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Global Existence in a General Stefan-like Problem

PETER KNABNER

Institut für Mathematik, Universität Augsburg, Memminger Str. 6 D-8900 Augsburg, West Germany

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We consider a one-dimensional two-phase Stefan-like free boundary problem for the heat equation with nonlinear Neumann boundary conditions. Local existence has been shown in [8], furthermore it is clear that cases exist, where global existence fails. The question arises, under which conditions the solution exists globally. In this paper it is shown that especially the natural sign conditions are sufficient for global existence.

1. INTRODUCTION

In the last decades free boundary problems for the heat equation have been intensively studied. The most general results for two-phase Stefan-like problems in one space dimension have been obtained by Fasano and Primicerio in [8]. Like them we will deal with a *general Stefan-like problem with nonlinear Neumann condition*. Using the notation,

$$\begin{aligned} D^-_{T_1T_2}(s) &:= \{(x,t) \mid T_1 < t \leqslant T_2, \quad 0 < x < s(t)\}, \\ D^+_{T_1T_2}(s) &:= \{(x,t) \mid T_1 < t \leqslant T_2, \quad s(t) < x < 1\}, \qquad (1.1) \\ D^-_T(s) &:= D^-_{0T}(s) \quad \text{and analogously } D^+_T(s), \end{aligned}$$

we will consider:

DEFINITION 1.1. Let $T>0$, $b \in (0,1)$, $c_i > 0$ and functions $q^{(i)}$, $h^{(i)}$, $g^{(i)}$, $i=1,2$, λ and μ be given.

Then a *solution* $(u^{(1)}, u^{(2)}, s)$ *in* $[0, T]$ must fulfill

$$u^{(1)}_{xx} - c_1 u^{(1)}_t = q^{(1)} \qquad \text{in } D^-_T(s), \quad (1.2)$$

$$u^{(1)}(x,0) = h^{(1)}(x), \qquad 0 \leqslant x \leqslant b = s(0), \quad (1.3)$$

$$u^{(1)}_x(0,t) = g^{(1)}[u^{(1)}(0,t), t], \qquad 0 < t \leqslant T, \quad (1.4)$$

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$$u^{(2)}_{xx} - c_2 u^{(2)}_t = q^{(2)} \qquad \text{in } D^+_T(s), \quad (1.5)$$

$$u^{(2)}(x, 0) = h^{(2)}(x), \qquad b \leqslant x \leqslant 1, \quad (1.6)$$

$$u^{(2)}_x(1, t) = g^{(2)}[u^{(2)}(1, t), t], \qquad 0 < t \leqslant T, \quad (1.7)$$

$$u^{(1)}(s(t), t) = u^{(2)}(s(t), t) = 0, \qquad 0 < t \leqslant T, \quad (1.8)$$

$$-u^{(1)}_x(s(t), t) + u^{(2)}_x(s(t), t) = \lambda(s(t), t)\, \dot{s}(t) + \mu(s(t), t), \qquad 0 < t \leqslant T, \quad (1.9)$$

and exhibit the following regularity:

$$s(t) \in (0, 1) \qquad \text{for all} \quad t \in [0, T], s \in C[0, T] \cap C^1(0, T],$$

$u^{(1)} \in C(\overline{D^-_T(s)})$, $u^{(1)}_x(x, t)$ is continuous in $t \in (0, T]$, $x \in [0, s(t)]$, $u^{(1)}_{xx}$, $u^{(1)}_t$ are continuous in $D^-_T(s)$, and analoguous properties for $u^{(2)}$.

To simplify the notation, we take the data to be defined for all $t \geqslant 0$. Under suitable regularity assumptions Fasano and Primicerio proved *local existence*. Let $(u^{(1)}, u^{(2)}, s)$ be a solution and

$$[0, T^*) \qquad \text{its maximal interval of existence}, \qquad (1.10)$$

i.e., for all $T < T^*(u^{(1)}, u^{(2)}, s)$ is a solution in $[0, T]$ and there is no continuation to $[0, T^*]$. For the *global behaviour* the following cases are possible:

(A) $T^* = \infty$.

(B) $T^* < \infty$, $0 < \liminf_{t \to T^*} s(t) \leqslant \limsup_{t \to T^*} s(t) < 1$.

(C) $T^* < \infty$, $\liminf_{t \to T^*} s(t) = 0$ or $\limsup_{t \to T^*} s(t) = 1$.

In general, all these cases actually can occur. Indeed, for the one-phase problem this is known since [14] and has been thoroughly studied by Fasano and Primicerio (e.g., [13]). On the other hand, in the *classical two-phase Stefan problem* ($q^{(i)} = \mu = 0$, $\lambda = 1$, $g^{(i)}[y, t] = g^{(i)}(t) \leqslant 0$, $(-1)^{i+1} h^{(i)} \geqslant 0$) case (B) cannot occur. This results dates back to [11], where an L^∞-estimate for u_x is proved, which allows the continuation of a solution beyond every $T^* < \infty$ in the case (B). Thus it is likely that the *sign restriction* $(-1)^{i+1} u^{(i)} \geqslant 0$ is a decisive property to exclude pathologies like (B). The approach of [11], however, seems not to be applicable to the general problem above, but due to the mentioned result about local existence (see Theorem 2.2), we need only to control some L^p-norm of $u^{(i)}_x(\cdot, t)$. This will be done by blending and extending of techniques, developed in [1, 4, 10]. We will end up with the following result:

THEOREM. *If* $u^{(1)} \geqslant 0$, $u^{(2)} \leqslant 0$, *then*

(1) *Case* (B) *cannot occur.*

(2) *In case* (C) *we have*

$$\lim_{t \to T^*} s(t) = 0 \quad \text{or} \quad = 1 \qquad \text{and} \qquad s \in C^{2/5}(0, T^*].$$

The disappearance of a phase cannot be excluded without information about the total energy supplied (cf. [2]), therefore this result—apart from the Hölder exponent—seems to be optimal. Throughout the paper, we will use the following:

REGULARITY ASSUMPTION A1. *Let* $\Omega_{T_1T_2} := (0, 1) \times (T_1, T_2)$, $\Omega_T := \Omega_{0T}$. *For* $T > 0$ *there exists some* $\gamma \in (0, 1]$ *such that*

$$q^{(i)} \in H^{1+\gamma,(1+\gamma)/2}(\Omega_T), \qquad q^{(i)} \quad \text{is bounded in } \bar{\Omega}_T,$$

$q^{(i)}$ *is Hölder continuous with respect to* x *uniformly in* $\bar{\Omega}_T$. (1.11)
$g^{(i)}$ *is continuous in* $\mathbb{R} \times \{t \in \mathbb{R} \mid t \geqslant 0\}$, *for each compact* $K \subset \mathbb{R}$ *there is an* $L > 0$ *such that*

$$|g^{(i)}[y_1, t] - g^{(i)}[y_2, t]| \leqslant L\, |y_1 - y_2| \qquad \text{for} \quad y_{1,2} \in K,\ t \geqslant 0,$$

there exist Y', Y'', G', G'' *such that*

$$\begin{aligned} (-1)^{i+1}\, g^{(i)}[y, t] &\geqslant G' \qquad \text{for} \quad y \geqslant Y',\ t \geqslant 0, \\ (-1)^{i+1}\, g^{(i)}[y, t] &\leqslant G'' \qquad \text{for} \quad y \leqslant Y'',\ t \geqslant 0. \end{aligned} \tag{1.12}$$

$\mu \in H^{\gamma}(\Omega_T) \cap C(\bar{\Omega}_T)$ *and* μ *is Lipschitz continuous with respect to* x *uniformly in* $\bar{\Omega}_T$. (1.13)

λ, λ_x, λ_t, $\lambda_{xx} \in C(\bar{\Omega}_T)$, *there exist* λ', $\lambda'' > 0$ *such that*

$$\lambda' \leqslant \lambda(x, t) \leqslant \lambda'' \qquad \text{for} \quad (x, t) \in \bar{\Omega}_T. \tag{1.14}$$

$h^{(i)}$ *is continuous in* $[0, b]$ (*resp.* $[b, 1]$) *and there are* $H > 0$, $\alpha \in (0, 1]$ *such that*

$$\begin{aligned} |h^{(1)}(x)| &\leqslant H(b - x)^{\alpha} \qquad \text{for} \quad x \in [0, b], \\ |h^{(2)}(x)| &\leqslant H(x - b)^{\alpha} \qquad \text{for} \quad x \in [b, 1]. \end{aligned} \tag{1.15}$$

Here and in the following we adopt the function space notation of [12], which we also apply to noncylindrical domains. The assumptions above basically coincide with those from [8]. At some places we have to strengthen to:

REGULARITY ASSUMPTION A2. (A1) *is fulfilled and additionally for* $0 < T_1 < T_2$

$$\text{there exists } \gamma \in (0, 1] \text{ such that } q^{(i)} \in H^{\gamma, \gamma/2}(\overline{\Omega_{T_1 T_2}}), \tag{1.16}$$

for compact $K \subset \mathbb{R}$ *there exists* $\gamma \in (\frac{1}{2}, 1]$ *such that*

$$|g^{(i)}[y, t_1] - g^{(i)}[y, t_2]| \leqslant |t_1 - t_2|^{\gamma} \quad \text{for} \quad t_{1,2} \in [T_1, T_2],\ y \in K. \tag{1.17}$$

In the following we will also use the abbreviations:

$$u(x, t) := \begin{cases} u^{(1)}(x, t) & \text{for } (x, t) \in D_T^-(s) \\ u^{(2)}(x, t) & \text{for } (x, t) \in D_T^+(s), \end{cases}$$

and defined in an analogous way $q(x, t)$ and $c(x, t)$ for $(x, t) \in D_T^-(s) \cup D_T^+(s)$.

2. LOCAL EXISTENCE OF CLASSICAL SOLUTIONS AND SMOOTHNESS UP TO THE BOUNDARY

We collect the auxiliary results, which will be needed in the following analysis. With the possible exception of Lemma 2.5, basically they are well known and given here for exact reference. We start with an L^{∞}-estimate independent of the boundary s:

LEMMA 2.1. *Let* $T > 0$, s: $[0, T] \to (0, 1)$ *be continuous*, $b := s(0)$, *and* $u^{(i)}$ *solutions of* (1.2)–(1.8). *Then a constant* U_T *exists, only dependent on* $\|q\|_{\infty, \Omega_T}$, $\|h\|_{\infty}$, $-G'(>0)$, $G''(>0)$, Y', $-Y''$ *and* T *in a monotone way such that* $|u^{(i)}(x, t)| \leqslant U_T$ *for* $(x, t) \in D_T^{\mp}(s)$ *respectively.*

Proof. It suffices to consider $i = 1$. Set $v := u^{(1)} - w$, where w solves (1.2) in $D_T^-(s)$ with homogeneous initial and Dirichlet boundary values. Set $Q := \|q\|_{\infty, \Omega_T}$. Because of

$$|w(x, t)| \leqslant QT, \qquad |w_x(0, t)| \leqslant 2/\pi^{1/2} Q T^{1/2},$$

only v needs further consideration, where we can apply the results of [5]. This proves the assertion. ∎

The local existence of solutions has been studied by Fasano and Primicerio in [8]. Although they only treat the case of Dirichlet boundary conditions, their analysis can be duplicated for nonlinear Neumann condition as considered here. A careful investigation of their proof shows the following result:

THEOREM 2.2. *Let $T>0$ and for some δ, $H>0$, $\alpha\in(0,1]$*

$$\delta\leqslant b\leqslant 1-\delta, \tag{2.1}$$

$$\begin{aligned} |h^{(1)}(x)| &\leqslant H(b-x)^{\alpha} \quad \text{for} \quad 0\leqslant x\leqslant b, \\ |h^{(2)}(x)| &\leqslant H(x-b)^{\alpha} \quad \text{for} \quad b\leqslant x\leqslant 1. \end{aligned} \tag{2.2}$$

Then there exists a $T_0>0$, only dependent on

$$T,\ \|q\|_{\infty,\Omega_T},\ H,\ \alpha,\ U_T \text{ (from Lemma 2.1)},\ \lambda',\ \|\mu\|_{\infty,\Omega_T}, \text{ and } \delta$$

such that in $[0, T_0]$ a solution $(u^{(1)}, u^{(2)}, s)$ of (1.2)–(1.9) exists.

The importance of Theorem 2.2 lies in the fact that an interval of existence independent of the initial time in some fixed time interval can be guaranteed as long as the distance of the starting point of s to the fixed boundaries and the Hölder behaviour there of the initial values can be controlled. We now attend to the smoothness of solutions of (1.2)–(1.9) to settle a basis for the calculations in Section 3. We will see that much less regularity will be sufficient. In the following let $(u^{(1)}, u^{(2)}, s)$ be a solution of (1.2)–(1.9) in $[0, T]$.

LEMMA 2.3. *Let $0<T_1<T_2\leqslant T$. There exists $\gamma\in(0,1]$ such that*

$$u_x^{(i)}(s(\cdot),\cdot)\in C^{\gamma}[T_1, T_2]. \tag{2.3}$$

Proof. Fix some $\varepsilon\in(0, T_1)$ and $\delta\in(0, \min\{s(t)\mid t\in[T_1-\varepsilon, T_2]\})$. Consider $u^{(1)}$ for $t\in[T_1-\varepsilon, T_2]$, $x\in[\delta, s(t)]$ and write it as $u^{(1)}=v+w$ similiar to the proof of Lemma 2.1. Now the Hölder continuity of $v_x(s(\cdot),\cdot)$ can be deduced from Theorem 2.2 in [1]. According to w, transformation to cylindrical domain makes IV, Theorem 9.1 in [12] applicable because of $s\in C^1[T_1-\varepsilon, T_2]$. This together with [12], II, Lemma 3.3 implies the assertion for $i=1$. The proof for $i=2$ follows the same lines. ∎

Define for $\delta>0$, $0\leqslant T_1<T_2$,

$$D^-_{T_1T_2,\delta}(s) := \{(x,t)\mid T_1<t<T_2,\ \delta<x<s(t)\} \tag{2.4}$$

and $D^+_{T_1T_2,\delta}(s)$ in an analogous way.

LEMMA 2.4. *Let $0<T_1<T_2\leqslant T$, $\delta>0$.* Then there exists $\gamma\in(0,1]$ such that

$$\begin{aligned} u^{(1)} &\in H^{2+\gamma,1+\gamma/2}(\overline{D^-_{T_1T_2,\delta}(s)}), \\ u^{(2)} &\in H^{2+\gamma,1+\gamma/2}(\overline{D^+_{T_1T_2,\delta}(s)}). \end{aligned} \tag{2.5}$$

Proof. In the following we only consider $u^{(1)}$, furthermore γ denotes a generic constant from $(0, 1]$. Fix some $\varepsilon \in (0, T_1)$, set $\bar{T}_1 := T_1 - \varepsilon$ and w.l.o.g. let $\delta < \min\{s(t) \mid t \in [0, T]\}$. Equation (1.9) together with Lemma 2.3, (A1) and the Lipschitz continuity of s in $[\bar{T}_1, T_2]$ imply

$$\dot{s} \in C^{\gamma}[\bar{T}_1, T_2]. \tag{2.6}$$

We apply the usual transformation $x \mapsto x/s(t)$, $t \mapsto t$, which maps $D_{\bar{T}_1 T_2, \delta}(s)$ into $(2\bar{\delta}, 1) \times (\bar{T}_1, T_2)$ for some $\bar{\delta} \in (0, 1/2)$. Let $\bar{u}(x, t) := u^{(1)}(xs(t), t)$, $(x, t) \in [0, 1] \times [0, T]$ be the solution of the transformed problem. It fulfills a boundary value problem in $(0, 1) \times (0, T)$, whose coefficients and right-hand side are in

$$H^{\gamma, \gamma/2}(\bar{Q}) \qquad \text{using } Q := (\bar{\delta}, 1) \times (\bar{T}_1, T_2), \tag{2.7}$$

because of (2.6) and (A1). In a first step, this implies

$$\bar{u} \in H^{2+\gamma, 1+\gamma/2}(Q) \tag{2.8}$$

by means of III, Theorem 12.1 from [12], as for the classical solution $\bar{u}$ a fortiori $\bar{u} \in V_2^{1,0}(Q)$.

In a second step, now (2.7), (2.8) enable us to apply IV, Theorem 10.1 from [12] and to conclude,

$$\bar{u} \in H^{2+\gamma, 1+\gamma/2}([2\bar{\delta}, 1] \times [T_1, T_2]). \tag{2.9}$$

Having in mind the construction of $\bar{\delta}$ and (2.6), the reversed transformation yields the assertion. ∎

To have the smoothness of (2.5) also up to the fixed boundaries, we need some information about the regularity of $u_x^{(1)}(0, \cdot)$ and $u_x^{(2)}(1, \cdot)$. This will be provided by

LEMMA 2.5. *Let $0 < T_1 < T_2 \leqslant T$ and* (A2) *be fulfilled. Then there exists $\gamma \in (1/2, 1]$ such that*

$$\begin{aligned} &g^{(1)}[u^{(1)}(0, \cdot), \cdot] \in C^{\gamma}[T_1, T_2], \\ &g^{(2)}[u^{(2)}(1, \cdot), \cdot] \in C^{\gamma}[T_1, T_2]. \end{aligned} \tag{2.10}$$

We omit a proof. Because of (1.12), (1.17) it is sufficient to proof the same property for $u^{(1)}(0, \cdot)$ and $u^{(2)}(1, \cdot)$. This can be done in a rather tedious proof based on heat potential theory and a version of the lemma of Gronwall in a similiar fashion as in the proofs of Lemmas 4.1, 4.2 and Theorem 4.2 in [7].

LEMMA 2.6. *Let* $0 < T_1 < T_2 \leqslant T$ *and* (A2) *be fulfilled. Then there exists* $\gamma \in (0, 1]$ *such that*

$$u^{(1)} \in H^{2+\gamma, 1+\gamma/2}(\overline{D^-_{T_1T_2}(s)}),$$
$$u^{(2)} \in H^{2+\gamma, 1+\gamma/2}(\overline{D^+_{T_1T_2}(s)}). \tag{2.11}$$

Proof. The proof is a straightforward continuation of the proof Lemma 2.4: We have to consider $\bar{u}$ also in $[0, \bar{\delta}] \times [\bar{T}_1, T_2]$ for some $\bar{\delta} \in (2\delta, 1)$. (2.10) and (1.16) are what we need for the application of the quoted theorems from [12]. ∎

3. A PRIORI ESTIMATES

In this section we prove an a priori estimate, from which the announced theorem can easily be deduced. Let $(u^{(1)}, u^{(2)}, s)$ be a solution of (1.2)–(1.9) in $[0, T]$ and $0 < T_1 < T_2 \leqslant T$ for the remainder of the section. Then the following two identities hold:

LEMMA 3.1. *Let* $\rho \in W_1^1[0, 1]$, $\operatorname{supp} \rho \subset [\delta, 1-\delta]$ *for some* $\delta \geqslant 0$ *and* (A2) *be fulfilled, if* $\delta = 0$. *Then*

$$\int_{T_1}^{T_2} \rho(s(t))^2 \, u_x^{(1)}(s(t), t)^2 \, dt = \int_{T_1}^{T_2} \rho(0)^2 \, u_x^{(1)}(0, t)^2 \, dt$$
$$+ 2 \iint_{D^-_{T_1T_2}(s)} (\rho^2 q^{(1)} u_x^{(1)})(x, t) \, dx \, dt + 2 \iint_{D^-_{T_1T_2}(s)} c_1(\rho^2 u_t^{(1)} u_x^{(1)})(x, t) \, dx \, dt$$
$$+ 2 \iint_{D^-_{T_1T_2}(s)} \rho(x) \, \rho'(x) \, u_x^{(1)}(x, t)^2 \, dx \, dt. \tag{3.1}$$

Proof. Let $\Omega := D^-_{T_1T_2}(s)$, $v := u^{(1)}$, then Lemma 2.4 (resp. Lemma 2.6) provides enough regularity to justify the following calculations:

$$\int_\Omega \rho^2 q v_x \, dx \, dt = \int_\Omega \rho^2 v_{xx} v_x \, dx \, dt - \int_\Omega \rho^2 c_1 v_t v_x \, dx \, dt$$

and using Stokes' theorem,

$$\int_\Omega \rho^2 v_{xx} v_x \, dx \, dt = \int_\Omega \frac{d}{dx}\left(\frac{1}{2} \rho^2 v_x^2\right) dx \, dt - \int_\Omega \rho \rho' v_x^2 \, dx \, dt$$
$$= \frac{1}{2} \int_{\partial\Omega} \rho^2 v_x^2 \, dt - \int_\Omega \rho \rho' v_x^2 \, dx \, dt;$$

$$\int_{\partial\Omega} \rho^2 v_x^2 \, dt = \int_{T_1}^{T_2} \rho^2(s(t))\, v_x(s(t), t)^2 \, dt - \int_{T_1}^{T_2} \rho^2(0)\, v_x(0, t)^2 \, dt$$

gives the assertion. ∎

LEMMA 3.2. *Let the assumptions of Lemma* 3.1. *be fulfilled. Then*

$$\begin{aligned}
&\iint_{\Omega_{T_1T_2}} (c\rho^2 u_t^2)(x, t)\, dx\, dt + \tfrac{1}{2}\int_0^1 \rho(x)^2\, u_x(x, T_2)^2\, dx \\
&\quad = \tfrac{1}{2}\int_0^1 \rho(x)^2\, u_x(x, T_1)^2\, dx + \tfrac{1}{2}\int_{T_1}^{T_2} \rho(s(t))^2 [u_x^{(2)}(s(t), t)^2 \\
&\qquad - u_x^{(1)}(s(t), t)^2]\, \dot{s}(t)\, dt + \int_{T_1}^{T_2} \rho(1)^2 (u_t^{(2)} u_x^{(2)})(1, t) \\
&\qquad - \rho(0)^2 (u_t^{(1)} u_x^{(1)})(0, t)\, dt - \iint_{\Omega_{T_1T_2}} (\rho^2 q u_t)(x, t)\, dx\, dt \\
&\qquad - 2\iint_{\Omega_{T_1T_2}} (\rho\rho' u_x u_t)(x, t)\, dx\, dt. \qquad (3.2)
\end{aligned}$$

Proof. Let $\Omega_\varepsilon := \{(x, t) \mid T_1 < t < T_2, \varepsilon < x < s(t) - \varepsilon\}$, $v := u^{(1)}$. Again we remember the regularity of v provided by section 2 and furthermore $v_{xt} \in L^\infty(\Omega_\varepsilon)$ (see, e.g., [12, III, Theorem 12.1]). Therefore the following calculations are justified:

$$-\int_{\Omega_\varepsilon} \rho^2 q v_t \, dx\, dt = -\int_{\Omega_\varepsilon} \rho^2 v_{xx} v_t \, dx\, dt + \int_{\Omega_\varepsilon} c_1 \rho^2 v_t^2 \, dx\, dt, \qquad (3.3)$$

$$\begin{aligned}
-\int_{\Omega_\varepsilon} \rho^2 v_{xx} v_t \, dx\, dt &= \int_{\Omega_\varepsilon} 2\rho\rho' v_t v_x \, dx\, dt + \int_{\Omega_\varepsilon} \rho^2 v_{xt} v_x \, dx\, dt \\
&\quad - \int_{\partial\Omega_\varepsilon} \rho^2 v_t v_x \, dt \qquad \text{by Stokes' theorem;}
\end{aligned} \qquad (3.4)$$

and in the same way

$$\int_{\Omega_\varepsilon} \rho^2 v_{xt} v_x \, dx\, dt = -\tfrac{1}{2}\int_{\partial\Omega_\varepsilon} \rho^2 v_x^2 \, dx. \qquad (3.5)$$

Substituting (3.5) and (3.4) in (3.3), we get an equation, for which we can let $\varepsilon \to 0$ yielding

$$\iint_{D^-_{T_1T_2}(s)} c_1\rho^2 v_t^2\,dx\,dt + \tfrac{1}{2}\int_0^{s(T_2)} \rho(x)^2\, v_x(x, T_2)^2\,dx$$
$$= \tfrac{1}{2}\int_0^{s(T_1)} \rho(x)^2\, v_x(x, T_1)^2\,dx + \int_{T_1}^{T_2} \rho(s(t))^2\,(v_t v_x)(s(t), t)\,dt$$
$$+ \tfrac{1}{2}\int_{T_1}^{T_2} \rho(s(t))^2\, v_x(s(t), t)^2\,\dot{s}(t)\,dt - \int_{T_1}^{T_2} \rho(0)^2 (v_t v_x)(0, t)\,dt$$
$$- \int_{D^-_{T_1T_2}(s)} \rho^2 q v_t\,dx\,dt - 2\int_{D^-_{T_1T_2}(s)} \rho\rho' v_t v_x\,dx\,dt.$$

Using $v_t(s(t), t) = -v_x(s(t), t)\,\dot{s}(t)$ and repeating the same considerations for $u^{(2)}$ implies the assertion. ∎

LEMMA 3.3. *Let* $\rho \in W_2^1[0, 1]$, supp $\rho \subset [\delta, 1-\delta]$ *for some* $\delta \geqslant 0$ *and* (A2) *be fulfilled, if* $\delta = 0$. *Then*

$$\iint_{D^-_{T_1T_2}(s)} \rho(x)^2\, u_x^{(1)}(x, t)^2\,dx\,dt + \int_0^{s(T_2)} c_1\rho(x)^2\, u^{(1)}(x, T_2)^2\,dx$$
$$\leqslant \int_0^{s(T_1)} c_1\rho(x)^2\, u^{(1)}(x, T_1)^2\,dx - 2\iint_{D^-_{T_1T_2}(s)} (\rho^2 u^{(1)} q^{(1)})(x, t)\,dx\,dt$$
$$+ 4\iint_{D^-_{T_1T_2}(s)} \rho'(x)^2\, u^{(1)}(x, t)^2\,dx\,dt - 2\int_{T_1}^{T_2} \rho(0)^2 (u^{(1)} u_x^{(1)})(0, t)\,dt, \tag{3.6}$$

and an analogous estimate holds for $u^{(2)}$ *with the sign reversed for the last term.*

Proof. Let $\Omega := D^-_{T_1T_2}(s)$, $v := u^{(1)}$. We get by calculations analogous to the previous proofs:

$$-2\int_\Omega \rho^2 q v\,dx\,dt = 2\int_\Omega \frac{d}{dx}(\rho^2 v)\, v_x\,dx\,dt - 2\int_{\partial\Omega} \rho^2 v v_x\,dt$$
$$- \int_{\partial\Omega} c_1\rho^2 v^2\,dx,$$

and thus using $v(s(t), t) = 0$:

$$2 \int_{\Omega} \rho^2 v_x^2 \, dx \, dt + \int_0^{s(T_2)} c_1 \rho(x)^2 v(x, T_2)^2 \, dx$$

$$= \int_0^{s(T_1)} c_1 \rho(x)^2 v(x, T_1)^2 \, dx - 2 \int_{\Omega} \rho^2 v q \, dx \, dt$$

$$- 2 \int_{T_1}^{T_2} \rho(0)^2 (v v_x)(0, t) \, dt - 4 \int_{\Omega} \rho v_x \rho' v \, dx \, dt.$$

An application of Young's inequality using $4 = 2^{1/2} 8^{1/2}$ yields the assertion. ∎

Remark 3.4. (1) For $\rho \equiv 1$, $q^{(i)} \equiv 0$ (3.2) already appears in [1] and [4], and (3.6) in [10], but for different regularity assumptions.

(2) (3.1), (3.2), (3.6) are valid for a broader class of $(u^{(1)}, u^{(2)})$.

THEOREM 3.5. *Let*

$$u_x^{(i)}(s(t), t) \leqslant 0 \qquad \textit{for} \quad t \in [T_1, T_2] \tag{3.7}$$

and

(1) $\rho \in W^1_\infty[0, 1]$ *and* (A2) *be fulfilled or*

(2) $\rho \in W^2_2[0, 1]$, $\eta \in W^1_2[0, 1]$ *such that* $(\rho\rho')(x) \leqslant \eta^2(x)$ *a.e. in* $[0, 1]$ *and* supp ρ, supp $\eta \Subset (0, 1)$.

Then there exist constants C_k, *only dependent on* λ', λ'', c_1, c_2, T, $\|\rho\|_\infty$, $\|\rho'\|_\infty$ *and for case* (2) *additionally on* $\|\rho''\|_2$, $\|\eta\|_2$, $\|\eta'\|_2$ *such that, using* $U := U_{T_2}$, U_T *from Lemma* 2.1,

$$\iint_{\Omega_{T_1 T_2}} (c\rho^2 u_t^2)(x, t) \, dx \, dt + \int_0^1 \rho(x)^2 u_x(x, T_2)^2 \, dx$$

$$+ \int_{T_1}^{T_2} \rho(s(t))^2 |\dot{s}(t)|^3 \, dt$$

$$\leqslant C_1 \int_{T_1}^{T_2} \rho(1)^2 (u_x^{(2)} u_t^{(2)})(1, t) - \rho(0)^2 (u_x^{(1)} u_t^{(1)})(0, t) \, dt$$

$$+ C_2 \left(\int_0^1 u_x(x, T_1)^2 \, dx + U^2 + \|q\|^2_{\infty, \Omega_{T_1 T_2}} + \int_{T_1}^{T_2} \mu(s(t), t)^6 \, dt \right)$$

in case (1):

$$+ C_3 \left(\int_{T_1}^{T_2} g^{(1)}[u^{(1)}(0, t), t]^2 \, dt + \int_{T_1}^{T_2} g^{(2)}[u^{(2)}(1, t), t]^2 \, dt \right).$$

Proof. In the following C denotes a generic constant, having only the claimed dependencies. Let

$$A_1 := \iint_{\Omega_{T_1T_2}} (c\rho^2 u_t^2)(x, t)\, dx\, dt + \frac{1}{2}\int_0^1 \rho(x)^2\, u_x(x, T_2)^2\, dx,$$

$$A_3 := \int_{T_1}^{T_2} \rho(1)^2\, (u_x^{(2)} u_t^{(2)})(1, t) - \rho(0)^2\, (u_x^{(1)} u_t^{(1)})(0, t)\, dt,$$

$$A_4 := \int_0^1 u_x(x, T_1)^2\, dx,$$

$$B_1 := \frac{1}{2}\int_{T_1}^{T_2} \rho(s(t))^2 [u_x^{(2)}(s(t), t)^2 - u_x^{(1)}(s(t), t)^2]\, \dot{s}(t)\, dt,$$

$$B_2 := -\iint_{\Omega_{T_1T_2}} (\rho^2 q u_t)(x, t)\, dx\, dt,$$

$$B_3 := -2\iint_{\Omega_{T_1T_2}} (\rho\rho' u_x u_t)(x, t)\, dx\, dt.$$

Then Lemma 3.2 implies

$$A_1 \leqslant A_3 + CA_4 + \sum_{i=1}^{3} B_i.$$

Furthermore from (1.9), $B_1 = D_1 + B_5$, where

$$D_1 := \tfrac{1}{2}\int_{T_1}^{T_2} \rho(s(t))^2\, \bar{\lambda}(t)\, \dot{s}(t)^2 (u_x^{(1)}(s(t), t) + u_x^{(2)}(s(t), t))\, dt,$$

$$B_5 := \tfrac{1}{2}\int_{T_1}^{T_2} \rho(s(t))^2\, \bar{\mu}(t)\, \dot{s}(t) (u_x^{(1)}(s(t), t) + u_x^{(2)}(s(t), t))\, dt,$$

using $\bar{\mu}(t) := \mu(s(t), t)$, $\bar{\lambda}(t) := \lambda(s(t), t)$. Because of (3.7):

$$-2D_1 \geqslant \int_{T_1}^{T_2} \rho(s(t))^2\, \lambda' \dot{s}(t)^2\, |\bar{\lambda}(t)\, \dot{s}(t) + \bar{\mu}(t)|\, dt$$

and thus for

$$A_2 := \int_{T_1}^{T_2} \rho(s(t))^2\, |\dot{s}(t)|^3\, dt,$$

$$B_4 := \frac{1}{2}\int_{T_1}^{T_2} \rho(s(t))^2\, \lambda' \dot{s}(t)^2\, |\bar{\mu}(t)|\, dt:$$

$$A_1 + \frac{\lambda'^2}{2} A_2 \leqslant A_3 + CA_4 + \sum_{i=2}^{5} B_i \tag{3.8}$$

Therefore it remains to estimate the B_i:

Let $\varepsilon > 0$, $Q := \|q\|_{\infty, \Omega_{T_1T_2}}$.

$$B_2 \leqslant C\varepsilon A_1 + C\varepsilon^{-1} Q^2,$$
$$B_4 \leqslant C\varepsilon A_2 + C\varepsilon^{-2} \|\bar{\mu}\|_3^3, \tag{3.9}$$

both by Young's inequality with $p = 2$ (resp. $p = \frac{3}{2}$).

$B_5 = B_6 + B_7 + B_8$ by (1.9), where

$$B_6 = \tfrac{1}{2} \int_{T_1}^{T_2} \rho(s(t))^2 \,\bar{\lambda}(t)\, \dot{s}(t)^2\, \bar{\mu}(t)\, dt$$
$$\leqslant C\varepsilon A_2 + C\varepsilon^{-2} \|\bar{\mu}\|_3^3, \tag{3.10}$$

$$B_7 = \tfrac{1}{2} \int_{T_1}^{T_2} \rho(s(t))^2 \,\dot{s}(t)\, \bar{\mu}(t)^2\, dt$$
$$\leqslant C\varepsilon A_2 + C\varepsilon^{-1\,2} \|\bar{\mu}\|_3^3, \tag{3.11}$$

$$B_8 \leqslant \tfrac{1}{2} \int_{T_1}^{T_2} \rho(s(t))^2\, u_x^{(1)}(s(t), t)^2\, dt + \tfrac{1}{2} \int_{T_1}^{T_2} \rho(s(t))^2\, \dot{s}(t)^2\, \bar{\mu}(t)^2\, dt$$

$=: B_9 + B_{10}$ always using Young's inequality

and

$$B_{10} \leqslant C\varepsilon A_2 + C\varepsilon^{-2} \|\bar{\mu}\|_6^6. \tag{3.12}$$

Lemma 3.1 gives a representation of B_9: $B_9 = \sum_{i=11}^{14} B_i$, where

$$B_{11} = \tfrac{1}{2} \int_{T_1}^{T_2} \rho(0)^2\, g^{(1)}[u^{(1)}(0, t), t]^2\, dt, \tag{3.13}$$

$$B_{12} = \iint_{D^-_{T_1T_2}(s)} (\rho^2 q^{(1)} u_x^{(1)})(x, t)\, dx\, dt$$
$$\leqslant CB_{15} + CQ^2, \tag{3.14}$$

where

$$B_{15} := \iint_{D^-_{T_1T_2}(s)} \rho^2(x)\, u_x^{(1)}(x, t)^2\, dx\, dt,$$

$$B_{13} = c_1 \iint_{D^-_{T_1T_2}(s)} (\rho^2 u_t^{(1)} u_x^{(1)})(x, t)\, dx\, dt$$
$$\leqslant C\varepsilon A_1 + C\varepsilon^{-1} B_{15}, \tag{3.15}$$

$$B_{14} = \iint_{D^-_{T_1T_2}(s)} \rho(x)\, \rho'(x)\, u_x^{(1)}(x, t)^2\, dx\, dt.$$

We estimate B_{15} by substituting of ρ by $\tilde{\rho}$, using $\tilde{\rho} := \|\rho\|_\infty$ in case (1) and $\tilde{\rho} := \rho$ in case (2) and then applying Lemma 3.3. This yields:

$$B_{15} \leqslant \int_0^{s(T_1)} c_1 \tilde{\rho}(x)^2 u^{(1)}(x, T_1)^2 \, dx - 2 \iint_{D^-_{T_1T_2}(s)} (\tilde{\rho}^2 u^{(1)} q^{(1)})(x, t) \, dx \, dt$$

$$+ 4 \iint_{D^-_{T_1T_2}} \tilde{\rho}'(x)^2 u^{(1)}(x, t)^2 \, dx \, dt$$

$$- 2 \int_{T_1}^{T_2} \tilde{\rho}(0)^2 (u^{(1)} u_x^{(1)})(0, t) \, dt =: \sum_{i=16}^{19} B_i. \tag{3.16}$$

We proceed in the usual way

$$\begin{aligned} &B_{16} \leqslant CU^2, \qquad B_{17} \leqslant CU^2 + CQ^2, \\ &B_{18} \leqslant CU^2, \qquad B_{19} = 0 \qquad \text{in case (2)}, \\ &B_{19} \leqslant C \int_{T_1}^{T_2} g^{(1)}[u^{(1)}(0, t), t]^2 \, dt + CU^2 \qquad \text{in case (1)}. \end{aligned} \tag{3.17}$$

To treat B_{14}, we now set $\tilde{\eta} := (\|\rho\|_\infty \|\rho'\|_\infty)^{1/2}$ in case (1) and $\tilde{\eta} := \eta$ in case (2) and estimate B_{14} by substitution of $\rho\rho'$ by $\tilde{\eta}^2$. Lemma 3.3 implies

$$B_{14} \leqslant \sum_{i=16}^{19} \tilde{B}_i, \tag{3.18}$$

where $\tilde{B}_i$ corresponds to B_i after substituting of $\tilde{\rho}$ by $\tilde{\eta}$. Therefore as above,

$$\begin{aligned} &\tilde{B}_{16} \leqslant CU^2, \qquad \tilde{B}_{17} \leqslant CU^2 + CQ^2, \\ &\tilde{B}_{18} \leqslant CU^2, \qquad \tilde{B}_{19} = 0 \quad \text{in case (2)}, \\ &\tilde{B}_{19} \leqslant C \int_{T_1}^{T_2} g^{(1)}[u^{(1)}(0, t), t]^2 \, dt + CU^2 \quad \text{in case (1)}. \end{aligned} \tag{3.19}$$

Thus finally we are left with an estimation of B_3:

$$B_3 \leqslant C\varepsilon A_1 + \varepsilon^{-1} \iint_{D^-_{T_1T_2}(s)} \rho'(x)^2 u_x^{(1)}(x, t)^2 \, dx \, dt$$

$$+ \varepsilon^{-1} \iint_{D^+_{T_1T_2}(s)} \rho'(x)^2 u_x^{(2)}(x, t)^2 \, dx \, dt =: C\varepsilon A_1 + B_{20} + B_{21}.$$

Again we use Lemma 3.3, setting $\varphi := \|\rho'\|_\infty$ in case (1) and $\varphi := \rho'$ in case (2), and get estimates for B_{20}, B_{21} analogously to (3.16), (3.17).

Summarizing all the estimates from (3.8) to (3.19) and taking ε sufficiently small proves the assertion. ∎

Remark 3.6. In [4] an estimate similar to Theorem 3.5 has been developed, but using $\delta < s(t) < 1-\delta$ for $t \in [0, T]$ and some fixed $\delta > 0$.

4. Global Existence of Classical Solutions

Theorem 3.5 will be the appropriate instrument to investigate the behaviour of a solution of (1.2)–(1.9) at $t = T^*$. Therefore in this section we have to assume for some $\bar{T} < T^*$,

$$u_x^{(i)}(s(t), t) \leqslant 0 \qquad \text{for all} \quad t \in [\bar{T}, T^*). \tag{4.1}$$

The natural sufficient condition for (4.1) is $u^{(1)} \geqslant 0$ and $u^{(2)} \leqslant 0$ in the corresponding domains. These conditions can be guaranteed in the following way:

Lemma 4.1. *Let* s: $[0, T] \to (0, 1)$ *be Lipschitz continuous,* $b := s(0)$, $q^{(1)} \leqslant 0$ *in* $D_T^-(s)$, $q^{(2)} \geqslant 0$ *in* $D_T^+(s)$, $h^{(1)} \geqslant 0$ *in* $[0, b]$, $h^{(2)} \leqslant 0$ *in* $[b, 1]$, $g^{(1)}[0, t] \leqslant 0$, $g^{(2)}[0, t] \leqslant 0$ *for* $t \in (0, T]$, *then solutions* $u^{(i)}$ *of* (1.2)–(1.8) *fulfill*

$$u^{(1)} \geqslant 0 \quad \text{in } D_T^-(s), \qquad u^{(2)} \leqslant 0 \quad \text{in } D_T^+(s).$$

Proof. It suffices to consider $i = 1$. Let u^* be a solution of the same boundary value problem as $u^{(1)}$, but with

$$g^*[y, t] := \begin{cases} g[y, t], & y \geqslant 0 \\ g[0, t], & y < 0 \end{cases}$$

instead of g. u^* exists uniquely [6, Theorem 3] and the strong minimum principle and the lemma of Viborny–Friedman (e.g., [9, pp. 34, 49]) imply $u^* \geqslant 0$ in $\overline{D_T^-(s)}$. Therefore u^* also fulfills the boundary value problem of u. As u is unique (again [6]), we are done. ▌

Now let $(u^{(1)}, u^{(2)}, s)$ be a solution of (1.2)–(1.9), $[0, T^*)$ its maximal interval of existence and let (4.1) be fulfilled. Then:

Theorem 4.2. *The case*

$$T^* < \infty, \quad 0 < \liminf_{t \to T^*} s(t) \leqslant \limsup_{t \to T^*} s(t) < 1 \tag{4.2}$$

cannot occur.

Proof. Inequality (4.2) is equivalent with the existence of some $\delta > 0$ such that

$$\delta \leqslant s(t) \leqslant 1-\delta \qquad \text{for} \quad t \in [0, T^*) \qquad \text{and} \qquad T^* < \infty. \tag{4.3}$$

Now choose $\rho \in W_2^2[0, 1]$, $\eta \in W_2^1[0, 1]$ such that $\rho\rho' \leqslant \eta^2$ a.e. in $[0, 1]$, supp ρ, supp $\eta \Subset (0, 1)$ and $\rho \equiv 1$ in $[\delta/2, 1-\delta/2]$. Their existence is obvious. Furthermore fix some $\bar{T} \in (0, T^*)$ fulfilling (4.1). From Theorem 3.5(2) we know the existence of a constant C, only dependent on $\lambda', \lambda'', c_1, c_2, T^*, \delta$ (by the norms of ρ and η), U_{T^*}, $\|q\|_{\infty,\Omega_{T^*}}$, $\|\mu\|_{\infty,\Omega_{T^*}}$ and $\|u_x(\cdot, \bar{T})\|_2$ such that

$$\int_0^1 \rho(x)^2\, u_x(x, t)^2\, dx \leqslant C \qquad \text{for all } t \in (\bar{T}, T^*). \tag{4.4}$$

Let $\tilde{t} \in (\bar{T}, T^*)$. Because of the choice of δ, (4.4) implies for

$$x \in [\delta/2, s(\tilde{t})]: \quad |u^{(1)}(x, \tilde{t})| \leqslant C^{1/2}\, |x - s(\tilde{t})|^{1/2}. \tag{4.5}$$

Furthermore because of (4.3), using $U := U_{T^*}$, for

$$x \in [0, \delta/2]: \quad |u^{(1)}(x, \tilde{t})| \leqslant U \leqslant U(2/\delta)^{1/2}\, |x - s(\tilde{t})|^{1/2}. \tag{4.6}$$

For $i = 2$ the same is true.

Now we consider $t = \tilde{t}$ as a starting time for the solution of the free boundary problem, i.e., $b := s(\tilde{t})$, $h^{(i)} := u^{(i)}(\cdot, \tilde{t})$, etc. Because of (4.5), (4.6), (2.2) is fulfilled for $H := \max(U(2/\delta)^{1/2}, C^{1/2})$ and $\alpha = \frac{1}{2}$. (2.1) is also satisfied; therefore Theorem 2.2 guarantees the existence of a $T_0 > 0$ independent of $\tilde{t}$ such that in $[\tilde{t}, T_0 + \tilde{t}]$ a solution exists. For $\tilde{t}$ close enough to T^* this leads to a contradiction. ∎

THEOREM 4.3. *If $T^* < \infty$, then*

$$s(T^*) := \lim_{t \to T^*} s(t) \qquad \textit{exists and} \quad s(T^*) \in \{0, 1\}. \tag{4.7}$$

Proof. To show the existence of $s(T^*)$ due to Theorem 3.5 we can repeat exactly the reasoning in the proof of Theorem 4.3 in [3, pp. 102, 103]. Theorem 4.2 then implies $s(T^*) \in \{0, 1\}$. ∎

Remark 4.4. Theorems 4.2 and 4.3 are independent of the boundary conditions at $x = 0$ and $x = 1$. They only rely on the local existence in the form of Theorem 2.2 and enough smoothness to justify the calculations of Section 3. They are valid especially for Dirichlet conditions.

Finally,

THEOREM 4.5. *Let* (A2) *be fulfilled. Then*

$$\text{if} \quad T^* < \infty, \text{ then } s \in C^{2/5}(0, T^*]. \tag{4.8}$$

Proof. Fix some $\bar{T} \in (0, T^*)$ fulfilling (4.1) and define

$$\rho(x) := \begin{cases} x, & x \in [0, \frac{1}{3}] \\ \frac{1}{3}, & x \in (\frac{1}{3}, \frac{2}{3}] \\ 1 - x, & x \in (\frac{2}{3}, 1]. \end{cases} \tag{4.9}$$

Then $\rho \in W^1_\infty[0, 1]$ and because of $\rho(0) = \rho(1) = 0$, Theorem 3.5(1) guarantees the existence of a constant C_1, which only depends on the data, U_{T^*} and $\|u_x(\cdot, \bar{T})\|_2$ such that

$$\int_{\bar{T}}^{t} \rho(s(\tau))^2 \, |\dot{s}(t)|^3 \, dt \leqslant C_1 \qquad \text{for all} \quad t \in (\bar{T}, T^*). \tag{4.10}$$

We now adopt a reasoning of [10], let

$$F(y) := \int_0^y \rho(\xi)^{2/3} \, d\xi, \qquad y \in [0, 1].$$

Then F is strictly monotone increasing and F^{-1} is Hölder continuous with exponent $\frac{3}{5}$, which can be seen by explicit calculation of F^{-1}. Thus for $\bar{T} \leqslant t_1 < t_2 < T^*$:

$$\begin{aligned} |s(t_1) - s(t_2)| &\leqslant C_2 \, |F(s(t_1)) - F(s(t_2))|^{3/5} \\ &= C_2 \left| \int_{t_1}^{t_2} \frac{d}{dt} F(s(t)) \, dt \right|^{3/5} \\ &= C_2 \left| \int_{t_1}^{t_2} \rho(s(t))^{2/3} \, \dot{s}(t) \, dt \right|^{3/5} \\ &\leqslant C_2 \left(\int_{t_1}^{t_2} \rho(s(t))^2 \, |\dot{s}(t)|^3 \, dt \right)^{1/5} |t_1 - t_2|^{2/5} \end{aligned}$$

the last estimate by Hölder's inequality. Therefore (4.10) implies the Hölder continuity of s in $[\bar{T}, T^*]$ with exponent $\frac{2}{5}$, which proves the assertion. ∎

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