The well-posedness of an $M/G/1$ queue with second optional service and server breakdown

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\textbf{ABSTRACT}

In this paper, the solution of an $M/G/1$ queue with second optional service and server breakdown is investigated. By using the method of functional analysis, especially, the linear operator theory and the $C_0$ semigroup theory on Banach space, we prove the well-posedness of the system, and show the existence of a positive solution.

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1. Introduction

Queueing systems with repeated attempts are characteristic by the phenomenon that a customer finding all the servers busy upon arrival is obliged to leave the service area and repeat his request for service after some random time. Between trials, the blocked customer joins a pool of unsatisfied customers called “orbit”. Retrial queues have been widely used to model many problem in telephone switching system, telecommunication networks and computers competing to gain service from a central processor unit.

A more practical retrial queue with feedback occurs in many practical situations: for instance, a multiple access telecommunication system, where messages turned out as errors are sent again, can be modelled as retrial queue with feedback [1–3].

A remarkable and unavoidable phenomenon in the service facility of a queueing system is its breakdown. Until the failed service facility is recovered, the waiting times for customers in the system increases, thereby resulting in impatience of customers [4–6].

One important fact that has been overlooked is that perfectly reliable servers are virtually nonexistent. In fact, the servers may well be subject to lengthy and unpredictable breakdowns while serving a customer. For example, in manufacturing systems and computer systems, the machine may break down due to the machine or job related problems. This results in a period of unavailable time until the servers are repaired. Such a system with repairable server has been studied as a queueing model and a reliability model by many authors, see [7–9] and references therein. Recently, an $M/G/1$ queue with second optional service and unreliable server is studied in Ref. [10]. Using a supplementary variable method, the author in [10] obtains transient and steady-state solutions for both queueing and reliability measures of interest.

Madan in [11] studied the same system with the further assumption that the server may be subject to breakdowns and repairs in the two service processes. The study was motivated by the fast-expanding area of tele-services, which prominently include telephone call centres and the emerging internet-based market (see [12]). Many call centres use interactive voice

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response (IVR) units in addition to providing the services of agents or CSRs (customer service representatives). These specialised computers allow customers to communicate their needs and to “self-serve” before they may speak to a CSR. This makes it much more convenient to provide optional services that generate new revenue. Meanwhile, it should be observed that in toll-free services, such as 1–800, holding times of customers (including ones that eventually abandon) are paid by service providers. With the explosive growth of toll-free services, these costs have become a major economic driver. Server breakdowns may also have a significant effect on a system’s performance. These results may be helpful to understand the phenomenon of server breakdowns and its role played in the whole system’s performance.

We consider the M/G/1 queueing system with the following assumptions.

1. Customers arrive at the system according to a Poisson process with rate $\lambda$.

2. The first essential service is needed for all arriving customers. Let $B(v)$ and $b(v)$, respectively, be the distribution function and the density function of the first service times with mean $\frac{1}{\mu_1}$ and let $\frac{1}{\mu_1(x)}$ be the hazard rate function.

3. As soon as the first service of a customer is completed, then with probability $1 - r$, he may opt for a second service, in which case his second service will immediately commence or else with probability $r$, he may opt to leave the system in which case another customer at the head of the queue (if any) is taken up for his first essential service.

4. The second service times are assumed to be exponential with service time $\frac{1}{\mu_2}$.

5. We assume that a server’s lifetime has exponential distribution with mean $\frac{1}{\alpha_1}$ in the first essential service. In the second optional service, the server fails at an exponential rate $\alpha_2$.

6. The server may break down when servicing customers, and when the server breaks down, it is sent for repair. The customer just being served before server breakdown waits for the server to complete its remaining service. The repair time distributions of both service phases are arbitrarily distributed with probability distribution function $G_1(x)$ and $G_2(x)$, respectively. Also, let $g_k(x)$, $\frac{1}{\beta_k}$, and $\beta_k(x)$, $k = 1, 2$, be the corresponding probability density functions, means, and hazard rates. Immediately after the server is fixed, it starts to serve customers, and the service time is cumulative.

7. Various stochastic processes involved in the system are assumed to be independent of each other.

Let $N(t)$ be the number of customers in the system at time $t$. To make it a Markov process, we introduce supplementary variables. Define $X(t)$ as the elapsed service time of the customer currently being served at time $t$, and $Y(t)$ the elapsed repair time of the failed server at time $t$. And define the state probabilities at time $t$ as follows.

1. $Q(t)$ is the probability that the server is idle at time $t$.

2. $P_n^1(t, x)\,dx$ is the joint probability that at time $t$ there are $n$ customers in the queue excluding the one being provided the first essential service, the server is up, and a customer is being served with the elapsed service time between $x$ and $x + dx(n > 0)$.

3. $P_n^2(t)$ is the joint probability that at time $t$ there are $n$ customers in the queue excluding the one being provided the second optional service ($n \geq 0$).

4. $R_n^1(t, x, y)\,dy$ is the joint probability that at time $t$ there are $n$ customers in the queue excluding the one being provided the first essential service, the elapsed service time for the customer under service is equal to $x$, and the server is being repaired with the elapsed repair time between $y$ and $y + dy(n > 0)$.

5. $R_n^2(t, y)\,dy$ is the joint probability that at time $t$ there are $n$ customers in the queue excluding the one being provided the second optional service, and the server is being repaired with the elapsed repair time between $y$ and $y + dy(n \geq 0)$.

From [10], the system of differential equations associated with the model is the following for $n = 0, 1, 2, \ldots$:

\[
\frac{dQ(t)}{dt} = -\lambda Q(t) + \mu_2 P_0^2(t) + (1 - r) \int_0^{+\infty} \mu_1(x) P_0^1(x, t)\,dx, \tag{1.1}
\]

\[
\frac{\partial P_n^1(x, t)}{\partial t} + \frac{\partial P_n^1(x, t)}{\partial x} = -[\lambda + \alpha_1 + \mu_1(x)] P_n^1(x, t) + \lambda P_{n-1}^1(x, t) + \int_0^{+\infty} \beta_1(y) R_n^1(x, y, t)\,dy, \tag{1.2}
\]

\[
\frac{dP_n^2(t)}{dt} = -[\lambda + \mu_2 + \alpha_2] P_n^2(t) + \lambda P_{n-1}^2(t) + r \int_0^{+\infty} \mu_1(x) P_n^1(x, t)\,dx + \int_0^{+\infty} \beta_2(y) R_n^2(y, t)\,dy, \tag{1.3}
\]

\[
\frac{\partial R_n^1(x, y, t)}{\partial t} + \frac{\partial R_n^1(x, y, t)}{\partial x} = -[\lambda + \beta_1(y)] R_n^1(x, y, t) + \lambda R_{n-1}^1(x, y, t), \tag{1.4}
\]

\[
\frac{\partial R_n^2(x, y, t)}{\partial t} + \frac{\partial R_n^2(x, y, t)}{\partial y} = -[\lambda + \beta_2(y)] R_n^2(y, t) + \lambda R_{n-1}^2(y, t), \tag{1.5}
\]

where $P_{-1}^1(x, t) = P_{-1}^2(t) = R_{-1}^1(x, y, t) = R_{-1}^2(y, t) = 0$, and with the boundary conditions:

\[
P_n^1(0, t) = (1 - r) \int_0^{+\infty} \mu_1(x) P_{n+1}^1(x, t)\,dx + \mu_2 P_{n+1}^2(t), \tag{1.6}
\]

\[
P_0^1(0, t) = (1 - r) \int_0^{+\infty} \mu_1(x) P_1^1(x, t)\,dx + \mu_2 P_1^2(t) + \lambda Q(t), \tag{1.7}
\]
\[ r_n^{(3)}(x, 0, t) = \alpha_1 p_n^{(1)}(x, t), \quad R_n^{(2)}(0, t) = \alpha_2 p_n^{(2)}(t). \]  

(1.8)

Eqs. (1.1)–(1.8) should be solved together with the normalizing condition

\[ Q(t) + \sum_{n=0}^{\infty} \left[ \int_{0}^{+\infty} p_n^{(1)}(x, t)dx + \int_{0}^{+\infty} R_n^{(2)}(y, t)dy \right] = 1 \]

(1.9)

and an initial conditions \( Q(0) = 1. \)

Note that the equations are very complicated, it is difficult to obtain an analysis solution for them. To discuss the properties of the system, [10] gives some results based on the following hypotheses:

1. There exists a unique non-negative solution \( Q(t), P^{(1)}(x, t), P^{(2)}(t), R^{(1)}(x, y, t), R^{(2)}(y, t) \) of this model;
2. The limits

\[ \lim_{t \to +\infty} Q(t) = Q, \quad \lim_{t \to +\infty} P^{(1)}(x, t) = P^{(1)}(x), \quad \lim_{t \to +\infty} P^{(2)}(t) = P^{(2)}, \]

\[ \lim_{t \to +\infty} R^{(1)}(x, y, t) = R^{(1)}(x, y), \quad \lim_{t \to +\infty} R^{(2)}(y, t) = R^{(2)}(y), \]

exist, where

\[ P^{(1)}(x, t) = \{P^{(1)}_{k}(x, t)\}_{k=0}^{\infty}, \quad P^{(2)}(t) = \{P^{(2)}_{k}(t)\}_{k=0}^{\infty}, \]

\[ R^{(1)}(x, y, t) = \{R^{(1)}_{k}(x, y, t)\}_{k=0}^{\infty}, \quad R^{(2)}(y, t) = \{R^{(2)}_{k}(y, t)\}_{k=0}^{\infty}, \]

and

\[ p^{(1)}(x) = \{P^{(1)}_{k}(x)\}_{k=0}^{\infty}, \quad p^{(2)}(t) = \{P^{(2)}_{k}(t)\}_{k=0}^{\infty}, \]

\[ R^{(1)}(x, y) = \{R^{(1)}_{k}(x, y)\}_{k=0}^{\infty}, \quad R^{(2)}(y) = \{R^{(2)}_{k}(y)\}_{k=0}^{\infty}. \]

It is well known that the above hypotheses not always hold and it is necessary to prove the correctness. Therefore, the existence of dynamic solution of Eqs. (1.1)–(1.8) is still unsolved problem. Motivated by this, we shall investigate the well-posedness of the system in the present paper. Furthermore, we show the existence of positive solution.

In this paper we assume that:

1. \( \lambda, \mu, r, \alpha_1, \alpha_2 \) are positive constants and \( 0 < r < 1; \)
2. \( \beta_i(y), \mu_1(x) \) are measurable functions and

\[ 0 < c_i = \inf_{y \in \mathbb{R}^+} \beta_i(y) < \sup_{y \in \mathbb{R}^+} \beta_i(y) = +\infty, \quad i = 1, 2, \]

\[ 0 < c_3 = \inf_{x \in \mathbb{R}^+} \mu_1(x) < \sup_{x \in \mathbb{R}^+} \mu_1(x) = +\infty. \]

2. The well-posedness of the system

Let \( \mathbb{R} \) be a real number set, \( \mathbb{R}^+ \) be non-negative real number set, \( \mathbb{N} \) be non-negative integer number set, respectively. Denote by \( \Omega = \mathbb{R}^+ \times \mathbb{N}, \Omega_0 = \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{N}, \)

\[ L^1(\Omega) = \left\{ f(x, n) : \int_{\Omega} |f(x, n)| dx \cdot dn < +\infty \right\}, \quad \|f\|_1 = \int_{\Omega} |f(x, n)| dx \cdot dn, \]

\[ L^1(\Omega_0) = \left\{ f(x, y, n) : \int_{\Omega_0} |f(x, y, n)| dx \cdot dy \cdot dn < +\infty \right\}, \quad \|f\|_0 = \int_{\Omega_0} |f(x, y, n)| dx \cdot dy \cdot dn. \]

Let \( X = \mathbb{R} \times L^1(\Omega) \times L^1(\Omega_0) \times L^1(\Omega). \) Well known, \( X \) is a Banach space with the norm for \( (P_0, P_1(x, n), P_2(n), R_1(x, y, n), R_2(y, n)) \in X, \)

\[ \|(P_0, P_1, P_2, R_1, R_2)\| = |P_0| + \int_{\Omega} |P_1(x, n)| dx + \sum_{n=0}^{\infty} |P_2(n)| + \int_{\Omega_0} |R_1(x, y, n)| dx \cdot dy \cdot dn + \int_{\Omega} |R_2(y, n)| dy \cdot dn. \]

Define the following operators: \( L f(x, n) = f(x, n - 1), \)

\[ A_0 P_0 = -\lambda P_0 + \mu_2 P_2(0), \]

\[ A_1 P_1(x, n) = -P_1(x, n) + [\lambda + \alpha_1 + \mu_1(x)] P_1(x, n), \]

\[ A_2 P_2(n) = -[\lambda + \alpha_2] P_2(n), \]

\[ A_3 R_1(x, y, n) = -\frac{\partial R_1(x, y, n)}{\partial y} - [\lambda + \beta_1(y)] R_1(x, y, n), \]

\[ A_4 R_2(y, n) = -R_2(y, n) - [\lambda + \beta_2(y)] R_2(y, n), \]
Theorem 2.1. The operator $\mathcal{A}$ is a densely defined and closed linear operator in $X$.

The proof of Theorem 2.1 is a direct verification, we omit the detail.

Let $X^*$ be the dual of $X$ and $\mathcal{A}^*_1$ be the dual of $\mathcal{A}_1$, then we have

$$X^* = \mathbb{R} \times L^\infty(\Omega) \times \ell^\infty \times L^\infty(\Omega_0) \times L^\infty(\Omega).$$
For any \( P = (P_0, P_1(x, n), P_2(n), R_1(x, y, n), R_2(y, n)) \in D(A_1) \), and
\[
Q = (q_0, q_1(x, n), q_2(n), r_1(x, y, n), r_2(y, n)) \in D(A_1^*),
\]
we have
\[
(A_1^* P, Q) = \left[ -\lambda_0 P_0 + \mu_2 P_2(0) + (1 - r) \int_0^{+\infty} \mu_1(x) P_1(x, 0) dx \right] q_0
\]
\[
+ \sum_{n=0}^{\infty} \int_0^{+\infty} \left\{ -P_1'(x, n) \left[ \lambda + \alpha_1 + \mu_1(x) \right] P_1(x, n) + \int_0^{+\infty} \beta_1(y) R_1(x, y, n) dy \right\} q_1(x, n) dx
\]
\[
+ \sum_{n=0}^{\infty} \int_0^{+\infty} \int_0^{+\infty} \left\{ -\partial R_1(x, y, n) \frac{\partial}{\partial y} - \left[ \lambda + \beta_1(y) \right] R_1(x, y, n) \right\} r_1(x, y, n) dy dx
\]
\[
+ \sum_{n=0}^{\infty} \int_0^{+\infty} \left\{ -R_2'(y, n) \left[ \lambda + \beta_2(y) \right] R_2(y, n) \right\} r_2(y, n) dy.
\]
where we have used the following equalities
\[
\sum_{n=0}^{\infty} \int_0^{+\infty} [-P_1'(x, n) q_1(x, n)] dx = \sum_{n=0}^{\infty} P_1(0, n) q_1(0, n) + \sum_{n=0}^{\infty} \int_0^{+\infty} q_1'(x, n) P_1(x, n) dx
\]
\[
= \left[ (1 - r) \int_0^{+\infty} \mu_1(x) P_1(x, 1) dx + \mu_2 P_2(1) + \lambda P_0 \right] q_1(0, 0) + \sum_{n=0}^{\infty} \int_0^{+\infty} q_1'(x, n) P_1(x, n) dx
\]
\[
+ \sum_{n=1}^{\infty} \left[ (1 - r) \int_0^{+\infty} \mu_1(x) P_1(x, n + 1) dx + \mu_2 P_2(n + 1) \right] q_1(0, n)
\]
\[
= \sum_{n=0}^{\infty} \left[ (1 - r) \int_0^{+\infty} \mu_1(x) P_1(x, n + 1) dx \right] q_1(0, n) + \sum_{n=0}^{\infty} \mu_2 P_2(n + 1) q_1(0, n)
\]
\[
+ \lambda P_0 q_1(0, 0) + \sum_{n=0}^{\infty} \int_0^{+\infty} q_1'(x, n) P_1(x, n) dx,
\]
\[
\sum_{n=0}^{\infty} \int_0^{+\infty} \int_0^{+\infty} \frac{\partial R_1(x, y, n)}{\partial y} r_1(x, y, n) dy dx
\]
\[
= \sum_{n=0}^{\infty} \left[ \int_0^{+\infty} R_1(x, 0, n) r_1(x, 0, n) dx + \int_0^{+\infty} \int_0^{+\infty} \frac{\partial r_1(x, y, n)}{\partial y} R_1(x, y, n) dy dx \right]
\]
\[
= \sum_{n=0}^{\infty} \left[ \int_0^{+\infty} \alpha_1 P_1(x, n) r_1(x, 0, n) dx + \int_0^{+\infty} \int_0^{+\infty} \frac{\partial r_1(x, y, n)}{\partial y} R_1(x, y, n) dy dx \right],
\]
\[
\sum_{n=0}^{+\infty} \int_0^{+\infty} -R_2'(y, n) r_2(y, n) dy = \sum_{n=0}^{+\infty} \left[ R_2(0, n) r_2(0, n) + \int_0^{+\infty} r_2'(y, n) R_2(y, n) dy \right]
\]
\[
= \sum_{n=0}^{+\infty} \left[ \alpha_2 P_2(n) r_2(0, n) + \int_0^{+\infty} r_2'(y, n) R_2(y, n) dy \right].
From \((A_1P, Q) = (P, A_1^*Q)\), we obtain

\[
A_1^* = \begin{pmatrix}
q_0 \\
q_1(x, 0) \\
q_1(x, n) \\
q_2(0) \\
q_2(n) \\
r_1(x, y, n) \\
r_2(y, n)
\end{pmatrix} = \begin{pmatrix}
\frac{\lambda[q_1(0, 0) - q_0]}{q_0} \\
q_1'(x, 0) + \mu_1(x)[(1 - r)q_0 + rq_2(0)] \\
-\left[\lambda + \alpha_1 + \mu_1(x)\right]q_1(x, 0) + \alpha_1 r_1(x, 0, 0) \\
q_1'(x, n) + [(1 - r)q_1(0, n - 1) + rq_2(n)]\mu_1(x) \\
-\left[\lambda + \alpha_1 + \mu_1(x)\right]q_1(x, n) + \alpha_1 r_1(x, 0, n) \\
\mu_2 q_0 - \left(\lambda + \mu_2 + \alpha_2\right) q_2(0) + \alpha_2 r_2(0, 0) \\
\mu_2 q_1(0, n - 1) - \left(\lambda + \mu_2 + \alpha_2\right) q_2(n) + \alpha_2 r_2(0, n) \\
\frac{\partial r_1(x, y, n)}{\partial y} + \beta_1(y) q_1(x, n) - \left[\lambda + \beta_1(y)\right] r_1(x, y, n) \\
r_2'(y, n) + q_2(n) \beta_2(y) - \left[\lambda + \beta_2(y)\right] r_2(y, n)
\end{pmatrix},
\] (2.4)

with domain

\[
D(A_1^*) = \left\{(q_0, q_1(x, n), q_2(n), r_1(x, y, n), r_2(y, n)) \in X^* : q_1'(x, n), r_2'(y, n) \in L^\infty(\Omega), \frac{\partial r_1(x, y, n)}{\partial y} \in L^\infty(\Omega_0) ; q_1(x, n), r_2(y, n), r_1(x, y, n) \text{ are absolutely continuous} \right\}
\] (2.5)

**Theorem 2.2.** 1 is not an eigenvalue of \(A_1^*\).

**Proof.** Let \(Q = (q_0, q_1(x, n), q_2(n), r_1(x, y, n), r_2(y, n)) \in X^*\), such that \(A_1^*Q = Q\), i.e.,

\[
\lambda[q_1(0, 0) - q_0] = q_0.
\] (2.6)

\[
q_1'(x, 0) + (1 - r) q_0 \mu_1(x) - \left[\lambda + \alpha_1 + \mu_1(x)\right] q_1(x, 0) + \alpha_1 r_1(x, 0, 0) = q_1(x, 0),
\] (2.7)

\[
q_1'(x, n) + [(1 - r) q_1(0, n - 1) + rq_2(n)] \mu_1(x) + \alpha_1 r_1(x, 0, n) - \left[\lambda + \alpha_1 + \mu_1(x)\right] q_1(x, n) = q_1(x, n), \quad n \geq 1,
\] (2.8)

\[
\mu_2 q_0 - \left(\lambda + \mu_2 + \alpha_2\right) q_2(0) + \alpha_2 r_2(0, 0) = q_2(0),
\] (2.9)

\[
\mu_2 q_1(0, n - 1) - \left(\lambda + \mu_2 + \alpha_2\right) q_2(n) + \alpha_2 r_2(0, n) = q_2(n), \quad n \geq 1,
\] (2.10)

\[
\frac{\partial r_1(x, y, n)}{\partial y} - \left[\lambda + \beta_1(y)\right] r_1(x, y, n) + \beta_1(y) q_1(x, n) = r_1(x, y, n),
\] (2.11)

\[
r_2'(y, n) - \left[\lambda + \beta_2(y)\right] r_2(y, n) + q_2(n) \beta_2(y) = r_2(y, n).
\] (2.12)

From (2.7), (2.8), (2.11) and (2.12), we get

\[
q_1(x, 0) = e^{\int_0^x [(1 - r) q_0 \mu_1(u) + \alpha_1 r_1(u, 0, 0)] e^{-\int_u^x [1 + \lambda + \alpha_1 + \mu_1(s)] ds} du} q_1(0, 0) - \int_0^x [(1 - r) q_0 \mu_1(u) + \alpha_1 r_1(u, 0, 0)] e^{-\int_u^x [1 + \lambda + \alpha_1 + \mu_1(s)] ds} du.
\] (2.13)

\[
q_1(x, n) = e^{\int_0^x [(1 - r) q_0 \mu_1(u) + \alpha_1 r_1(u, 0, 0)] e^{-\int_u^x [1 + \lambda + \alpha_1 + \mu_1(s)] ds} du} q_1(0, n - 1) + \int_0^x (1 - r) q_1(0, n - 1) + rq_2(n) \mu_1(u) e^{-\int_u^x [1 + \lambda + \alpha_1 + \mu_1(s)] ds} du,
\] (2.14)

\[
r_1(x, y, n) = e^{\int_0^x [(1 - r) q_0 \mu_1(u) + \alpha_1 r_1(u, 0, 0)] e^{-\int_u^x [1 + \lambda + \alpha_1 + \mu_1(s)] ds} du} q_1(x, n) - \int_0^y \beta_1(u) e^{-\int_u^y [1 + \lambda + \beta_1(s)] ds} du.
\] (2.15)

\[
r_2(y, n) = e^{\int_0^y [(1 - r) q_0 \mu_1(u) + \alpha_1 r_1(u, 0, 0)] e^{-\int_u^y [1 + \lambda + \beta_2(s)] ds} du} q_2(n) - \int_0^y \beta_2(u) e^{-\int_u^y [1 + \lambda + \beta_2(s)] ds} du.
\] (2.16)

Since \(q_1(x, n), r_1(x, y, n), r_2(y, n) \in L^\infty, n \in \mathbb{N}\), we have

\[
\lim_{x \to +\infty} q_1(x, n) e^{-\int_0^y [1 + \lambda + \alpha_1 + \mu_1(s)] ds} = 0,
\]

\[
\lim_{y \to +\infty} r_1(x, y, n) e^{-\int_0^x [1 + \lambda + \beta_1(s)] ds} = 0,
\]

\[
\lim_{y \to +\infty} r_2(y, n) e^{-\int_0^y [1 + \lambda + \beta_2(s)] ds} = 0,
\]

and hence

\[
q_1(0, 0) = (1 - r) q_0 \mu(E) + \alpha_1 l_0(E),
\] (2.17)

\[
q_1(0, n) = (1 - r) q_1(0, n - 1) \mu(E) + rq_2(n) \mu(E) + \alpha_1 l_n(E), \quad n = 1, 2, \ldots,
\] (2.18)

\[
r_1(x, 0, n) = q_1(x, n) \beta_1(E), \quad n = 0, 1, 2, \ldots,
\] (2.19)

\[
r_2(0, n) = q_2(n) \beta_2(E), \quad n = 0, 1, 2, \ldots.
\] (2.20)
where
\[
\mu(E) = \int_0^{+\infty} \mu_1(u) e^{-\int_0^u (1 + \lambda + \sigma_1 + \mu_1(s)) ds} du,
\]
\[
l_0(E) = \int_0^{+\infty} r_1(u, 0, n) e^{-\int_0^u (1 + \lambda + \sigma_1 + \mu_1(s)) ds} du, \quad n \geq 0.
\]
\[
\beta_i(E) = \int_0^{+\infty} \beta_i(u) e^{-\int_0^u (1 + \lambda + \beta_i(s)) ds} du, \quad n \geq 0, \; i = 1, 2.
\]

Set \( Z_n = (q_1(0, n), q_2(n), r_1(x, 0, n), r_2(0, n)), \; n \geq 0, \) from (2.17)-(2.20), we define operator \( F \) on \( D(A^*_1) \):
\[
F \begin{pmatrix} Z_0 \\ Z_1 \\ \vdots \\ Z_n \end{pmatrix} = \begin{pmatrix} G_0 & 0 & 0 & \cdots & 0 & \cdots \\ 0 & G_1 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & G_n & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} Z_0 \\ Z_1 \\ \vdots \\ Z_n \end{pmatrix},
\]
\[
(2.21)
\]

where
\[
G_0 = \begin{pmatrix} 0 & r_1 \mu(E) & \alpha_1 l_0(E) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1(E) & 0 \\ 0 & 0 & 0 & \beta_2(E) \end{pmatrix}, \quad G_n = \begin{pmatrix} (1 - r)_\mu(E) & r_1 \mu(E) & \alpha_1 l_n(E) & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1(E) & 0 \\ 0 & 0 & 0 & \beta_2(E) \end{pmatrix}.
\]

Since
\[
\int_0^{+\infty} [1 + \lambda + \alpha_1 + \mu_1(u)] e^{-\int_0^u (1 + \lambda + \alpha_1 + \mu_1(s)) ds} du = 1,
\]
\[
\int_0^{+\infty} [1 + \lambda + \beta_i(u)] e^{-\int_0^u (1 + \lambda + \beta_i(s)) ds} du = 1, \quad i = 1, 2,
\]
then
\[
0 < r_0 = 1 - \int_0^{+\infty} (1 + \lambda) e^{-\int_0^u (1 + \lambda + \alpha_1 + \mu_1(s)) ds} du < 1,
\]
\[
0 < r_i = 1 - \int_0^{+\infty} (1 + \lambda) e^{-\int_0^u (1 + \lambda + \beta_i(s)) ds} du < 1, \quad i = 1, 2.
\]

Thus from
\[
\| r_1 \mu(E) q_2(0) + \alpha_1 l_0(E) \|_\infty \leq r_0 \| Z_0 \|_\infty,
\]
\[
\| (1 - r)_\mu(E) q_1(0, n - 1) + r_1 \mu(E) q_2(n) + \alpha_1 l_n(E) \|_\infty \leq r_0 \| Z_n \|_\infty,
\]
\[
\| \beta_i(E) q_1(x, n) \|_\infty \leq r_1 \| Z_n \|_\infty, \quad \| \beta_i(E) q_2(n) \|_\infty \leq r_2 \| Z_n \|_\infty,
\]
we get \( \| F \|_\infty < 1 \), so (2.17)-(2.20) only have the zero solution. Combining (2.6) and (2.13)-(2.16), we know that \( Q = 0 \). This means that 1 is not an eigenvalue of \( A^*_1 \). The proof is then complete. \( \Box \)

**Theorem 2.3.** The operator \( A_1 \) generates a \( C_0 \) semigroup of contraction.

**Proof.** Firstly, we show that \( A_1 \) is a dissipative operator in \( X \). In fact, for any \( P = (P_0, P_1(x, n), P_2(n), R_1(x, y, n), R_2(y, n)) \in D(A_1), \) we choose \( Q = (q_0, q_1(x, n), q_2(n), r_1(x, y, n), r_2(y, n)) \in X^* \), where
\[
q_i = \| P \| \| \text{sgn}(P_i) \|, \quad i = 0, 1, 2; \quad r_i = \| P \| \| \text{sgn}(R_i) \|, \quad i = 1, 2,
\]
then \( (P, Q) = \| P \| \| Q \| \). In addition, we have
\[
(A_1 P, Q) = \| P \| \left[ -\lambda P_0 + \mu_2 P_2(0) + (1 - r) \int_0^{+\infty} \mu_1(x) P_1(x, 0) dx \right] \text{sgn}(P_0)
\]
\[
+ \sum_{n=0}^{\infty} \int_0^{+\infty} \left[ -P'_1(x, n) - (\lambda + \alpha_1 + \mu_1(x)) P_1(x, n) \right] \text{sgn}(P_1(x, n)) dx
\]
\[
+ \sum_{n=0}^{\infty} \int_0^{+\infty} \beta_1(y) R_1(x, y, n) dy \right] \text{sgn}(P_1(x, n)) dx
\]

\[
\text{sgn}(P_0) = \begin{cases} 1, & P_0 > 0; \\ -1, & P_0 < 0. \end{cases}
\]

\[
\text{sgn}(P_1(x, n)) = \begin{cases} 1, & P_1(x, n) > 0; \\ 0, & P_1(x, n) = 0; \\ -1, & P_1(x, n) < 0. \end{cases}
\]

\[
\text{sgn}(R_1(x, y, n)) = \begin{cases} 1, & R_1(x, y, n) > 0; \\ 0, & R_1(x, y, n) = 0; \\ -1, & R_1(x, y, n) < 0. \end{cases}
\]
\[ + \sum_{n=0}^{\infty} \left\{ -\beta_2(n) P_2(n) \right\} + r \int_{0}^{+\infty} \beta_1(x) P_1(x, n) \, dx + \int_{0}^{+\infty} \beta_2(y) R_2(y, n) \, dy \right\} \text{sgn}(P_2(n)) \]

\[ + \sum_{n=0}^{\infty} \int_{0}^{+\infty} \int_{0}^{+\infty} \left\{ -\frac{\partial R_1(x, y, n)}{\partial y} - (\alpha + \beta_1(y)) R_1(x, y, n) \right\} \text{sgn}(R_1(x, y, n)) \, dx \, dy \]

\[ + \sum_{n=0}^{\infty} \left\{ -\frac{\partial R_2(y, n)}{\partial y} - (\alpha + \beta_2(y)) R_2(y, n) \right\} \text{sgn}(R_2(y, n)) \, dy \right\} \]

\[ = ||P|| \left\{ -\lambda |P_0| + \mu_2 P_2(0) \text{sgn}(P_0) + (1 - r) \int_{0}^{+\infty} \mu_1(x) P_1(x, 0) \text{sgn}(P_0) \, dx \right\} \]

Using the boundary condition we get

\[ (A_1 P, Q) = ||P|| \left\{ -\lambda |P_0| + \mu_2 P_2(0) \text{sgn}(P_0) + (1 - r) \int_{0}^{+\infty} \mu_1(x) P_1(x, 0) \text{sgn}(P_0) \, dx \right\} \]

Therefore, \(A_1\) is dissipative and hence \(R(I - A_1)\) is a closed subspace of \(X\). Furthermore, we have \(R(I - A_1) = X\). If it is not true, then there exists a \(Q \in X^*\), such that for any \(F \in R(I - A_1)\), \((F, Q) = 0\). Hence for any \(P \in D(A_1)\), \(r \neq 0\), \(r \neq 0\), i.e., for any \(P \in D(A_1)\), \((P, (I - A_1)^* Q) = 0\). Observing that \(D(A_1)\) is dense in \(X\), thus \(A_1^* Q = 0\), which means that 1 is an eigenvalue of \(A_1^*\), which contradicts with Theorem 2.1. Hence \(R(I - A_1) = X\). So the Lumer–Phillips Theorem (see, [13]) asserts that \(A_1\) generates a \(C_0\) semigroup of contraction.  

\[ \square \]

**Theorem 2.4.** The operator \(A\) generates a \(C_0\) semigroup on \(X\) and hence the system (2.3) is well-posed.
Proof. Obviously, \( \mathcal{B} \) is a bounded linear operator on \( X \), from the perturbation theory of semigroup \([13]\) we know that the operator \( \mathcal{A} \) generates a \( C_0 \) semigroup on \( X \). Therefore, the system \((2.3)\) is well-posed. □

Since the system \((1.1)-(1.8)\) describes a practical physical state, an important problem is the existence of positive solution.

**Definition 2.1** ([14]). Let \( Y \) be a Banach lattice, \( Y_+ \) be a positive cone of \( Y \) and \( \mathcal{T} \) be a linear operator in \( Y \). Denote
\[
G(x) = \{ \varphi \in Y_+ : (x, \varphi) = \|x\|^2 = \|\varphi\|^2, \}
\]
if, for any \( x \in D(\mathcal{T}) \), there exists a \( \varphi \in G(x) \) such that \((\mathcal{T}x, \varphi) \leq 0\), then \( \mathcal{T} \) is called the dispersive operator.

From Ref. [14] we known that the following result is true.

**Lemma 2.1.** Let \( Y \) be a Banach lattice and \( \mathcal{T} \) be a linear closed defined operator on \( Y \). Then \( \mathcal{T} \) generates a positive contractive semigroup if and only if \( \mathcal{T} \) is a dispersive operator and \( \mathcal{R}(I - \mathcal{T}) = Y \).

**Theorem 2.5.** The operator \( \mathcal{A} \) generates a positive \( C_0 \) contractive semigroup on \( X \).

**Proof.** It is well known that \( X \) is a Banach lattice. According to Lemma 2.1, it is sufficient to prove that \( \mathcal{A} \) is a dispersive operator.

For any \( P = (P_0, P_1(x, n), P_2(n), R_1(x, y, n), R_2(y, n)) \in D(\mathcal{A}) \), we choose \( Q = \|P\|([P_0]^+, [P_1(x, n)]^+, [P_2(n)]^+, [R_1(x, y, n)]^+, [R_2(y, n)]^+) \in X^* \), where
\[
[P_1(x, n)]^+ = ([P_1(x, 0)]^+, [P_1(x, 1)]^+, \ldots), [R_2(y, n)]^+ = ([R_2(y, 0)]^+, [R_2(y, 1)]^+, \ldots),
\]
and \([a]^+ = a \) for \( a > 0 \), \([a]^+ = 0 \) for \( a \leq 0 \). Obviously \( Q \in G(P) \), and
\[
(AP, Q) = \|P\| \left\{ -|\lambda|P_0 + \mu_2P_2(0)P_0 + (1 - r) \int_0^{+\infty} \mu_1(x)P_1(x, 0)P_0^+ dx \right.
\]
\[
+ \sum_{n=0}^{+\infty} \int_0^{+\infty} \left[ -P_1'(x, n) - (\lambda + \alpha_1 + \mu_1(x))P_1(x, n) \right] P_1(x, n)^+ dx
\]
\[
+ \sum_{n=0}^{+\infty} \int_0^{+\infty} \left[ \lambda P_1(x, n - 1) + \int_0^{+\infty} \beta_1(y)R_1(x, y, n)dy \right] P_1(x, n)^+ dx
\]
\[
+ \sum_{n=0}^{+\infty} \left[ -(\lambda + \mu_2 + \alpha_2)P_2(n) + \lambda P_2(n - 1) \right] P_2(n)^+
\]
\[
+ \sum_{n=0}^{+\infty} \left[ r \int_0^{+\infty} \mu_1(x)P_1(x, n)dx + \int_0^{+\infty} \beta_2(y)R_2(y, n)dy \right] P_2(n)^+
\]
\[
+ \sum_{n=0}^{+\infty} \int_0^{+\infty} \left[ -\frac{\partial R_1(x, y, n)}{\partial y} - (\lambda + \beta_1(y))R_1(x, y, n) \right] R_1(x, y, n)^+ dxdy
\]
\[
+ \sum_{n=0}^{+\infty} \int_0^{+\infty} \left[ \frac{\partial R_2(y, n)}{\partial y} - (\lambda + \beta_2(y))R_2(y, n) + \lambda R_2(y, n - 1) \right] R_2(y, n)^+ dy
\]
\[
\leq \|P\| \left\{ -|\lambda|P_0 + \|P_2(0)\| + (1 - r) \int_0^{+\infty} \mu_1(x)|P_1(x, 0)||P_0| + \sum_{n=0}^{+\infty} |P_0(0, n)| \right.
\]
\[
+ \sum_{n=0}^{+\infty} \int_0^{+\infty} \left[ -|\alpha_1 + \mu_1(x)||P_1(x, n)| |P_0| + \int_0^{+\infty} \beta_1(y)|R_1(x, y, n)||P_0| \right] dxdy
\]
\[
+ \sum_{n=0}^{+\infty} \left[ -(\mu_2 + \alpha_2)|P_2(n)| + r \int_0^{+\infty} \mu_1(x)|P_1(x, n)||P_0| + \int_0^{+\infty} \beta_2(y)|R_2(y, n)| |P_0| \right] dxdy
\]
\[
+ \sum_{n=0}^{+\infty} \left[ |R_1(x, 0, n)||P_0| + \int_0^{+\infty} \beta_1(y)|R_1(x, y, n)||P_0| \right] dxdy \right\}
Theorem 2.6. Let \( T(t) \) be a positive contractive semigroup with generator \( A \), then \( T(t) \) satisfies positive conserve property, i.e., for any \( H_0 \in D(A) \) and \( H_0 > 0 \), \( \| T(t) H_0 \| = \| H_0 \|, \ t \geq 0 \).

**Proof.** Since \( H_0 \in D(A) \) and \( H_0 > 0 \), then \( T(t) H_0 \in D(A) \) is the classical solution of the system (2.3). Let

\[
P(t) = (P_0(t), P_1(x, t), P_2(t), R_1(x, y, t), R_2(y, t)) = T(t) H_0 > 0,
\]

then \( P(t) \) satisfies (1.1)–(1.8). Note that

\[
\frac{d}{dt} \| P(t) \| = \frac{d}{dt} \| T(t) H_0 \| = \frac{d}{dt} P(0(t), P_1(x, 0), P_2(t), R_1(x, y, 0), R_2(y, 0)) = T(t) H_0 > 0,
\]

then \( P(t) \) satisfies (1.1)–(1.8). Note that

\[
\frac{d}{dt} \| P(t) \| = \frac{d}{dt} \| T(t) H_0 \| = \frac{d}{dt} P(0(t), P_1(x, 0), P_2(t), R_1(x, y, 0), R_2(y, 0)) = T(t) H_0 > 0,
\]

The desired result follows from Lemma 2.1. \( \square \)
3. Concluding remark

In this paper, using the operator theory, we prove that the system is well-posed in the Banach space $X$. The result ensures that the system (2.3) has a positive solution for any $H_0 \in X_+$. Especially, for $P(0) = (1, 0, 0, 0, 0)$, from the Theorem 2.6, we assert that the solution of (1.1)–(1.8) satisfies the condition (1.9). This coincides with the practice problem.

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References