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On the Structure of the Steinberg Group St(Λ)

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1. INTRODUCTION

Following [6], Milnor defined a group St(Λ) in [4], where Λ is an arbitrary associative ring with unit. In this paper, we give a normal form for elements of St(Λ). This normal form, a cousin of the Bruhat decomposition, was originally inspired by the geometry of pseudo-isotopy theory. However, the present treatment is purely algebraic, and can be regarded as an extension of the idea embodied in the Whitehead identity (cf. [7, p. 5] or [1, p. 226]).

Let L and U be the subgroups of St(Λ) corresponding to the lower and upper triangular matrices, with 1's down the diagonal, and let P be the subgroup corresponding to the permutation matrices (cf. Section 2 for precise definitions). Our first main result is the:

DECOMPOSITION THEOREM FOR St(Λ). $\mathrm{St}(\Lambda) = LPLU$.

A proof of this is given in Section 3. We announced this theorem, without proof, in [3, p. 248]. It is interesting to note that there is in general no unstable analogue of Theorem 3.4 in any dimension. Indeed any $n \times n$ *matrix* of the form given, with $\Lambda = \mathbb{Z}$, has one of the entries of the first row congruent to ± 1 modulo the ideal generated by the previous entries in the first row. But $E(n, \mathbb{Z})$ contains elements not satisfying this for every $n > 1$.

Theorem 3.4 does not address the question of the uniqueness of the decomposition of elements in the form indicated. Just as in the case of the Bruhat decomposition (cf. [2, p. 28]), our decomposition is not unique. For example, if $l' = p^{-1}lp \in L$, where $l \in L$, $p \in P$, then $lp = pl'$ shows the non-uniqueness. This is the first of three kinds of relations R_1, R_2, R_3 among normal forms, all in the same spirit, which are described in Section 4. Our second main result (Theorem 4.3) is the:

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STRUCTURE THEOREM FOR St(Λ). *The decomposition of elements of* St(Λ) *given by the decomposition theorem is unique, up to the changes generated in a certain way by the relations* R_1, R_2, *and* R_3.

The proof of this goes farther, and gives the action of left multiplication of St(Λ) on the set of normal forms; thus one can write down a closed formula for the normal form of the product of two normal forms.

In Section 8 we give some extensions, in certain cases, of the structure theorem to $E(\Lambda)$ and $GL(\Lambda)$.

Finally, we remark that in [5] we gave an analogous result for the "non-hyperbolic part" of the Unitary Steinberg group. The results of this paper can be used to complete the description in the Unitary case.

2. THE MATRIX NOTATION FOR ELEMENTS OF St(Λ)

The following standard notation agrees with that of [4].

Λ — an arbitrary ring with identity 1.

$E(\Lambda)$ — the subgroup of $GL(\Lambda)$ generated by the elementary matrices e_{ij}^{λ}, $\lambda \in \Lambda$, i, j distinct positive integers.

St(Λ) — the Steinberg group, generated by symbols x_{ij}^{λ} subject to the relations

$$x_{ij}^{\lambda} x_{ij}^{\lambda} = x_{ij}^{\lambda+\mu},$$

$$[x_{ij}^{\lambda}, x_{kl}^{\mu}] = 1 \qquad \text{if} \quad i \neq l, \quad j \neq k$$

$$= x_{il}^{\lambda\mu} \qquad \text{if} \quad i \neq l, \quad j = k.$$

ϕ: St(Λ) $\to E(\Lambda)$ — the homomorphism sending $x_{ij}^{\lambda} \mapsto e_{ij}^{\lambda}$.

L — the subgroup of St(Λ) generated by x_{ij}^{λ}, $\lambda \in \Lambda$, $i > j$.

U — the subgroup of St(Λ) generated by x_{ij}^{λ}, $\lambda \in \Lambda$, $i < j$.

P — the subgroup of St(Λ) generated by the elements

$$w_{ij} = x_{ij}^{1} x_{ji}^{-1} x_{ij}^{1}.$$

The corresponding unstable notions are denoted by $E(n, \Lambda)$, St(n, Λ), ϕ, L_n, U_n, P_n. Moreover 1, or sometimes 1_n if we wish to emphasize the dimension, will denote the identity in any of these groups.

We shall now discuss the "matrix-like" notation for elements of St(Λ) which we shall use in the sequel. Roughly, the idea is that certain elements of St(Λ) are canonically determined by their images in $E(\Lambda)$, and hence may be denoted by matrices. We develop this idea into a modest calculus, using square brackets for matrices denoting elements of St(Λ) and round brackets for ordinary matrices. This device, and the lemmas given without proof below, appear in [5, Sect. 1]. We include them here for the convenience of the reader.

LEMMA 2.1. *ϕ induces isomorphisms $L \simeq \phi L$ and $U \simeq \phi U$.*

This lemma enables us to determine elements of L and U by their matrix images in $E(\Lambda)$. We can extend this mildly by using:

LEMMA 2.2. *The map $L \times U \to LU$ sending $(\lambda, \mu) \mapsto \lambda\mu$ is a bijection.*

Proof. $\lambda\mu = \lambda'\mu'$ implies $\lambda'^{-1}\lambda = \mu'\mu^{-1} \in L \cap U = 1$ by Lemma 2.1. Thus $\lambda = \lambda'$ and $\mu = \mu'$.

Lemma 2.2 allows us to determine elements of LU by their matrix images in $E(\Lambda)$.

Since every automorphism α of St(Λ) preserves the center, which is ker ϕ, it induces an automorphism $\bar{\alpha}$ of $E(\Lambda)$. We have:

LEMMA 2.3. *The homomorphism* Aut St(Λ) $\to$ Aut $E(\Lambda)$ *sending $\alpha \to \bar{\alpha}$ is an isomorphism.*

Lemma 2.3 allows us to make sense of expressions of the form $x^a = a^{-1}xa$, where $a \in GL(\Lambda)$, $x \in \mathrm{St}(\Lambda)$, by regarding it as the image of x under the automorphism corresponding to conjugation by a.

Let us say that $x \in \mathrm{St}(\Lambda)$ has dimension n if it lies in the image of St(n, Λ). (Perhaps we should say "dimension at most n.") Then if $x, y \in \mathrm{St}(\Lambda)$, with x of dimension n, we denote by $x \oplus y$, or the "matrix" $\left[\begin{smallmatrix} x & \\ & y \end{smallmatrix}\right]$, the expression xy^{π}, where $\pi \in GL(\Lambda)$ is any permutation matrix sending $n + i \mapsto i$ for $i = 1, 2, \ldots, \dim y$. This depends on the choice of dim x but not on dim y or π as may be seen from the following:

LEMMA 2.4. *If $x \in \mathrm{St}(\Lambda)$ with $\phi(x) \in E(n, \Lambda)$, then $axa^{-1} = x$ for $a = \left(\begin{smallmatrix} 1_n & \\ & b \end{smallmatrix}\right) \in GL(\Lambda)$.*

Next we give several commutation relations for St(Λ).

LEMMA 2.5.

$$\text{(a)} \quad \begin{bmatrix} x & \\ & y \end{bmatrix}\begin{bmatrix} 1 & \\ z & 1 \end{bmatrix} = \begin{bmatrix} 1 & \\ yzx^{-1} & 1 \end{bmatrix}\begin{bmatrix} x & \\ & y \end{bmatrix},$$

$$\text{(b)} \quad \begin{bmatrix} x & \\ & y \end{bmatrix}\begin{bmatrix} 1 & z \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & xzy^{-1} \\ & 1 \end{bmatrix}\begin{bmatrix} x & \\ & y \end{bmatrix}.$$

Thus expressions like $\left[\begin{smallmatrix} x & \\ z & y \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} x & z \\ & y \end{smallmatrix}\right]$ make good sense if $x, y \in \mathrm{St}(\Lambda)$ and z is a matrix of appropriate size. We shall sometimes abuse this notation by using $z \in \mathrm{St}(\Lambda)$ for which one should read $\phi(z)$.

DEFINITION. $u \in U$ and $l \in L$ are *permutable* if both lie in pLp^{-1} for some $p \in P$.

LEMMA 2.6. *If $u \in U$ and $l \in L$ are permutable, then $ul \in LU$.*

Proof. The argument of [2, p. 27–28] applies to any ring to show that $L = (L \cap pLp^{-1}) \cdot (L \cap pUp^{-1})$. Hence

$$p^{-1}Lp = (p^{-1}Lp \cap L) \cdot (p^{-1}Lp \cap U) \subset LU.$$

Finally, we set

$$\omega_{2,1}(a) = \begin{bmatrix} 1_n & \\ a & 1_n \end{bmatrix} \begin{bmatrix} 1_n & -a^{-1} \\ & 1_n \end{bmatrix} \begin{bmatrix} 1_n & \\ a & 1_n \end{bmatrix} \in \mathrm{St}(\Lambda),$$

where $a \in GL(n, \Lambda)$. Note that

$$\phi(\omega_{2,1}(a)) = \begin{pmatrix} & -\phi a^{-1} \\ \phi a & \end{pmatrix} \qquad \text{and} \qquad \omega_{2,1}(a) = a^{-1}\omega_{2,1}(1_n)a.$$

We widen this notation by: abbreviating $\omega_{2,1}(-1_n)$ to $\omega_n \in P$; extending in the obvious way to include $\omega_{i,j}(a) \in \mathrm{St}(\Lambda)$; and abusing it by writing $\omega_{i,j}(x)$ (where $x \in \mathrm{St}(\Lambda)$ is of dimension n) in place of $\omega_{i,j}(\phi x)$.

3. THE REDUCTION IDENTITY IN St(Λ)

We are concerned here with the following identity in $GL(\Lambda)$:

$$ABCD = \begin{pmatrix} 1 & \\ (AB)^{-1} & 1 \end{pmatrix} \cdot \begin{pmatrix} A & \\ & 1 \end{pmatrix} \begin{pmatrix} & 1 \\ -1 & \end{pmatrix} \begin{pmatrix} C & \\ & 1 \end{pmatrix}$$
$$\cdot \begin{pmatrix} D & \\ BCD & B \end{pmatrix} \cdot \begin{pmatrix} 1 & -(CD)^{-1} \\ & 1 \end{pmatrix}. \tag{3.1}$$

Of course, this identity may be verified by direct multiplication. If we take $A, C \in \bar{P}$ and $B, D \in \bar{L}$ (where we write $\phi(P) = \bar{P}$, etc.), then (3.1) shows that $\overline{PLPL} \subset \overline{LPLU}$, and hence the latter subset of $E(\Lambda)$ is stable under left multiplication by $\bar{P}$. But it is obviously also stable under left multiplication by $\bar{L}$, and since $\bar{P}$ and $\bar{L}$ generate $E(\Lambda)$, we obtain

$$E(\Lambda) = \bar{L}\bar{P}\bar{L}\bar{U}. \tag{3.2}$$

This equation is the analogue in $E(\Lambda)$ of our first main theorem:

DECOMPOSITION THEOREM 3.3. $\mathrm{St}(\Lambda) = LPLU$.

If we regard A, B, C, D as elements of $\mathrm{St}(\Lambda)$ then the factors of (3.1) all make sense as elements of St(Λ), via the matrix notation of Section 2. Here we regard $\left[\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right]$ as $\omega_{2,1}(-1)$. Indeed the proof given above for the equality (3.2) applies to give (3.3) in virtue of:

Proposition 3.4 (The Reduction Identity for St(Λ)). *Equation* (3.1) *holds in* St(Λ).

Proof. Manipulating by the rules of Section 2 we have

$$\begin{aligned} ABCD &= ABCD \begin{bmatrix} 1 & (CD)^{-1} \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ -CD & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ CD & 1 \end{bmatrix} \begin{bmatrix} 1 & -(CD)^{-1} \\ & 1 \end{bmatrix} \\ &= AB \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & \\ -1 & 1 \end{bmatrix} \begin{bmatrix} C & \\ & 1 \end{bmatrix} \begin{bmatrix} D & \\ CD & 1 \end{bmatrix} \begin{bmatrix} 1 & -(CD)^{-1} \\ & 1 \end{bmatrix} \\ &= AB \begin{bmatrix} 1 & \\ 1 & 1 \end{bmatrix} \omega_{2,1}(-1) \begin{bmatrix} C & \\ & 1 \end{bmatrix} \begin{bmatrix} D & \\ CD & 1 \end{bmatrix} \begin{bmatrix} 1 & -(CD)^{-1} \\ & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \\ (AB)^{-1} & 1 \end{bmatrix} \begin{bmatrix} AB & \\ & 1 \end{bmatrix} \omega_{2,1}(-1) \begin{bmatrix} C & \\ & 1 \end{bmatrix} \\ &\quad \times \begin{bmatrix} D & \\ CD & 1 \end{bmatrix} \begin{bmatrix} 1 & -(CD)^{-1} \\ & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \\ (AB)^{-1} & 1 \end{bmatrix} \begin{bmatrix} A & \\ & 1 \end{bmatrix} \omega_{2,1}(-1) \begin{bmatrix} C & \\ & 1 \end{bmatrix} \\ &\quad \times \begin{bmatrix} D & \\ BCD & B \end{bmatrix} \begin{bmatrix} 1 & -(CD)^{-1} \\ & 1 \end{bmatrix}. \end{aligned}$$

4. The Structure Theorem: Statement and Sketch of Proof

In this section we give a description of all possible ways of writing an element of St(Λ) as an element of $LPLU$, and give an outline of the proof, which is analogous to the proof of Matsumoto's theorem (cf. [4, Sects. 11 and 12]).

Let $\tilde{A} = L \times P \times LU$, and let R_0 be the equivalence relation on it generated by the following elementary relations:

R_1: $(\lambda l, \pi, \lambda'\mu) \sim (\lambda, \pi, l^{\pi}\lambda'\mu)$, where $l, l^{\pi} \in L$.

R_2: $(\lambda, \pi p, \lambda'\mu) \sim (\lambda, \pi, p\lambda'\mu)$, where $p = lul' \in P$, $l, l' \in L$, $u \in U$ and u and $l'\lambda'$ are permutable (cf. Section 2).

R_3: $(\lambda, \pi, \lambda'\mu) \sim (\lambda^p, \pi^p, (\lambda'\mu)^p)$, where $p \in P$ has the form $\left(\begin{smallmatrix} 1_n & \\ & q \end{smallmatrix}\right)$, $\phi(\lambda\pi\lambda'\mu) \in E(n, \Lambda)$ and $\lambda^p, \lambda'^p \in L$, $\mu^p \in U$.

Next we consider two maps:

(4.1) $\cdot : L \times \tilde{A} \to \tilde{A}/R_0$

$$l \cdot (\lambda, \pi, \lambda'\mu) = (l\lambda, \pi, \lambda'\mu),$$

(4.2) $\cdot : P \times \tilde{A} \to \tilde{A}/R_0$

$$p \cdot (\lambda, \pi, \lambda'\mu) = \left(\begin{bmatrix} 1 & \\ (p\lambda)^{-1} & 1 \end{bmatrix}, p\omega_n \pi, \begin{bmatrix} \lambda' & \\ \lambda\pi\lambda' & \lambda \end{bmatrix} \begin{bmatrix} \mu & -(\pi\lambda')^{-1} \\ & 1 \end{bmatrix} \right).$$

The latter map is well defined, independently of n, since increasing n (i.e., increasing the block size) has the effect of conjugating each "matrix" by a permutation which fixes the first n coordinates. By virtue of R_3, this is no change at all in $\tilde{A}/R_0$. Now let R be the smallest equivalence relation on $\tilde{A}$ containing R_0 and having the property $xRy \Rightarrow a \cdot xRa \cdot y$ for all $a \in L$ or P. We set $A(\Lambda) = \tilde{A}/R$. Our second main result is:

STRUCTURE THEOREM 4.3. *The map* $\tilde{A} \to \mathrm{St}(\Lambda)$ *sending* $(\lambda, \pi, \lambda'\mu) \to \lambda\pi\lambda'\mu$ *induces a bijection of sets* $\psi : A(\Lambda) \to \mathrm{St}(\Lambda)$.

In view of Lemma 2.4 and Proposition 3.4, ψ is well defined; moreover, the decomposition Theorem 3.3 shows ψ is surjective. The procedure for showing it is injective is to construct a *transitive left action* of $\mathrm{St}(\Lambda)$ on $A(\Lambda)$ such that ψ becomes equivariant with respect to this action and the action of left multiplication of $\mathrm{St}(\Lambda)$ on itself. Suppose we have such an action. Then if $\psi x = \psi y$ write $y = \alpha x$ for $\alpha \in \mathrm{St}(\Lambda)$. Thus $\psi(x) = \psi(\alpha x) = \alpha\psi(x)$ and so $\alpha = 1$ and $y = x$ and ψ is injective. It remains to construct the action. It is built up in stages. In Section 5 we describe the actions of L and P on $A(\Lambda)$ which arise from formulas 4.1 and 4.2. In Section 6 we put these actions together to obtain an action of $\mathrm{St}(\Lambda)$ on $A(\Lambda)$, and show that ψ is equivariant. In Section 7 we finish the proof of Theorem 4.1 by showing the action is transitive.

5. THE ACTIONS OF L AND P ON $A(\Lambda)$

We begin by noting that the condition $xRy \Rightarrow \alpha \cdot xR\alpha \cdot y$ for $\alpha \in L$ or P means that formulas (4.1) and (4.2) induce maps

(5.1) $\cdot : L \times A(\Lambda) \to A(\Lambda)$,

(5.2) $\cdot : P \times A(\Lambda) \to A(\Lambda)$.

It is perhaps remarkable that in the sequel we shall not use the relation R directly, but rather we shall use the existence of these two maps, together with the elementary relations R_1, R_2 and R_3. Indeed R_3 will be used only once more, in the proof of Proposition 5.5.

Now the map (5.1) obviously gives a left action of L on $A(\Lambda)$. However, the fact that (5.2) yields an action must be proved.

PROPOSITION 5.3. *Formula* (4.2) *yields a left action of* P *on* $A(\Lambda)$.

Proof. First we show that $1 \cdot (\lambda, \pi, \lambda'\mu) = (\lambda, \pi, \lambda'\mu)$. The left-hand side is

$$\left(\begin{bmatrix} 1 & \\ \lambda^{-1} & 1 \end{bmatrix}, \omega_n\pi, \begin{bmatrix} \lambda' & \\ \lambda\pi\lambda' & \lambda \end{bmatrix}\begin{bmatrix} \mu & -(\pi\lambda')^{-1} \\ & 1 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} \lambda & \\ 1 & 1 \end{bmatrix}, \pi \begin{bmatrix} 1 & \\ -\pi & 1 \end{bmatrix}\begin{bmatrix} 1 & \pi^{-1} \\ & 1 \end{bmatrix}\begin{bmatrix} 1 & \\ -\pi & 1 \end{bmatrix}, \begin{bmatrix} \lambda' & \\ \pi\lambda' & 1 \end{bmatrix}\begin{bmatrix} \mu & -(\pi\lambda')^{-1} \\ & 1 \end{bmatrix}\right) \quad \text{by } R_1$$

$$= \left(\begin{bmatrix} \lambda & \\ 1 & 1 \end{bmatrix}, \pi, \begin{bmatrix} 1 & \\ -\pi & 1 \end{bmatrix}\begin{bmatrix} \lambda'\mu & \\ & 1 \end{bmatrix}\right) \quad \text{by } R_2$$

$$= (\lambda, \pi, \lambda'\mu) \quad \text{by } R_1.$$

Next we show that $p \cdot (q \cdot (\lambda, \pi, \lambda'\mu)) = (pq) \cdot (\lambda, \pi, \lambda'\mu)$. The left-hand side is

$$\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ p^{-1} & & 1 & \\ (-pq\lambda)^{-1} & 1 & & 1 \end{bmatrix}, p\omega_{2n}q\omega_n\pi, \begin{bmatrix} \lambda' & & & \\ \lambda\pi\lambda' & \lambda & & \\ q\lambda\pi\lambda' & q\lambda & 1 & \\ & 1 & (q\lambda)^{-1} & 1 \end{bmatrix}\right.$$

$$\left. \times \begin{bmatrix} \mu & -(\pi\lambda')^{-1} & & (\pi\lambda')^{-1} \\ & 1 & -(q\lambda)^{-1} & -1 \\ & & 1 & \\ & & & 1 \end{bmatrix}\right)$$

$$= \left(\begin{bmatrix} 1 & & & \\ (pq\lambda)^{-1} & 1 & & \\ p^{-1} & & 1 & \\ & 1 & & 1 \end{bmatrix}, p\omega_{2n}q\omega_n\pi, \begin{bmatrix} \lambda' & & & \\ \lambda\pi\lambda' & \lambda & & \\ q\lambda\pi\lambda' & q\lambda & 1 & \\ -\pi\lambda' & & & 1 \end{bmatrix}\right.$$

$$\left. \times \begin{bmatrix} \mu & -(\pi\lambda')^{-1} & & (\pi\lambda')^{-1} \\ & 1 & -(q\lambda)^{-1} & -1 \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) \quad \text{by } R_1$$

Now we use

$$p\omega_{2n}q\omega_n\pi = pq\omega_n\pi \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -q & 1 & \\ \pi & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & -\pi^{-1} \\ & 1 & q^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -q & 1 & \\ \pi & & & 1 \end{bmatrix}$$

to get

$$= \left(\begin{bmatrix} 1 & & & \\ (pq\lambda)^{-1} & 1 & & \\ p^{-1} & & 1 & \\ & 1 & & 1 \end{bmatrix}, pq\omega_n\pi, \begin{bmatrix} 1 & & & \\ & 1 & & \\ & -q & 1 & \\ \pi & & & 1 \end{bmatrix} \begin{bmatrix} \lambda' & \\ \lambda\pi\lambda' & \lambda \end{bmatrix} \begin{bmatrix} \mu & -(\pi\lambda')^{-1} \\ & 1 \end{bmatrix} \right) \quad \text{by } R_2$$

$=$ the right-hand side by R_1.

We end this section by giving two propositions which relate the effects of the actions of L and P on $A(\Lambda)$.

PROPOSITION 5.4. *If $p \in P$ and $l \in L$ satisfy $l^p \in L$, then $l^p \cdot a = p^{-1} \cdot l \cdot p \cdot a$ for all $a \in A(\Lambda)$.*

Proof. Setting $a = (\lambda, \pi, \lambda'\mu)$, the right-hand expression is

$$\left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ l^{-1}p & & 1 & \\ -\lambda^{-1}p^{-1}l^{-1}p & 1 & & 1 \end{bmatrix}, p^{-1}\omega_{2n}p\omega_n\pi, \begin{bmatrix} \lambda' & & & \\ \lambda\pi\lambda' & \lambda & & \\ lp\lambda\pi\lambda' & lp\lambda & l & \\ & 1 & (p\lambda)^{-1} & 1 \end{bmatrix} \right.$$

$$\left. \times \begin{bmatrix} \mu & -(\pi\lambda')^{-1} & & (\pi\lambda')^{-1} \\ & 1 & -(p\lambda)^{-1} & -1 \\ & & 1 & \\ & & & 1 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} p^{-1}lp & & & \\ & 1 & & \\ p & & 1 & \\ -\lambda^{-1} & 1 & & 1 \end{bmatrix}, p^{-1}\omega_{2n}p\omega_n\pi, \begin{bmatrix} \lambda' & & & \\ \lambda\pi\lambda' & \lambda & & \\ p\lambda\pi\lambda' & p\lambda & 1 & \\ & 1 & (p\lambda)^{-1} & 1 \end{bmatrix} \right.$$

$$\left. \times \begin{bmatrix} \mu & -(\pi\lambda')^{-1} & & (\pi\lambda')^{-1} \\ & 1 & -(p\lambda)^{-1} & -1 \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) \quad \text{by } R_1.$$

Now we use $p^{-1}\omega_{2n}p\omega_n\pi=\pi\sigma_1\sigma_2\sigma_3$ where $\sigma_1=\omega_{4,1}(\pi)$, $\sigma_2=\omega_{4,2}(-1)$, and $\sigma_3=\omega_{3,2}(-p)$ to reduce to

$$\left(\begin{bmatrix} p^{-1}lp & & & \\ & 1 & & \\ p & & 1 & \\ -\lambda^{-1} & 1 & & 1 \end{bmatrix}, \pi\sigma_1\sigma_2, \begin{bmatrix} \lambda' & & & \\ \lambda\pi\lambda' & \lambda & & \\ -p\lambda\pi\lambda' & -p\lambda & 1 & \\ & 1 & & 1 \end{bmatrix}\right.$$

$$\left.\times\begin{bmatrix} \mu & -(\pi\lambda')^{-1} & & (\pi\lambda')^{-1} \\ & 1 & & -1 \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) \qquad \text{by } R_2$$

$$=\left(\begin{bmatrix} p^{-1}lp\lambda & & & \\ & 1 & & \\ & & 1 & \\ -1 & 1 & & 1 \end{bmatrix}, \pi\sigma_1\sigma_2, \begin{bmatrix} \lambda' & & & \\ \pi\lambda' & 1 & & \\ & & 1 & \\ & 1 & & 1 \end{bmatrix}\right.$$

$$\left.\times\begin{bmatrix} \mu & -(\pi\lambda')^{-1} & & (\pi\lambda')^{-1} \\ & 1 & & -1 \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) \qquad \text{by } R_1$$

$$=\left(\begin{bmatrix} p^{-1}lp\lambda & & & \\ & 1 & & \\ & & 1 & \\ -1 & 1 & & 1 \end{bmatrix}, \pi\sigma_1, \begin{bmatrix} \lambda' & & & \\ & 1 & & \\ & & 1 & \\ -\pi\lambda' & -1 & & 1 \end{bmatrix}\right.$$

$$\left.\times\begin{bmatrix} \mu & -(\pi\lambda')^{-1} & & (\pi\lambda')^{-1} \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}\right) \qquad \text{by } R_2$$

$$=\left(\begin{bmatrix} p^{-1}lp\lambda & & & \\ & 1 & & \\ & & 1 & \\ -1 & 1 & & 1 \end{bmatrix}, \pi, \begin{bmatrix} \lambda' & & & \\ & 1 & & \\ & & 1 & \\ \pi\lambda' & -1 & & 1 \end{bmatrix}^{\mu}\right) \qquad \text{by } R_2$$

$$=p^{-1}lp\cdot(\lambda,\pi,\lambda'\mu) \qquad \text{by } R_1.$$

Proposition 5.5. *Let* $\omega=w_{ij}\in P$ *and* $x=x_{ij}^1\in L$. *Then* $x\cdot\omega^{-1}\cdot x\cdot\omega\cdot x\cdot a=\omega\cdot a$ *for all* $a\in A(\Lambda)$.

Proof. It suffices to show that $x \cdot \omega \cdot x \cdot a = \omega \cdot x^{-1} \cdot \omega \cdot a$. We set $a = (\lambda, \pi, \lambda'\mu)$ so that

$$\omega \cdot x^{-1} \cdot \omega \cdot a = \left(\begin{bmatrix} 1 & & & \\ & 1 & & \\ x\omega^{-1} & & 1 & \\ -(\omega x^{-1}\omega\lambda)^{-1} & 1 & & 1 \end{bmatrix}, \omega\omega_{2n}\omega\omega_n\pi, \right.$$

$$\left. \begin{bmatrix} \lambda' & & & \\ \lambda\pi\lambda' & \lambda & & \\ x^{-1}\omega\lambda\pi\lambda' & x^{-1}\omega\lambda & x^{-1} & \\ & 1 & (\omega\lambda)^{-1} & 1 \end{bmatrix} \begin{bmatrix} \mu & -(\pi\lambda')^{-1} & & (\pi\lambda')^{-1} \\ & 1 & -(\omega\lambda)^{-1} & -1 \\ & & 1 & \\ & & & 1 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 1 & & & \\ (\omega x^{-1}\omega\lambda)^{-1} & 1 & & \\ x\omega^{-1} & & 1 & \\ & 1 & & 1 \end{bmatrix}, \omega\omega_{2n}\omega\omega_n\pi, \begin{bmatrix} \lambda' & & & \\ \lambda\pi\lambda' & \lambda & & \\ x^{-1}\omega\lambda\pi\lambda' & x^{-1}\omega\lambda & x^{-1} & \\ -\pi\lambda & & & 1 \end{bmatrix} \right.$$

$$\left. \times \begin{bmatrix} \mu & -(\pi\lambda')^{-1} & & (\pi\lambda')^{-1} \\ & 1 & -(\omega\lambda)^{-1} & -1 \\ & & 1 & \\ & & & 1 \end{bmatrix} \right) \quad \text{by } R_1.$$

Now we write $\omega\omega_{2n}\omega\omega_n\pi = \omega\omega_n\pi\omega_{3,2}(-\omega)(1_{2n} \oplus \omega)\, \omega_{4,1}(\pi)$ and use R_2 to get

$$= \left(\begin{bmatrix} 1 & & & \\ (\omega x^{-1}\omega\lambda)^{-1} & 1 & & \\ x\omega^{-1} & & 1 & \\ & 1 & & 1 \end{bmatrix}, \omega\omega_n\pi\omega_{3,2}(-\omega)(1_{2n} \oplus \omega), \right.$$

$$\left. \begin{bmatrix} \lambda' & & & \\ \lambda\pi\lambda' & \lambda & & \\ x^{-1}\omega\lambda\pi\lambda' & x^{-1}\omega\lambda & x^{-1} & \\ \pi\lambda' & & & 1 \end{bmatrix} \begin{bmatrix} \mu & -(\pi\lambda')^{-1} & & \\ & 1 & -(\omega\lambda)^{-1} & \\ & & 1 & \\ & & & 1 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} 1 & & \\ (\omega x^{-1}\omega\lambda)^{-1} & 1 & \\ x\omega^{-1} & & 1 \end{bmatrix}, \omega\omega_n\pi\omega_{3,2}(-\omega)(1_{2n} \oplus \omega), \right.$$

$$\left. \begin{bmatrix} \lambda' & & \\ \lambda\pi\lambda' & \lambda & \\ x^{-1}\omega\lambda\pi\lambda' & x^{-1}\omega\lambda & x^{-1} \end{bmatrix} \begin{bmatrix} \mu & -(\pi\lambda')^{-1} & \\ & 1 & -(\omega\lambda)^{-1} \\ & & 1 \end{bmatrix} \right) \quad \text{by } R_1.$$

Now use the fact that $\omega x^{-1} = x x^{-1t}$ and R_2 to obtain

$$= \left(\begin{bmatrix} 1 & & \\ (\omega x^{-1}\omega\lambda)^{-1} & 1 & \\ x\omega^{-1} & & 1 \end{bmatrix}, \omega\omega_n \pi \omega_{3,2}(-\omega), \right.$$

$$\left. \begin{bmatrix} \lambda' & & \\ \lambda\pi\lambda' & \lambda & \\ x x^{-1t}\omega\lambda\pi\lambda' & x x^{-1t}\omega\lambda & x \end{bmatrix} \begin{bmatrix} \mu & -(\pi\lambda')^{-1} & \\ & 1 & -(\omega\lambda)^{-1} \\ & & x^{-1t} \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} x & & \\ (x^{-1}\omega x^{-1}\omega\lambda)^{-1} & 1 & \\ x\omega^{-1}x & & 1 \end{bmatrix}, \omega\omega_n \pi \omega_{3,2}(-\omega), \right.$$

$$\left. \begin{bmatrix} \lambda' & & \\ \lambda\pi\lambda' & \lambda & \\ x^{-1t}\omega\lambda\pi\lambda' & x^{-1t}\omega\lambda & 1 \end{bmatrix} \begin{bmatrix} \mu & -(\pi\lambda')^{-1} & \\ & 1 & -(\omega\lambda)^{-1} \\ & & x^{-1t} \end{bmatrix} \right) \qquad \text{by } R_1.$$

Next we use the fact that $x^{-1}\omega x^{-1}\omega = \omega x$ in $E(\Lambda)$, together with conjugation by $1_{2n} \oplus \omega$ and the relation R_3 to obtain

$$= \left(\begin{bmatrix} x & & \\ (\omega x\lambda)^{-1} & 1 & \\ (\omega x)^{-1} & & 1 \end{bmatrix}, \omega\omega_n \pi \omega_{3,2}(-1_n), \right.$$

$$\left. \begin{bmatrix} \lambda' & & \\ \lambda\pi\lambda' & \lambda & \\ x\lambda\pi\lambda' & x\lambda & x \end{bmatrix} \begin{bmatrix} \mu & -(\pi\lambda')^{-1} & \\ & 1 & -\lambda^{-1} \\ & & 1 \end{bmatrix} \right)$$

$$= \left(\begin{bmatrix} x & & \\ (\omega x\lambda)^{-1} & 1 & \\ (\omega x)^{-1} & & x^{-1} \end{bmatrix}, \omega\omega_n \pi \omega_{3,2}(-1_n), \right.$$

$$\left. \begin{bmatrix} \lambda' & & \\ x\lambda\pi\lambda' & x\lambda & \\ x\lambda\pi\lambda' & x\lambda & x \end{bmatrix} \begin{bmatrix} \mu & -(\pi\lambda')^{-1} & \\ & 1 & -\lambda^{-1} \\ & & 1 \end{bmatrix} \right) \qquad \text{by } R_1$$

$$= \left(\begin{bmatrix} x & & \\ (\omega x\lambda)^{-1} & 1 & \\ (\omega x)^{-1} & & x^{-1} \end{bmatrix}, \omega\omega_n \pi, \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \right.$$

$$\left. \times \begin{bmatrix} \lambda' & & \\ x\lambda\pi\lambda' & x\lambda & \\ & & x \end{bmatrix} \begin{bmatrix} \mu & -(\pi\lambda')^{-1} & \\ & 1 & \\ & & 1 \end{bmatrix} \right) \qquad \text{by } R_2$$

$$= \left(\begin{bmatrix} x & \\ (\omega x \lambda)^{-1} & 1 \end{bmatrix}, \omega\omega_n \pi, \begin{bmatrix} \lambda' & \\ x\lambda\pi\lambda' & x\lambda \end{bmatrix} \begin{bmatrix} \mu & -(\pi\lambda')^{-1} \\ & 1 \end{bmatrix} \right) \qquad \text{by } R_1$$

$$= x \cdot \omega \cdot x \cdot a.$$

6. The Action of St(Λ) on $A(\Lambda)$

We first construct an action of U on $A(\Lambda)$. If $u \in U$, there always exist elements $p \in P$ such that $u^p \in L$. We define

$$u \cdot a = p \cdot u^p \cdot p^{-1} \cdot a \quad \text{for} \quad a \in A(\Lambda). \tag{6.1}$$

Of course (6.1) is independent of the choice of p by virtue of Proposition 5.4, and hence gives an action of U on $A(\Lambda)$.

Now we combine the actions of L and U to obtain an action of $L*U$ (the free product) on $A(\Lambda)$. Let N be the kernel of the canonical epimorphism $L*U \to \mathrm{St}(\Lambda)$.

Proposition 6.2. *The action of $L*U$ on $A(\Lambda)$ is trivial on N, and passes down to an action of* St(Λ) *on $A(\Lambda)$ extending the action of P.*

Proof. $N \subset L*U$ is generated by elements of the form

$$\begin{aligned} &[x_{ij}^{\lambda}, x_{kl}^{\mu}], && i \neq l,\ j \neq k, \text{ where } \lambda, \mu \in \Lambda \text{ (cf. Section 2)}, \\ &[x_{ij}^{\lambda}, x_{jk}^{\mu}]\, x_{ik}^{\lambda\mu}, && i \neq l. \end{aligned} \tag{6.3}$$

Of course, when the factors x_{ij}^{λ}, etc. in an expression of one of the forms given by (6.3) all lie in L, or all lie in U, then the expression is identically 1, and so acts trivially. When the factors are of mixed type, we can always choose an element $\pi \in P$, such that by conjugating each factor by π we obtain an expression all of whose factors lie in L. Thus it suffices to show that $\prod_{q=1}^{s} x_{i_q j_q}^{\lambda_q} \in L*U$ acts like $\pi\{\prod_{q=1}^{s} (x_{i_q j_q}^{\lambda_q})^{\pi}\}\, \pi^{-1} \in L*P$, where we assume $(x_{i_q j_q}^{\lambda_q})^{\pi} \in L$ for all q. This follows by induction on s, once we note that it holds for $s = 1$, either by Definition 6.1 or by Proposition 5.4. Finally we note that the formula given in Proposition 5.5 shows that the action of the generators of P via the action induced from St(Λ) is the same as their action via the original action of P given by (4.2).

We finish this section by showing that the map $\psi: A(\Lambda) \to \mathrm{St}(\Lambda)$ induced by sending $(\lambda, \pi, \lambda'\mu) \to \lambda\pi\lambda'\mu$ is equivariant with respect to the left action of St(Λ) on $A(\Lambda)$ described above and the canonical left action of St(Λ) on itself. We need only check the equivariance for elements of P and L since these generate St(Λ). For elements of L, this is obvious, and for elements of P this follows from the Reduction Identity (3.4) applied to the definition of the action of P.

7. The Action is Transitive

In this section we complete the proof of the structure theorem by showing the action of St(Λ) on $A(\Lambda)$ is transitive. This we do in a series of lemmas.

Lemma 7.1. *If $p \in P$, then $p \cdot (1, 1, \lambda'\mu) = (1, p, \lambda'\mu)$.*

Proof.

$$\begin{aligned} p \cdot (1, 1, \lambda'\mu) &= \left(\begin{bmatrix} 1 & \\ p^{-1} & 1 \end{bmatrix}, p\omega_n, \begin{bmatrix} \lambda' & \\ \lambda' & 1 \end{bmatrix} \begin{bmatrix} \mu & -\lambda'^{-1} \\ & 1 \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} 1 & \\ p^{-1} & 1 \end{bmatrix}, p, \begin{bmatrix} \lambda' & \\ -\lambda' & 1 \end{bmatrix} \begin{bmatrix} \mu & \\ & 1 \end{bmatrix} \right) \qquad \text{by } R_2 \\ &= \left(\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, p, \begin{bmatrix} \lambda' & \\ & 1 \end{bmatrix} \begin{bmatrix} \mu & \\ & 1 \end{bmatrix} \right) \qquad \text{by } R_1 \\ &= (1, p, \lambda'\mu). \end{aligned}$$

Lemma 7.2. *If $\lambda \in L$ then $\lambda \cdot (1, 1, \lambda'\mu) = (1, 1, \lambda\lambda'\mu)$.*

Proof. $\lambda \cdot (1, 1, \lambda'\mu) = (\lambda, 1, \lambda'\mu) = (1, 1, \lambda\lambda'\mu)$ by R_1.

Lemma 7.3. $\left[\begin{smallmatrix} 1 & u \\ & 1 \end{smallmatrix}\right] \cdot (1, 1, \mu) = (1, 1, \left[\begin{smallmatrix} \mu & u \\ & 1 \end{smallmatrix}\right])$.

Proof. One easily verifies that

$$\begin{bmatrix} 1 & u \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & -u & 1 \end{bmatrix} \omega_{3,1}(-1) \begin{bmatrix} 1 & & \\ & 1 & \\ & u & 1 \end{bmatrix} \omega_{3,1}(-1)^{-1} \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix}.$$

Now by Lemmas 7.1 and 7.2 we have

$$\begin{aligned} &\begin{bmatrix} 1 & & \\ & 1 & \\ & u & 1 \end{bmatrix} \omega_{3,1}(-1)^{-1} \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix} \cdot (1, 1, \mu) \\ &= \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & u & 1 \end{bmatrix}, \omega_{3,1}(-1)^{-1}, \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix} \mu \right) \\ &= \left(\begin{bmatrix} 1 & & \\ & 1 & \\ & u & 1 \end{bmatrix}, 1, \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & -1 \\ & 1 & \\ & & 1 \end{bmatrix} \mu \right) \qquad \text{by } R_2 \\ &= \left(1, 1, \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & u & 1 \end{bmatrix} \begin{bmatrix} 1 & & -1 \\ & 1 & \\ & & 1 \end{bmatrix} \mu \right) \qquad \text{by } R_1. \end{aligned}$$

Hence

$$\begin{bmatrix} 1 & u \\ & 1 \end{bmatrix} \cdot (1, 1, 1)$$

$$= \left(\begin{bmatrix} 1 & & \\ & 1 & \\ 1 & -u & 1 \end{bmatrix}, \omega_{31}(-1), \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & u & 1 \end{bmatrix} \begin{bmatrix} 1 & & -1 \\ & 1 & \\ & & 1 \end{bmatrix} \mu \right)$$

$$= \left(\begin{bmatrix} 1 & & \\ & 1 & \\ 1 & -u & 1 \end{bmatrix}, 1, \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & u & 1 \end{bmatrix} \begin{bmatrix} 1 & u & \\ & 1 & \\ & & 1 \end{bmatrix} \mu \right) \quad \text{by } R_2$$

$$= \left(1, 1, \begin{bmatrix} \mu & u & \\ & 1 & \\ & & 1 \end{bmatrix} \right) \quad \text{by } R_1.$$

COROLLARY 7.4. *If* $u \in U$, *then* $u \cdot (1, 1, 1) = (1, 1, u)$.

Proof. If u is 2×2 this follows from Lemma 7.3. If u is $n \times n$ write it as $u = \left[\begin{smallmatrix} 1 & u_1 \\ & 1 \end{smallmatrix}\right] \mu$, where μ is $n-1 \times n-1$. Then

$$\begin{bmatrix} 1 & u_1 \\ & 1 \end{bmatrix} \mu \cdot (1, 1, 1) = \begin{bmatrix} 1 & u_1 \\ & 1 \end{bmatrix} \cdot (1, 1, \mu) \quad \text{by induction}$$

$$= \left(1, 1, \begin{bmatrix} 1 & u_1 \\ & 1 \end{bmatrix} \mu \right) \quad \text{by 7.3.}$$

PROPOSITION 7.5. $\mathrm{St}(\Lambda)$ *acts transitively on* $A(\Lambda)$.

Proof. The preceding lemmas combine to yield $\lambda\pi\lambda'\mu \cdot (1, 1, 1) = (\lambda, \pi, \lambda'\mu)$.

8. REMARKS ON SOME EXTENSIONS OF THE STRUCTURE THEOREM

As we noted in Section 3, the decomposition theorem certainly holds in $E(\Lambda)$. What can be said about the structure theorem in that context? The difficulty arises in trying to control the kernel of $\phi = K_2(\Lambda)$ as we pass from $\mathrm{St}(\Lambda)$ to $E(\Lambda)$. If we are not careful, it will give rise to relations in $E(\Lambda)$ which are *not* consequences of our standard relations R_1, R_2, R_3 via the maps (4.1) and (4.2). However, if $K_2(\Lambda)$ arises in P itself (as it does in the case $\Lambda = Z$) we have very good control. This suggests that we try extend the structure theorem using a larger group than P, in order to have more chance to control $K_2(\Lambda)$.

Let $V \subset \Lambda$ be a group of units and let P_V be the subgroup of $\mathrm{St}(\Lambda)$ generated by $w_{ij}(v) = x_{ij}^{v} x_{ji}^{-v^{-1}} x_{ij}^{v}$. The image $\bar{P}_V = \phi(P_V)$ is a group of generalized permutation matrices. Our previous arguments extend almost verbatim to this context, and yield a *structure theorem* identical to Theorem 4.3 except that P is replaced by P_V throughout.

Now if $K_2(\Lambda) \subset P_V$, then there is a structure theorem for $E(\Lambda)$ identical to (4.1) except that L, P, U are replaced by $\bar{L}, \bar{P}_V, \bar{U}$. To see this, suppose $\bar{\lambda}\bar{\pi}\bar{\lambda}'\bar{\mu} = \bar{\lambda}_1 \bar{\pi}\bar{\lambda}'_1 \bar{\mu}_1$. Then $\lambda\pi\lambda'\mu = \lambda_1(c\pi)\lambda'_1\mu$, for some $c \in K_2(\Lambda) \subset P_V$. Hence $(\lambda, \pi, \lambda'\mu)$ is equivalent to $(\lambda_1, c\pi, \lambda'_1\mu_1)$ and so $(\bar{\lambda}, \bar{\mu}, \bar{\lambda}'\bar{\mu})$ is equivalent to $(\bar{\lambda}_1, \bar{\pi}_1, \bar{\lambda}'_1\bar{\mu}_1)$.

If we assume not only $K_2(\Lambda) \subset P_V$ but also $V \subset GL(1, \Lambda) \to K_1(\Lambda)$ is onto, then we get a structure theorem for $GL(\Lambda)$, based on $\bar{P}'_V$, the group of generalized permutation matrices generated by $\bar{P}$ and $V \subset GL(1, \Lambda) \subset GL(\Lambda)$. Indeed if $\lambda\pi\lambda'\mu = \lambda_1\pi_1\lambda'_1\mu_1$, choose $v \in V$ so that $\pi \equiv v^{-1} \bmod E(\Lambda)$. Then $\lambda(\pi \oplus v)\lambda'\mu = \lambda_1(\pi_1 \oplus v)\lambda'_1\mu_1$ in $E(\Lambda)$ and hence $(\lambda, \pi \oplus v, \lambda'\mu)$ is equivalent to $(\lambda_1, \pi_1 \oplus v, \lambda'\mu)$. Multiplying by $1 \oplus v^{-1} \in \bar{P}'_V$ we obtain $(\lambda, \pi, \lambda'\mu)$ equivalent to $(\lambda_1, \pi_1, \lambda'\mu)$.

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