} b_i^{(2)} = j - 2m$ (and $P_0 = 0$); and

$$S_j = \max_{1 \leqslant i \leqslant j} \bigl(\max\bigl(b_i^{(2)}, 0\bigr) + P_{i-1} - a_i^{(2)}\bigr).$$

Now $\max(b_i^{(2)}, 0) + P_{i-1} - a_i^{(2)}$ is equal to 1 when $i = 1$ and is negative for $i > 1$, and hence $S_j = 1$ for all $j$. Thus $a_{2m}^{(3)} = \max(P_{2m}, S_{2m}) = 1$, $b_{2m}^{(3)} = S_{2m} - P_{2m} = 1$, and for $1 \leqslant j < 2m$,

$$\begin{aligned} a_j^{(3)} &= \max\bigl(P_j, a_{j+1}^{(2)} + S_j\bigr) = \max\bigl(j - 2m, a_{j+1}^{(2)} + 1\bigr) = a_{j+1}^{(2)} + 1 \\ &= \begin{cases} j + 1, & 1 \leqslant j \leqslant m - 1, \\ 2m + 1 - (j + 1) + 1, & m \leqslant j \leqslant 2m - 1, \end{cases} \end{aligned}$$

$$b_j^{(3)} = b_{j+1}^{(2)} + S_j - S_{j+1} = b_{j+1}^{(2)} = 1.$$

Hence $(a^{(3)}, b^{(3)}) = (a, b)$ as required. □

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