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CATEGORIES OF EMBEDDINGS

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Abstract. We present a categorical generalisation of the notion of domains, which is closed under (suitable) exponentiation. The goal was originally to generalise Girard's model of polymorphism to F_ω. If we specialise this notion in the poset case, we get new cartesian closed categories of domains.

Introduction

The purpose of this paper is to present an abstract and general framework where we deal with categories of "structures" and embeddings, which are closed under directed colimits, and such that any "structure" is the directed colimit of its finite substructures. This is a categorical generalisation of the notion of domain used in denotational semantics.

Examples of such categories are the category of sets and injections, the category of graphs and embeddings, the category of linear orders and strictly increasing maps, any category of domains with embedding-projection pairs as morphisms, but also any domain, seen as a category. We show that if we have two such categories, then the category of continuous (resp. continuous and stable, i.e. pull-back preserving [22]) functors with natural transformations (resp. cartesian natural transformations) as morphisms is still a category of this kind. Therefore, the category of all such categories is a cartesian closed category (since the existence of products does not raise any problems). The situation is similar to the one in domain theory, where the poset of continuous (resp. continuous and stable) functions between two domains is still a domain.

The original motivation of this work comes from the denotational semantics of polymorphism [6, 2, 3]. A direct application is indeed the extension of the models of F_2 described in [6, 2, 3] to models of F_ω. This work shows that there is a large class of models for F_ω where, as in Girard's model [6], types are interpreted as domains. There may be other areas in computer science where one wants to give a denotation as an object in a category instead of as an element of a poset.

Such a general concept seems also to play a role in the theory of dilators of Girard and it may be interesting to look for applications in this framework.

As pointed out by Gunter, if we restrict these definitions to posets, we do not get exactly the known notion of Scott domains (resp. dI-domains), but a generalisation of it. This new category of domains is still a cartesian closed category, and seems to be interesting in itself (we have a stable corresponding notion that generalises the cartesian closed category of dI-domains). This new category of domains has been recently discovered by Jung [11]. The author was unaware of his work, until Gunter remarked that if we restrict our definition in the poset case, we obtain exactly the same notion of domains [8]. In the extensional case, the drawback of these domains is that they are not countably based in general. If we restrict ourselves to profinite such domains, we get the "short" domains of Gunter [7]. In the stable case, we obtain a new category of domains.

The reader who is not familiar with stable functions may skip the stable part (since the ideas are almost the same in both cases). Actually, one may even read this paper by considering only the poset case, since some results are (we hope) new in that special case.

In the first part we recall some definitions and basic properties about domains and categories. It is then shown how certain posets of sections of a Grothendieck (co-)fibration of a functor form a domain. After the definition of the categories of embeddings (in the extensional and stable cases), this fact is directly used to show that this notion of category is closed under suitable exponentiation (thus this work is really an example of the use of the notion of dependent family and dependent products in domain theory). A brief account of how this can be used to build a full model of polymorphism is then outlined.

1. About some notions in category theory

1.1. Generalities and notations

Since we will deal with very large categories, we have to be careful with foundational problems. For this, the notion of Grothendieck universe [18] seems convenient. We work in ZFC + the axiom of universes, and we write U_0, U_1, U_2 for the first universes. Any category is a set (there are no classes), but this set may be in different universes. We say that a set is small if and only if its cardinality belongs to U_0.

If $\mathscr{C}$ is a category, then $\mathscr{C}_0$ denotes the set of objects of $\mathscr{C}$ and $\mathscr{C}_1$ the set of morphisms of $\mathscr{C}$. We have the domain dom and codomain codom maps from $\mathscr{C}_1$ to $\mathscr{C}_0$. If $a, b \in \mathscr{C}_0$, then $\mathscr{C}(a, b)$ denotes the set of all $f \in \mathscr{C}_1$ such that $\mathrm{dom}(f) = a$ and $\mathrm{codom}(f) = b$. As usual, we say that $\mathscr{C}$ is *locally small* if, and only if, for any objects a and b of $\mathscr{C}$, the set $\mathscr{C}(a, b)$ is small. We work only with locally small categories. If S a subset of $\mathscr{C}_0$, we say that S is *essentially small* if and only if there exists a small subset of S so that any element of S is isomorphic to one object in this small subset.

We will need the concept of filtered colimits (see [15, 18], it is important to note that the empty category is not filtered, see [9, p. 24]). It is the categorical analogue of the notion of directed sups: a category is filtered if and only if any finite diagram (the empty diagram as well) may be completed into a cone (see [15, p. 67], this cone is not necessarily a colimit one). We say that a functor is *Scott-continuous* or simply continuous if and only if it preserves [15] filtered colimits. This is the categorical generalisation of the notion of "Scott-continuous" maps in denotational semantics. Another important property has been considered in denotational semantics, that is the preservation of meets of compatible elements, and we will also need its categorical version: a functor is said to be stable if and only if it preserves pull-backs.

In any category $\mathscr{C}$, we can consider, for any object $a \in \mathscr{C}$, the poset of equivalence classes of maps of codomain a, associated to the pre-order of subobjects of a. We write $\text{Sub}(a)$ this poset, and allow ourselves to say that f belongs to $\text{Sub}(a)$ instead of "the equivalence class defined by f belongs to $\text{Sub}(a)$". One says that $\mathscr{C}$ is *well-powered* if and only if each $\text{Sub}(a)$ is of small cardinality (see [15, p. 126]). The poset $\text{Sub}(a)$ is equivalent to the category $\text{Mon}(a)$ of monics into a.

We have for any object a a functor $F_a : \text{Mon}(a) \to \mathscr{C}$ defined on objects by $F_a(f : b \to a) = b$ and on morphisms by $F_a(g : (f_1 : b_1 \to a) \to (f_2 : b_2 \to a)) = g$. For any category $\mathscr{C}$, we have that F_a creates arbitrary colimits (see [15, p. 108]). Hence, if $\mathscr{C}$ has filtered colimits, each $\text{Mon}(a)$ has filtered colimits, and F_a preserves filtered colimits. We deduce that each $\text{Sub}(a)$ has directed sups if $\mathscr{C}$ has filtered colimits.

If we suppose furthermore that all morphisms in $\mathscr{C}$ are monomorphisms, then to each map $f \in \mathscr{C}(a, b)$ is associated the direct image functor $f_! : \text{Sub}(a) \to \text{Sub}(b)$, which is the composition with f. If $\mathscr{C}$ also has pull-backs, for $f \in \mathscr{C}(a, b)$, we have that $f^* : \text{Sub}(b) \to \text{Sub}(a)$, pull-backs along f, is a right adjoint to $f_! : \text{Sub}(a) \to \text{Sub}(b)$, which is the direct image functor. Hence, the image functor $f_! : \text{Sub}(a) \to \text{Sub}(b)$, which is a left adjoint, preserves arbitrary sups, and hence is continuous. But in general, f^* will not be continuous. If f^* is continuous, then $(f_!, f^*)$ is an embedding-projection pair between the two cpos $\text{Sub}(a)$ and $\text{Sub}(b)$.

But there is more. First, both f^* and $f_!$ are stable (since $\mathscr{C}$ has pull-backs, each $\text{Sub}(a)$ has pull-backs as a poset, that is infs of compatible elements). Secondly, $(f_!, f^*)$ is a rigid embedding. That is $f^* \circ f_! = \text{id}, f_! \circ f^* \leq \text{id}$ for the *stable* ordering, which is to say that $f_! \circ f^* \leq \text{id}$ for the extensional ordering and $f_! \circ f^*(x) = f_! \circ f^*(y) \wedge x$ if $x \leq y$ (see [1, 22, 6] for more about the notion of rigid embedding). One intuition behind this concept is that there are no "holes" in the image of $f_!$. Indeed we can check that if $x \leq f_!(y)$, then there exists $z \leq y$ such that $x = f_!(z)$ (namely $z = f^*(x)$).

The category $\mathscr{P}$ of posets with directed sups and pull-backs, and rigid embeddings as morphisms, is a category with pull-backs and small directed colimits. The proposition that follows, which summarizes our discussion, is a motivation for the notion of stable functors.

Proposition 1.1. *Let $\mathscr{C}$ be a category where all morphisms are monomorphisms. Then, $\mathscr{C}$ has pull-backs if and only if each posets* Sub(a), $a \in \mathscr{C}_0$, *has infs. Suppose that $\mathscr{C}$ has pull-backs, small filtered colimits, is well-powered, and that, for any $f \in \mathscr{C}(a, b)$, the pull-back functor f^* from* Sub (b) *into* Sub(a) *is continuous. Then* Sub *defines a* continuous *and* stable *functor from the category $\mathscr{C}$ into the category $\mathscr{P}$.*

Note that the fact that $\mathscr{C}$ has all filtered colimits is used twice: for the definition of Sub as a functor from $\mathscr{C}$ to **Dom**, and then for proving that this functor is continuous. For proving this last part, use the characterisation of directed colimits in the category $\mathscr{P}$: A is the directed colimit of $(A_i, (f_i^L, f_i^R))$ if and only if $\bigvee f_i^L \circ f_i^R = \mathrm{id}$ (same proof as in [17]).

From now on, we shall work mainly with well-powered categories where all morphisms are monomorphisms. This is not a restriction with respect to our goal, which is to give a formalisation of the notion of substructure. However, it is likely that our notions may be generalised to the case where morphisms are not always monomorphisms. For instance, we would like to have a notion that also covers the notion of locally finitely presentable category, see [10]. Our framework is enough for the main application we have in mind, the extension of the semantics proposed in [6] to a model of F_ω.

In categories where all morphisms are monomorphisms, we can say that b is a directed colimit of a directed system (a_i, f_i), with $f_i \in \mathscr{C}(a_i, b)$ instead of saying that (b, f_i) is the directed colimit of the system $(a_i, (f_{ij}))$, since the maps f_{ij} are uniquely determined by the equations $f_j \circ f_{ij} = f_i$. In that case, we can also work with directed colimit instead of filtered colimit. However, we will state some definitions in the general case.

Definition 1.2. Let $\mathscr{C}$ be a category. An object $a \in \mathscr{C}_0$ is a *finitely presentable* if and only if $\mathscr{C}(a, -)$ commutes with filtered colimits.

For instance, in the category of sets with injections (or in any reasonable category of domains with embeddings), this definition of finiteness coincides with the set-theoretic notion of finiteness. In the category of groups with embeddings, this notion coincides with the notion of being finitely generated.

We will say sometimes, instead of finitely generated, that a is *finite*, since it seems a good generalisation of being finite (and we will see that in the stable case, we have indeed very strong finiteness properties). In general, perhaps "isolated" or "compact" would be a better terminology.

In the case where morphisms are monomorphisms, an object A is finite if and only if any morphism of A into a directed colimit, factors through one of the canonical maps of the directed colimit. This is, however, not true in general.

1.2. Ind-*completion*

We assume known the notion of the ind-completion of any small category (see [18, 9]), and we write ind($\mathscr{C}$) for the ind-completion of $\mathscr{C}$. This is a direct generalisation of the concept of ideal-completion of a poset. Intuitively, we add formally all filtered colimits. This is the full subcategory of the pre-sheaf category over $\mathscr{C}$ of functors $\mathscr{C}^{\mathrm{opp}} \to U_0$ that are filtered colimits of representable functors (which corresponds to the ideals, i.e. downward closed and filtered subsets, of a poset).

Definition 1.3. A category $\mathscr{C}$ is *algebraic* if and only if the full subcategory $\mathscr{C}_{\mathrm{fin}}$ of finitely presentable objects of $\mathscr{C}$ is essentially small and for any object a in $\mathscr{C}$, the comma category $\mathscr{C}_{\mathrm{fin}}\downarrow a$ is filtered, and has a for colimit.

We have the general result [18].

Proposition 1.4. *Let $\mathscr{C}$ be a category. Then $\mathscr{C}$ is algebraic if and only if $\mathscr{C}_{\mathrm{fin}}$ is essentially small and $\mathscr{C}$ is the* ind-*completion of $\mathscr{C}_{\mathrm{fin}}$. The* ind-*completion of an arbitrary small category $\mathscr{C}$ is algebraic, its category of finite elements being the Karoubi envelope of $\mathscr{C}$.*

Actually, we will need only this result in the case where all morphisms in $\mathscr{C}$, $\mathscr{C}$ small category, are monomorphisms, and in that case $(\mathrm{ind}(\mathscr{C}))_{\mathrm{fin}}$ is equivalent to $\mathscr{C}$. In the case of posets, this proposition says that an algebraic lattice is actually isomorphic to the poset we obtain by adding formal directed colimits from the subposet of its finite elements.

For an algebraic category $\mathscr{C}$, we have a stratification (C_κ), κ cardinal, of the objects of $\mathscr{C}$, by saying that an object is in C_κ, or is generated by less than κ elements if and only if this object is a filtered colimit over a filtered family of cardinality less than κ. One fundamental remark is then that, in an algebraic category, for any (small) cardinal κ, the set C_κ is essentially a small set.

1.3. About finiteness

Let us say that $a \in \mathscr{C}_0$ is finite relative to $f \in \mathscr{C}(a, b)$ if and only if f is a finite element of Sub(b). Then, there are two notions of finiteness: one "global", and another "local", that do not coincide in general. However, we can state the following.

Proposition 1.5. *Let $\mathscr{C}$ be a category where all morphisms are monomorphisms, with directed colimit, pull-backs, and such that for any $f \in \mathscr{C}(a, b)$, the pull-back functor f^* from the slice category $\mathscr{C}/b$ into the slice category $\mathscr{C}/a$ is continuous. Let a be an object of $\mathscr{C}$. We can consider $F_a : \mathscr{C}/a \to \mathscr{C}$, defined on object by $F_a(f : b \to a) = b$, and on morphisms by $F_a(g : (f_1 : b_1 \to a) \to (f_2 : b_2 \to a)) = g$. Then f is finite in $\mathscr{C}/a$ if and only if $F_a(f)$ is finite in $\mathscr{C}$.*

Proof. We have here, for any object b of $\mathscr{C}$ that b is finite in $\mathscr{C}$ if and only if id_b is finite in $\mathscr{C}/b$. This is a direct consequence of the fact that each f^* preserves filtered colimits.

Let us consider $f:b\to a$ in $\mathscr{C}/a$. We know that $f_!$ is an embedding. Hence, $f=f_!(\mathrm{id}_b)$ is finite in $\mathscr{C}/a$ if and only if id_b is finite in $\mathscr{C}/b$, if and only if $b=F_a(f)$ is finite in $\mathscr{C}$. □

Thus, in that case, to be finite has both a "local" and "global" meaning, and these meanings coincide, i.e. the following conditions are equivalent

- a is finite,
- $\mathrm{id}\in\mathscr{C}(a,a)$ is a finite element of $\mathrm{Sub}(a)$,
- there exists $f\in\mathscr{C}(a,b)$ which defines a finite element of $\mathrm{Sub}(b)$.

2. Domain of sections

We recall in this section some results about domains of section of a Grothendieck fibration of a functor from a small category into a category of domains with embeddings. There are two cases. In the extensional case, the category of domains is the category of complete algebraic lattices and embeddings. In the stable case, the functor must be from a small category with pull-backs, into the category of complete dI-domains with rigid embeddings (the complete dI-domains are the posets that are algebraic complete, completely distributive, and that satisfy the property of finiteness, called property I, that every finite element dominates only a finite number of elements, see [22]).

Let us first give the result in the case of complete algebraic lattices. First, let us recall that if we have a functor F from a small category $\mathscr{C}$ to the category **Dom** of complete algebraic lattices and embedding-projection pairs as morphisms, then the Grothendieck (co-)fibration of F (see [18, 3]), written $\Sigma(F)$ is the category whose objects are the pairs (X,x), X objects of $\mathscr{C}$ and x element of $F(X)$, and a morphism $f:(X,x)\to(Y,y)$ is $f\in\mathscr{C}(X,Y)$ such that $F(f)^L(x)\leq y$. We then have a cofibration [18] $p:\Sigma(F)\to\mathscr{C}$ which is the first projection. The category of *sections* of this cofibration is here the poset of family (t_X) such that $t_X\in F(X)$ and $F(f)^L(t_X)\leq t_Y$ if $f\in\mathscr{C}(X,Y)$ ordered by the pointwise ordering.

Theorem 2.1. *Let $\mathscr{C}$ be a small category, and F be an arbitrary functor from $\mathscr{C}$ to* **Dom**. *The poset $\Pi(F)$ of all sections of the Grothendieck cofibration of F is a complete algebraic lattice.*

Proof. The fact that it is a complete lattice follows by straightforward manipulation. We can actually even describe explicitly the finite elements of $\Pi(F)$. Let S be a finite set of pairs of the form (A,a), where A is an object of $\mathscr{C}$, and a a finite element of $F(A)$. Then the family u^S defined by

$$u^S_X=\bigvee\{F(f)^L(a)\mid(A,a)\in S\ \&\ f\in\mathscr{C}(A,X)\}$$

is a finite element, and any finite element is of this form. The fact that u^S is indeed finite comes from the equivalence, for any section (t_X), of $u^S \leq t$ and $a \leq t_A$ for all $(A, a) \in S$. The collection of all u^S is directed, since $u^{S_1} \vee u^{S_2} = u^{S_1 \cup S_2}$ (notice that the sup in the poset of sections is computed pointwise). Furthermore, a continuous section (t_X) is the directed sup of all sections u^S, where S is a finite set of pairs (A, a) such that $a \leq t_A$. □

In the stable case, we need to have pull-backs, and the proof is a little trickier. Here, **Dom** is the category of complete dI-domains with rigid embeddings (it would be possible to take the category of complete atomic boolean algebras and rigid embeddings).

Theorem 2.2. *Let $\mathscr{C}$ be a small category with pull-back, and F be a stable (i.e. pull-back preserving) functor from $\mathscr{C}$ to* **Dom**. *The poset $\Pi(F)$ (for the stable ordering) of all* stable *sections of the Grothendieck cofibration of F is a* dI-*domain,* not *complete in general.*

Proof. First, we have indeed the fact that the Grothendieck cofibration $\Sigma(F)$ of F has pull-backs, so we can speak about stable sections and one stable ordering. This corresponds to the notion of cartesian natural transformation. That is, $\eta : F \to G$ is a cartesian natural transformation if and only if it is a natural transformation and furthermore whenever $f: X \to Y$, the diagram defined by $(F(f), \eta_X, G(f), \eta_Y)$ is a pull-back diagram.

We can compute explicitly the pull-backs of $f \in (X, x) \to (Z, z)$, $g \in (Y, y) \to (Z, z)$ (that is $f \in \mathscr{C}(X, Z)$ and $F(f)^L(x) \leq z$, and $g \in \mathscr{C}(Y, Z)$ and $F(g)^L(y) \leq z$). Take the pull-back of f and g to be $u \in \mathscr{C}(T, X)$ and $v \in \mathscr{C}(T, Y)$. Then the pull-back in $\Sigma(F)$ is $(T, F(u)^R(x) \wedge F(v)^R(y))$.

The stable ordering on sections is nothing but the ordering defined by cartesian natural transformations. A simple calculation, using the form of the pull-back in the Grothendieck cofibration of F, shows that we have $u \leq t$ for the stable ordering, if and only if for any $f \in \mathscr{C}(X, Y)$, $u_X = t_X \wedge F(f)^R(u_Y)$. Notice that, in general, it does not coincide with the extensional ordering (see the remark below).

Here again, the fact that $\Pi(F)$ is a conditionally complete completely distributive cpo follows from a formal argument. We only show that it is algebraic, and that the property of finiteness I is satisfied.

Let us fix an element t of $\Pi(F)$. We show that t is indeed the directed sup of finite elements that satisfy the property I. Notice the *important difference* with the extensional case: here we can only give effectively finite elements less than one given element, even if each $F(X)$ is a complete domain. This seems to be a general phenomenon: in the extensional case, one can describe directly the finite elements of the function space (finite sups of steps functions), but in the stable case, one can only describe the finite elements less than one given element.

A proof that $\Pi(F)$ is a dI-domain could be given by using the notion of event structure of Winskel (see [22, 2]), instead we will sketch a purely domain theoretic proof. We consider the set of pairs (A, a) with $A \in \mathscr{C}_0$ and $a \leq t_A$. We define the relation $(A, a) \leq (B, b)$ on that set by "there exists $f \in \mathscr{C}(A, B)$ such that $F(f)^L(a) \leq b$" (notice that it is nothing else than the pre-order associated with the Grothendieck cofibration of F). For any finite set S of pairs (A, a), a finite element of $F(A)$ and $a \leq t_A$, we consider the family (u_Y^S), where u_Y^S is defined as

$$\bigvee \{F(\varphi)^L(u) \mid (A, a) \in S \;\&\; \varphi \in \mathscr{C}(X, Y) \;\&\; (X, x) \sqsubseteq (A, a)\}.$$

We claim that this family is a stable section of F, that it is finite, that all finite elements have this form, and that it dominates only a finite number of elements.

In order to keep the argument simple, we consider only the case where S is the singleton $\{(A, a)\}$, and we write u instead of u^S. We show that $u \leq t$ for the stable ordering, that is $u_X \leq t_X$ for all X and if $g \in \mathscr{C}(Y, Z)$, then $u_Y = t_Y \wedge F(g)^R(u_Z)$. This is a direct consequence of F preserving pull-backs, and that the category $\mathscr{C}$ has pull-backs. Indeed, we show that for p prime (that is, if p is less than a sup, it is less than one of the element of the sup, see [22]), if $p \leq t_Y$ and $F(g)^L(p) \leq u_Z$ then $p \leq t_Y$, and that is enough since, in a dI-domain, all element is the sup of the primes below it (see [22, 2]). From this, it is possible to derive that u is indeed a stable section.

If $u \leq v$ and $v = \bigvee v_i$ (for the *stable* ordering), then there is a i_0 such that $a \leq v_{i_0}(A)$, and it is then formal to check that $u \leq v_{i_0}$ for the stable ordering.

Finally, we must prove that u dominates only a finite number of elements. But if $v \leq u$ for the stable ordering, then we have that v_Y is equal to

$$\bigvee \{F(\varphi)^L(x) \mid x \leq t_X \;\&\; \varphi \in \mathscr{C}(X, Y) \;\&\; (X, x) \sqsubseteq (A, a \wedge v_A)\},$$

by looking at the prime elements below v_Y. Hence the result, since there are only a finite number of elements below a. □

Notice that $\Pi(F)$ has one "top" section: the section always equal to the top element, but this section is *not* the greatest element of $\Pi(F)$ for the stable ordering. The fact that $\Pi(F)$ is not a complete lattice in general is not a problem, since we will only need to consider the lattice of element less than this special section, and this poset is indeed, a complete dI-domain.

The next remark explains why Girard's definition in [6] of the product of a family of type seems different from the definition given here.

Remark 2.3. *Let us say that a category $\mathscr{C}$ satisfies Moggi's condition if and only if for any $f \in \mathscr{C}(X, Y)$, then we can find $u, v \in \mathscr{C}(Y, Z)$ such that (f, f, u, v) is a pull-back diagram. Then as shown in* [2], *a family $(t_X) \in \Pi_X(F(X))$ is a stable section of F if and only if it is a uniform section of F, that is, $f \in \mathscr{C}(X, Y)$ implies $t_X = F(f)^R(t_Y)$, and, in that case, the stable and the pointwise ordering are equivalent.*

Proof. Let (t_X) be a stable section of F over a category that satisfies Moggi's condition. By the explicit computation of the pull-back in $\Sigma(F)$ and by stability of (t_X), we obtain, for $f \in \mathscr{C}(X, Y)$, $t_X = F(f)^R(t_Y) \wedge F(f)^R(t_Y) = F(f)^R(t_Y)$. Thus, (t_X) is uniform. □

3. Categories of embeddings

First, we present a definition in the extensional framework.

Let us call a *small* category, a *category of information* if and only if all morphisms are monomorphisms and each slice is a sup-semi-lattice (it is even enough to say only that each slice is a sup-semi-lattice, since this implies that all morphisms are monomorphisms). Note that the ordinary notion of partial sup-semi-lattice (poset of finite elements of a Scott domain) falls into this framework, so that we can see this notion as a categorical version of the notion of information system (see [19]).

Definition 3.1. A *category of embeddings* is a category equivalent to the ind-completion of a category of information.

Note that in the case where the category is a poset, then we obtain a generalisation of partial sup-semi-lattices, and the ind-completions of these posets are algebraic domains that are more general than Scott domains, where (see below) all we ask is that any element dominates a complete algebraic lattice.

Let us give at once an equivalent definition for the notion of category of embeddings.

Proposition 3.2. *Let $\mathscr{C}$ be a category. Then $\mathscr{C}$ is a category of embeddings if and only if $\mathscr{C}$ is an algebraic category such that*

- *all morphisms are monomorphisms,*
- *$\mathscr{C}$ has all filtered (or directed, since all morphisms are monomorphisms) colimits,*
- *for any object A,* Sub(A) *(which is the slice category over A) is a complete algebraic lattice.*

Examples of categories of embeddings, which show that this notion is quite general, are:

- the category of sets with injections (here Sub is the subset functor),
- the category of graphs, with embeddings,
- the category **Dom** itself of complete algebraic lattices with embeddings as morphisms,
- the category of linear order, with strictly increasing functions as morphisms (example used in the theory of dilators),
- any Scott domain, seen as a category.

One intuition is that a category of embeddings is a category of structure of a certain kind, with embeddings as morphisms.

Let us give still another equivalent definition of categories of embeddings, which is convenient in order to check that something is indeed a category of this kind.

Proposition 3.3. *Let $\mathscr{C}$ be a category. Then $\mathscr{C}$ is a category of embeddings if and only if*

- *all morphisms are monomorphisms,*
- *$\mathscr{C}$ has all filtered (or directed, since all morphisms are monomorphisms) colimits,*
- *for any object A,* Sub(A) *(which is the slice category over A) is a complete algebraic lattice,*
- *by the previous conditions, $\mathscr{C}$ has pull-backs, and we ask that the pull-back functors f^** : Sub(B) → Sub(A), *for any $f \in \mathscr{C}(A, B)$, preserves directed colimits,*
- *the last condition entails that to be finite has an intrinsic meaning, and we ask that the set of finite objects is essentially small.*

The fact that these two last definitions are equivalent comes from the fact that, in the case where the pull-back morphisms preserve directed colimits, the two possible notions of finiteness (global and local) coincide (by Proposition 1.5). Now, if we take a category that satisfies the second definition, it is (equivalent to) the ind-completion of its subcategory of its finite objects, which is small by hypothesis, and the verification of the second condition is nothing other than the fact that the set of finite elements of a complete algebraic lattice is stable under finite sups.

We thus see, that a poset is a category of embeddings if and only if it is closed under directed sups (but not necessarily with a least element) and any element in that poset dominates a complete algebraic lattice (thus, this is more general than Scott domains: such a poset is not necessarily consistently complete).

Let **Dom** be the category of complete algebraic lattices and *rigid* embeddings. We have for any category of embeddings $\mathscr{C}$, a function Sub: $\mathscr{C}_0 \to \mathbf{Dom}_0$, so that Sub($A$) is a complete algebraic lattice that is in U_0 and isomorphic to the poset of subobjects of A. We will identify this isomorphic lattice with the real poset of subobjects of A.

As an application of Proposition 1.1, we have the following result.

Proposition 3.4. *The application* Sub: $\mathscr{C}_0 \to \mathbf{Dom}_0$ *is the object part of a continuous and* stable *functor* Sub: $\mathscr{C} \to \mathbf{Dom}$, *where* Sub$(f)^L$ *is the direct image mapping $f_!$ and* Sub$(f)^R$ *the inverse image mapping f^*.*

Let us look at what happens with the first definition of "category of embeddings". If A is an object of the category $\mathscr{C}$ we start with, and we complete by ind-completion, then A "becomes" a finite object of ind($\mathscr{C}$), and $\text{Sub}_{\text{ind}(\mathscr{C})}(A)$ is really the completion of Sub(A). Thus we have $\text{Sub}_{\text{ind}(\mathscr{C})}(A) = \text{ind}(\text{Sub}_{\mathscr{C}}(A))$. In particular, if $\text{Sub}_{\mathscr{C}}(A)$ was finite, then it will not be "perturbed" and made infinite by the ind-completion, and that is what happens in the stable case.

In the general case, A is a filtered colimit of objects A_i in $\mathscr{C}$, and Sub(A) is the filtered colimit (in the category of cpos with embeddings) of the posets ind($\mathrm{Sub}_{\mathscr{C}}(A_i)$).

Theorem 3.5. *The category of continuous functors between two categories of embeddings, with morphisms the natural transformation, is a category of embeddings.*

Proof. Let $\mathscr{C}$ and $\mathscr{D}$ be two categories of embeddings. We want to show that $\mathscr{C} \to \mathscr{D}$, category of continuous functors, is still a category of embeddings. Since $\mathscr{C}$ is an ind-completion, we first reduce the problem to the one of showing that if $\mathscr{C}$ is a category of information and $\mathscr{D}$ a category of embeddings, then the functor category is a category of embedding. We prove actually that if $\mathscr{C}$ is any small category, and $\mathscr{D}$ a category of embeddings, then the category of functors from $\mathscr{C}$ to $\mathscr{D}$ is a category of embeddings.

Let us check that this functor category satisfies the conditions of the third definition. By [15, p. 112], we see that $\mathscr{C} \to \mathscr{D}$ has all filtered colimits, has pull-backs, which commute with filtered colimits, and that all morphisms in this category are monomorphisms. We thus reduce the problem to show that each Sub(F) is a complete algebraic lattice, and that the set of finite functors is essentially small. The first point is a consequence of the following computation of Sub(F).

Lemma 3.6. *Let $\mathscr{D}$ be a category of embeddings, $\mathscr{C}$ be a small category and F an object of $\mathscr{C} \to \mathscr{D}$. The subobjects of F are precisely the sections of the functor* Sub $\circ$ F.

We have thus $\mathrm{Sub}_{\mathscr{C}\to\mathscr{D}}(F) = \Pi(\mathrm{Sub} \circ F)$. And we know that this poset is a complete algebraic lattice by the previous section.

For showing the second point, we need to know what is the shape of the finite functors. For this, we need only to know what is the shape of finite sub-functors of a given functor F (since we are in the good case where pull-backs commute with directed colimits). If we translate the proof of the fact that $\Pi(\mathrm{Sub} \circ F)$ is a complete algebraic lattice we find the following description. Let S be a finite set of pairs (A, f), where A is an object of $\mathscr{C}$, **dom**(f) a finite object of $\mathscr{D}$, and **codom**$(f) = F(A)$. We define a functor $\Phi_{F,S}$ together with a natural transformation $\tau : \Phi_{F,S} \to F$, by

$$\tau(X) = \bigvee \{F(\varphi)^L(f) \mid \varphi \in \mathscr{C}(A, X) \;\&\; (A, f) \in S\},$$

so that τ is a subobject of F. Then, $\Phi_{F,S}$ is a finite functor, and F is the directed sups of all the functors $\Phi_{F,S}$. This is actually a mere reformulation of the result about the domain of sections (Theorem 2.1). If F is a finite functor, then there exists S such that $\Phi_{F,S}$ is isomorphic to F. Hence, we have a description of finite functors of the functor category.

All we have now to prove is that the set of all the functors $\Phi_{F,S}$ is essentially small. The difficulty is that this family is indexed over the set of all functors in $\mathscr{C} \to \mathscr{D}$, which is not small in general. Since $\mathscr{C}$ is small, and locally small, there is a infinite small cardinal κ which is a bound for the cardinality of all sets $\mathscr{C}(A, X)$.

The key remark is then that $\Phi_{F,S}(X)$ is a directed colimit of finite objects over a system of cardinality $\leqslant\kappa$.

Indeed, by definition, $\tau(X)$ is a sup of finite subobjects of $F(X)$ over a family of cardinality $\leqslant\kappa$. By taking the directed family of finite sups of this family, we see that $\tau(X)$ is a directed sup of finite subobjects of $F(X)$ over a family of cardinality less than κ. Since $\mathscr{D}$ has filtered colimits, and pull-backs that commute with filtered colimits, this entails that $\Phi_{F,S}(X)$ is a filtered colimit of finite objects over a family of cardinality less than κ. Thus the functor $\Phi_{F,S}$ sends an object in $\mathscr{C}$ into the essentially small set $\mathscr{D}_\kappa$, of filtered colimits of finite objects of $\mathscr{D}$ over a family of cardinality at most κ. This shows that there are only (up to isomorphisms) a small number of functors like $\Phi_{F,S}$, and so, a small number of finite functors. □

Corollary 3.7. *The category of categories of embeddings, with continuous functors as morphisms, is a cartesian closed category.*

In the case where all sets $\mathscr{C}(A, B)$ are *finite*, we can be more precise: $\Phi_{F,S}(X)$ is a directed colimit of finite objects over a finite system, hence every $\Phi_{F,S}(X)$ is finite. We have then that a finite functor sends a finite object to a finite object.

But there is more, if $\mathscr{C}$ and $\mathscr{D}$ have the property that each hom-set between finite objects is a finite set, then the exponential category $\mathscr{C}\to\mathscr{D}$ still has this property. Indeed, a morphism from $\Phi_{F,S}$ to G determines a set of morphisms from $\mathbf{dom}(f)$ into $G(A)$ for each $(A,f)\in S$, and is completely determined by this data. Since we know that G sends a finite object to a finite object, we see that there are only finitely many morphisms from F to G.

If we restrict ourselves to posets, we get a cartesian closed category of domains that properly contains the category of Scott domains (the domains are not consistently complete any more in general). Note that this category is not included in the category of SFP, though this category is known as the largest cartesian closed category of domains [21]! This is because we do not insist that the basis of our domains is countable. If one wants a cartesian closed category where all domains are countably based, then one has to consider the intersection of the present category of posets with the collection of profinite domains. This cartesian closed category of domains was already known and is presented in [7] (the corresponding stable category, however, seems new).

Proposition 3.8. *The category of posets, closed by directed sups, and such that any element dominates a complete algebraic lattice, with continuous functions as morphisms, is a cartesian closed category.*

In this case, the proof of "cartesian closedness" is considerably simplified, since there are no problems of size any more, and the theorem that we obtain a cartesian closed category of domains may be seen as a nice application of the proposition that a dependent product of a family of domains over a poset is a domain (exercise of [16]). Indeed, let D_1 and D_2 two such domains. Then $D_1 \to D_2$ is closed by nonempty directed sups, since they are computed pointwise. If $f \in D_1 \to D_2$, then $\mathrm{Sub}(f) = \Pi(\mathrm{Sub} \circ f)$ where $\mathrm{Sub}: D_2 \to \mathbf{Dom}$ is the continuous functor $\mathrm{Sub}(x) = \{y \in D_2 \mid y \leqslant x\}$. Thus, $\mathrm{Sub}(f)$ is a dependent product of a family cf complete algebraic lattice, and so is a complete algebraic lattice.

3.1. Reformulation with stable functors

If we want to work with *stable* functors as in [6], then the previous results hold almost without change. We take now for **Dom** the category of complete dI-domains, with rigid embeddings as morphisms. One can take the category of complete coherent spaces of Girard [6], with rigid embeddings as morphisms, that is complete atomic boolean algebras as an alternative. The fact that there is no cartesian closed categorical structure on this collection of objects does not matter here.

We call a *category of events* a category that is small, where all morphisms are monomorphisms, and each slice category is a finite distributive lattice. The poset of finite elements of a dI-domain is a particular instance of this concept.

Definition 3.9. A *stable category of embeddings* is the ind-completion of a category of events.

Equivalently, one can state the following.

Proposition 3.10. *Let $\mathscr{C}$ be a category. Then that $\mathscr{C}$ is a stable category of embeddings if and only if $\mathscr{C}$ is an algebraic category such that*

- *all morphisms are monomorphisms,*
- *$\mathscr{C}$ has all filtered (or directed, since all morphisms are monomorphisms) colimits,*
- *for any object A,* Sub(A) *(which is the slice category over A) is a complete* dI-*domain.*

Or else, the following proposition holds.

Proposition 3.11. *Let $\mathscr{C}$ be a category. Then $\mathscr{C}$ is a stable category of embeddings if and only if $\mathscr{C}$ is such that*

- *all morphisms are monomorphisms,*
- *$\mathscr{C}$ has all filtered (or directed, since all morphisms are monomorphisms) colimits,*
- *for any object A,* Sub(A) *(which is the slice category over A) is a complete* dI-*domain,*
- *by the previous conditions, $\mathscr{C}$ has pull-backs, and we ask that the pull-back functors $f^*: \mathrm{Sub}(B) \to \mathrm{Sub}(A)$, for any $f \in \mathscr{C}(A, B)$, preserves directed colimits,*
- *the last condition entails that to be finite has an intrinsic meaning, and we ask that the set of finite objects is essentially small.*

All these definitions are equivalent. We now have the very strong properties of finite elements as follows.

Proposition 3.12. *Let $\mathscr{C}$ be a stable category of embeddings. Then A is a finite object of $\mathscr{C}$ if and only if* Sub(A) *is finite, as a set.*

We have the stability and continuity of Sub, as in the extensional case.

Proposition 3.13. *The application* $\mathrm{Sub}: \mathscr{C}_0 \to \mathbf{Dom}_0$ *is the object part of a continuous and stable functor* $\mathrm{Sub}: \mathscr{C} \to \mathbf{Dom}$, *where* $\mathrm{Sub}(f)^L$ *is the composition with f and* $\mathrm{Sub}(f)^R$ *the pull-back with f.*

We will need the following characterisation of Sub(F).

Lemma 3.14. *Let $\mathscr{D}$ be a stable category of embeddings, $\mathscr{C}$ a small category with pull-backs, and F a stable functor of $\mathscr{C} \to \mathscr{D}$. The subobjects of F are precisely the stable sections of the functor* $\mathrm{Sub} \circ F$ *that are less than (for the stable ordering) the "top" section of* $\mathrm{Sub} \circ F$.

And we can now state the following.

Theorem 3.15. *The category of stable and continuous functors between two stable categories of embeddings, with as morphisms the cartesian natural transformation, is a stable category of embeddings.*

Corollary 3.16. *The category of stable categories of embeddings, with stable and continuous functor as morphisms, is a cartesian closed category.*

Since a category of embeddings is an ind-completion, this reduces to showing that $\mathscr{C} \to \mathscr{D}$, the category of stable functors and cartesian natural transformations, is a category of embeddings if $\mathscr{C}$ is small and has pull-backs, and $\mathscr{D}$ is a category of embeddings. As in the extensional case, we are reduced to proving that for any functor F in the category $\mathscr{C} \to \mathscr{D}$ the poset Sub(F) is a complete dI-domain, and that there are only a small number of finite objects. It is a complete dI-domain by the last lemma and the previous section.

The fact that there are only a small number of finite elements goes through in the same way as in the extensional case.

In the case where each hom-sets is finite in the category $\mathscr{C}$, then a finite functor sends every object to a finite element of $\mathscr{C}$. As in the extensional case, if $\mathscr{C}$ and $\mathscr{D}$ have the property that each hom-set between finite objects is finite, then $\mathscr{C} \to \mathscr{D}$ has this property.

If we restrict ourselves to posets, we obtain a new notion of domains that form a cartesian closed category (yet another one!), which contains the category of dI-domains [1, 22], and that satisfy the strong condition that a finite element dominates only a finite number of elements.

If we want a category where all domains are countably based, we must impose two further conditions on our domains: that they are countably based, and that there are profinite (or alternatively, that any finite set has a finite collection of minimal upper bounds, that are pairwise incompatible). This is the stable analogue of the notion of short domains presented in [7]. This category is a new extension of the category of dI-domains.

Proposition 3.17. *The category of posets that are closed by nonempty directed sups, and such that any element dominates a complete* dI-*domain* (*resp. a complete atomic boolean algebra*), *with continuous stable functions as morphisms, is a cartesian closed category. Any finite element dominates only a finite number of elements. The full subcategory of profinite such domains, alternatively domains such that any finite subset as a finite collection of minimal upper bounds, is a cartesian closed category.*

4. Application to the semantics of F_ω

In [3], a general framework has been given for building models of polymorphism, that shows that Girard's ideas may apply as well on other notions of domains than qualitative domains used in [6]. We show here how to extend these models to models of F_ω.

The main remark is simply that since we have a cartesian closed category of categories of embeddings, we can interpret the ordinary typed λ-calculus, where types become categories of embeddings. But here, this cartesian closed category is used to interpret the *orders* (see [20]) of F_ω. Intuitively speaking, F_ω is a two-level system, and the first level (the level of orders) is like simply typed λ-calculus, and is interpreted by the cartesian closed category of categories of embeddings. Now, we have to find a suitable interpretation of the order of truth values Ω (with the notation of [20]). Following the "Curry-Howard" analogy between propositions and types, we take for it any category of domains, with embedding-projection pair as morphisms. This shows that we obtain a wide range of models of F_ω since we can use almost any category of domains for this, for instance complete algebraic lattices, or Scott domains (we still do not know, however, if it is possible to use the category of SFP domains). The fact that the posets of sections form a domain shows that we do have an interpretation of the universal quantification.

To check the details, a possibility is to use the work of [4], which shows in a concrete example how to build a model of F_ω (by using ideas from [20]). Actually, all we need to say is that we can use this construction, by replacing everywhere lfp categories by categories of embeddings (resp. stable categories of embeddings, and continuous functors by stable and continuous functors), and the category of complete algebraic lattices with left adjoints becomes the category of Scott domains with embeddings (resp. category of dI-domains with rigid embeddings, resp. category of atomic dI-domains with rigid embeddings, resp. category of coherent spaces with rigid embeddings).

Let us so outline only the extensional case. We interpret Ω, the objects of truth-values of higher-order logic, as being the category of Scott domains with embedding-projection pairs as morphisms. We know that this is a category of embeddings, and we know [3], that for any category of embeddings $\mathscr{C}$, one can define a continuous functor $\Pi:(\mathscr{C}\to\Omega)\to\Omega$. This will be the interpretation of the universal quantification. For the interpretation of the implication, we take $\Rightarrow$: $\mathbf{Dom}\to\mathbf{Dom}\to\mathbf{Dom}$, where $D_1\Rightarrow D_2$ is the poset of continuous functions from D_1 to D_2 (see [17]). The rest is only a matter of checking some equations, as in [4]. The formal structure is the same in all these models.

Notice that we cannot include the notion of lfp categories as a special case of the concept of categories of embeddings, because of our restriction on morphisms to be monomorphisms. It would be interesting to generalise the definition of category of embeddings so that it includes this case (that is associated with the problem of removing the condition that all morphisms are monomorphisms).

These models are actually models of a richer type system than F_ω since Ω is actually in this model a full subcategory of the category of types.

5. Conclusion

We have proposed two possible axiomatisations of the notion of category of structure, so that any structure is the directed colimit of its finite substructures, and we have shown that these notions are preserved by (suitable) exponentiation. Our work may so be seen as a step towards the categorical generalisation of the notion of domain. Two questions (at least) seem interesting in that respect: what is the categorical analogue to the notion of profinite domains (directed colimit of finite posets), and what is the connection of the notion of category of structure described here with the categorical version of the notion of models of a theory, such as points of a topos. This is known for lfp categories, since, if $\mathscr{C}$ is a category with finite colimits, the category ind($\mathscr{C}$) is (equivalent to) the category of points of the topos of pre-sheaves over $\mathscr{C}^{\mathrm{op}}$. It would be interesting to generalise this property. One very promising research direction is to connect the present notions with the work of Lamarche (see [12, 13]), who was inspired by the work of Diers (see for instance [5]), in order to generalise Girard's model. As pointed out by Lamarche, it is not the case that all category of embeddings is a Diers category: the category of algebraically closed fields is a category of embedding, but has no multi-initial family. It is actually possible to generalise the notion of Diers category in order to subsume the case of category of embeddings [14].

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