MATHEMATICS

ON THE DETERMINATION OF THE BURMESTER POINTS FOR FIVE POSITIONS OF A MOVING PLANE

BY

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1. A plane V^1 moves with respect to the coinciding fixed plane V. If we consider five positions of V^1 then an arbitrary point Q of V^1 takes five positions $Q_i (i=0, 1, \ldots, 4)$ in V.

A classical theorem of BURMESTER states that there are four points in V^1 (not necessarily real) such that the five positions of each of them are on a circle in V. We shall call the four points in V^1 the Burmester points and the centres of the four circles the Burmester centres belonging to the five positions; the combination of a point and the corresponding centre shall be called a Burmester pair [1]).

A particular case is that of five infinitesimally separated positions; we are dealing then with instantaneous kinematics and a Burmester point is now a point of V^1 the path of which in V has five coinciding points of intersection with its osculating circle at the moment under consideration; the Burmester centres are the centres of these circles.

The determination of the Burmester points is an important problem of the synthesis of mechanisms for it enables the construction of a four-bar linkage the coupler plane of which coincides during its motion with five given positions. Until recently the methods used took place along the following lines. One considered first *four* given positions and derived the locus of the centres of those circles in V which pass through the four positions of a moving point. This locus, known as the centre-point curve, is a circular cubic. The Burmester centres are accordingly found as the intersections of two curves of this kind. Of the nine intersections five are parasites, slipped in by the imperfections of the procedure: two are the isotropic points of V, three others are relative centres of rotation of the given positions. The remaining ones are the Burmester centres proper. An excellent analytic treatment of this method was recently given by FREUDENSTEIN and SANDOR [2]). Making use of a complex-number form

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1) The terminology is not uniform; Burmester points and Burmester centres in our sense are sometimes called circle-points and Burmester points respectively.

2) FREUDENSTEIN and SANDOR, On the Burmester points of a plane, Journal of Applied Mechanics, vol. 28, 1961, 41–49, where a bibliography on the subject is added.

of the centre-point curve they were able to derive a procedure to locate the Burmester points numerically by means of an automatic computer. Several examples are given and many special cases considered. For the points asked for they reduce the problem to an equation of the fifth degree and in view of this the uncertainty of an extra root has to be accepted.

In the second part of their paper, however, the authors make a succesful direct attack on the problem and still making use of complex numbers they succeed in developing a method, again suitable for numerical applications, by which the Burmester pairs are determined in such a way that no quasi solutions have to be rejected afterwards.

In what follows we give also a direct analytic method to determine the Burmester pairs [1]). From the start all five positions are considered simultaneously, so that the centre-point curve does not come into the picture. By introducing a set of new unknowns the procedure leads to a system of linear equations and moreover two quadratic equations and thus, speaking geometrically the Burmester points are found as the intersections of two conics. The method leads to an equation of the fourth (or, if wanted, of the third) degree and may be made fit to give numerical solutions by means of a computer. Special cases are dealt with in a simple way. The method may be used without modification for the instantaneous problem and for situations in which a mixture of distinct and instantaneous positions is given.

It yields some general statements of which we mention a generalization of MÜLLER's theorem on the collinearity of Burmester points.

2. In the fixed plane V and in the moving plane V^1 we take (for the time being arbitrarily) the cartesian frames OXY and oxy respectively. A position of V^1 with respect to V is given analytically by

$$(1)\qquad \begin{cases} X = x\cos\varphi - y\sin\varphi + a \\ Y = x\sin\varphi + y\cos\varphi + b \end{cases}$$

and depends on the three numbers a, b and φ. This position will be denoted by $D(a, b, \varphi)$.

We consider now in V^1 the point $Q(x_0, y_0)$ and in V the circle Ω with the equation

$$(2)\qquad A_0(X^2+Y^2)-2A_1X-2A_2Y+A_3 = 0.$$

[1]) The method developed here was given by the author in a series of lectures on the geometry of mechanisms, delivered in a scientific seminar at Yale University, July 1963.

He was informed at that time by Prof. Freudenstein that some results given here *viz.* the extension of MÜLLER's theorem (§ 4) and the special case of three parallel positions (§ 5) occur in a paper by FREUDENSTEIN, SANDOR and PRIMROSE, the manuscript of which was courteously shown to the author. The paper will be published in due course in the Journal of Applied Mechanics.

If $A_0 \neq 0$ then Ω has a finite radius and its centre is the point $X_0 = A_1/A_0$, $Y_0 = A_2/A_0$. But in view of what follows we do not exclude $A_0 = 0$; in that case Ω is a straight line and the centre is a point at infinity.

We derive now the condition (which will be fundamental) that in the position $D(a, b, \varphi)$ the point Q is on the circle Ω.

It follows easily from (1) and (2):

$$(3)\quad \left\{ \begin{aligned} & A_0\{(x_0 \cos\varphi - y_0 \sin\varphi + a)^2 + (x_0 \sin\varphi + y_0 \cos\varphi + b)^2\} \\ & \quad - 2A_1(x_0 \cos\varphi - y_0 \sin\varphi + a) - 2A_2(x_0 \sin\varphi + y_0 \cos\varphi + b) + A_3 = 0 \end{aligned} \right.$$

or

$$(4)\quad \left\{ \begin{aligned} & (a^2+b^2)A_0 - 2aA_1 - 2bA_2 + 2(a\cos\varphi + b\sin\varphi)A_0x_0 \\ & \qquad + 2(-a\sin\varphi + b\cos\varphi)A_0y_0 + 2(1-\cos\varphi)(A_1x_0 + A_2y_0) \\ & \qquad + 2\sin\varphi(A_1y_0 - A_2x_0) + A_0(x_0^2+y_0^2) - 2A_1x_0 - 2A_2y_0 + A_3 = 0. \end{aligned} \right.$$

We introduce the abbreviations

$$(5)\quad \left\{ \begin{aligned} a^1 &= -a\cos\varphi - b\sin\varphi \\ b^1 &= a\sin\varphi - b\cos\varphi. \end{aligned} \right.$$

a^1 and b^1 have a simple geometrical meaning: from (1) it follows that they are the coordinates of the origin O in the system oxy, or in other words the inverse displacement of $D(a, b, \varphi)$ is $D(a^1, b^1, -\varphi)$.

Furthermore we put

$$(6)\quad \left\{ \begin{aligned} & Z_0 = A_0,\ \ Z_1 = A_1,\ \ Z_2 = A_2,\ \ Z_3 = A_0x_0,\ \ Z_4 = A_0y_0, \\ & \quad Z_5 = A_1x_0 + A_2y_0,\ Z_6 = A_1y_0,\ Z_7 = A_0(x_0^2+y_0^2) - 2A_1x_0 - 2A_2y_0 + A_3. \end{aligned} \right.$$

Hence (4) reads

$$(7)\quad \tfrac{1}{2}(a^2+b^2)Z_0 - aZ_1 - bZ_2 - a^1Z_3 - b^1Z_4 + (1-\cos\varphi)Z_5 + \sin\varphi Z_6 + \tfrac{1}{2}Z_7 = 0.$$

This is a homogeneous *linear* relation between the quantities Z_j (which depend on the point Q and the circle Ω), the coefficients of which are functions of the position parameters a, b and φ.

Obviously two homogeneous relations exist between the Z_j, for they are eight in number and they depend on A_0, A_1, A_2, A_3 and on x_0, y_0. Eliminating these from (6) we find

$$(8a)\quad Z_0Z_5 = Z_1Z_3 + Z_2Z_4$$

$$(8b)\quad Z_0Z_6 = Z_1Z_4 - Z_2Z_3$$

which are *quadratic* relations.

We suppose now that five positions $D(a_i, b_i, \varphi_i)$, $i = 0, 1, \ldots, 4$, of V^1 are given which means that the (fifteen) numbers a_i, b_i, φ_i are known.

We ask for a point Q in V^1 and a circle Ω in V such that in all five positions Q lies on Ω; that means that the unknown numbers x_0, y_0, A_0, A_1, A_2 and A_3 satisfy the five equations we obtain if in (4) we substitute a_i, b_i, φ_i for a, b, φ. Or making use of the new unknowns Z we ask for a set of numbers $Z_j (j=0, 1, \dots, 7)$, not all being zero, which satisfy five linear equations (7):

$$(9) \quad \begin{cases} \frac{1}{2}(a_i^2+b_i^2)Z_0 - a_iZ_1 - b_iZ_2 - a_i^1Z_3 - b_i^1Z_4 + (1-\cos\varphi_i)Z_5 \\ \qquad\qquad + \sin\varphi_i Z_6 + \frac{1}{2}Z_7 = 0 \qquad (i=0, 1, \dots 4) \end{cases}$$

and moreover the two equations (8).

The seven homogeneous equations (8) and (9) have in general four solutions for the ratios of Z_j. They can be all real or there may be one or two pairs of conjugate complex solutions.

If, for a real solution, the condition $Z_0 \neq 0$ is satisfied (which will be the general case), we have $A_0 \neq 0$ which means that Ω is a circle with a finite radius. From (6) it follows that the coordinates of the Burmester point are $x_0 = Z_3/Z_0$, $y_0 = Z_4/Z_0$ and those of the corresponding centre $X_0 = Z_1/Z_0$, $Y_0 = Z_2/Z_0$. In the special case that that a real solution exists for which $Z_0 = 0$, the conclusion is not valid. We have a singular situation which will be dealt with below.

4. The method to determine the Burmester pairs has now essentially been given. For the application we are able to simplify it by making a suitable choice for the until now arbitrary cartesian frames in V and V^1. First of all we take OXY and oxy such that they coincide in the position $i=0$; that means $a_0 = b_0 = \varphi_0 = 0$ and therefore $a_0^1 = b_0^1 = 0$.

From the first equation (9) we draw the conclusion $Z_7 = 0$, which is moreover obvious if we consider the definition of Z_7 as given in (6).

Hence the set of equations (9) reduces to

$$(10) \quad \begin{cases} \frac{1}{2}(a_i^2+b_i^2)Z_0 - a_iZ_1 - b_iZ_2 - a_i^1Z_3 - b_i^1Z_4 + (1-\cos\varphi_i)Z_5 + \sin\varphi_i Z_6 = 0 \\ \qquad\qquad (i=1, 2, 3, 4) \end{cases}$$

which has to be combined with (8), which is unchanged because it does not contain the unknown Z_7. Our problem is reduced now to the solution of a set of four linear and two quadratic equations for the seven homogeneous unknowns $Z_j (j=0, 1, \dots, 6)$.

In this stage we want to give an extension to our definition of the variables Z_j. The Burmester centres are denoted by their homogeneous coordinates (A_0, A_1, A_2) but the Burmester points until now by (x_0, y_0). This imperfect dualism is removed by introducing $\alpha_0, \alpha_1, \alpha_2$ such that $x_0 = \alpha_1/\alpha_0$, $y_0 = \alpha_2/\alpha_0$; the points Q are then given by the homogeneous co-ordinates $(\alpha_0, \alpha_1, \alpha_2)$.

We have now, in view of (6) the more symetric formulas:

$$(11) \quad \begin{cases} Z_0 = A_0\alpha_0,\, Z_1 = A_1\alpha_0,\, Z_2 = A_2\alpha_0,\, Z_3 = A_0\alpha_1,\, Z_4 = A_0\alpha_2, \\ \qquad\qquad Z_5 = A_1\alpha_1 + A_2\alpha_2,\, Z_6 = A_1\alpha_2 - A_2\alpha_1. \end{cases}$$

If we consider the inverse motion, that is if we interchange the planes V and V^1 then in (10) a_i and a_i^1, b_i and b_i^1 are interchanged and φ_i is replaced by $-\varphi_i$; furthermore $a_i^2+b_i^2=a_i^{1\,2}+b_i^{1\,2}$, both sides being the square distance of O and o. Therefore in (10) the unknowns must be replaced respectively by $Z_0, Z_3, Z_4, Z_1, Z_2, Z_5$ and $-Z_6$, but in view of (11) this is the same thing as interchanging α_i and A_i. All this verifies that considering the inverse motion we have to interchange Burmester's points and Burmester's centres, a well-known fact.

In order to determine the Burmester pairs by means of (8) and (10) we proceed as follows. From (10) we may in general solve the unknowns Z_1, Z_2, Z_5 and Z_6 in terms of the remaining ones. Such a procedure is possible if the determinant

$$\Delta_1 = |a_i \quad b_i \quad 1-\cos\varphi_i \quad \sin\varphi_i| \tag{12}$$

is not zero.

We obtain in this case

$$\begin{cases} Z_1 = c_{11}Z_0+c_{12}Z_3+c_{13}Z_4 \\ Z_2 = c_{21}Z_0+c_{22}Z_3+c_{23}Z_4 \\ Z_5 = c_{31}Z_0+c_{32}Z_3+c_{33}Z_4 \\ Z_6 = c_{41}Z_0+c_{42}Z_3+c_{43}Z_4 \end{cases} \tag{13}$$

and if we substitute the result in (8) we have two homogeneous quadratic equations for Z_0, Z_3 and Z_4.

The trivial solution $Z_0=Z_3=Z_4=0$ leads to nothing because it would imply that $Z_j=0$ for all values of the index. Therefore in view of (11) $A_0\neq 0$ and thus we have two homogeneous quadratic equations for α_0, α_1 and α_2, or speaking geometrically, we find two conics K_1 and K_2 in the moving plane V^1; their intersections are the Burmester points. The configuration of these points depends on the way the two conics are situated one to the other and we may expect all kinds of cases: four distinct real points, two distinct real points, two coinciding and two distinct real points and so on.

It may be that an intersection of K_1 and K_2 is at infinity. In that case we have a solution of our equations for which $\alpha_0=0$, which implies $Z_0=Z_1=Z_2=0$. But then as a consequence the determinant

$$\Delta_2 = |a_i^1 \quad b_i^1 \quad 1-\cos\varphi_i \quad \sin\varphi_i| \tag{14}$$

must be zero.

If on the other hand $\Delta_2\neq 0$ we are able to solve from (10) the unknowns Z_3, Z_4, Z_5 and Z_6 in terms of the remaining ones:

$$\begin{cases} Z_3 = d_{11}Z_0+d_{12}Z_1+d_{13}Z_2, \\ Z_4 = d_{21}Z_0+d_{22}Z_1+d_{23}Z_2, \text{ etc.} \end{cases} \tag{15}$$

Then $\alpha_0\neq 0$ and we obtain two conics K_1^1 and K_2^1 in the fixed plane, the intersections of which are the Burmester centres, which are finite in view of $\Delta_1\neq 0$.

Hence the theorem: *if $\Delta_1 \neq 0$, $\Delta_2 \neq 0$ all Burmester points and all Burmester centres are finite points.*

Moreover we have (in view of $A_0\alpha_0 \neq 0$) from (13) and (15) for a Burmester point (x_0, y_0) and the corresponding centre (X_0, Y_0):

$$(16) \qquad \begin{cases} X_0 = c_{11}+c_{12}x_0+c_{13}y_0, \;\; Y_0 = c_{21}+c_{22}x_0+c_{23}y_0, \\ \text{and} \\ x_0 = d_{11}+d_{12}X_0+d_{13}Y_0, \;\; y_0 = d_{21}+d_{22}X_0+d_{23}Y_0 \end{cases}$$

from which it follows: *the configuration of the Burmester points is related to that of the corresponding Burmester centres by a* (non-singular) *affinity.* Therefore, for instance, if $Q^{(1)}$, $Q^{(2)}$, $Q^{(3)}$ and $Q^{(4)}$ are the (real and distinct) Burmester points and B_1, B_2, B_3 and B_4 the corresponding centres, then the diagonals $Q^{(1)}Q^{(3)}$ and $Q^{(2)}Q^{(4)}$ divide one another in the same ratios as the diagonals B_1B_3 and B_2B_4.

So much for the general case. Suppose now e.g. that $\Delta_1 = 0$, $\Delta_2 \neq 0$. Then Z_3, Z_4, Z_5, Z_6 may be solved in terms of Z_0, Z_1, Z_2 and hence the conics $K_1{}^1$ and $K_2{}^1$ exist. In view of $\Delta_1 = 0$, however, one of their intersections, B_1 say, is at infinity. The Burmester point $Q^{(1)}$ corresponding to this centre is such that its five positions $Q_i{}^{(1)} (i = 0, 1, \ldots, 4)$ are on a straight line. (Making use of the terminology introduced for instantaneous kinematics by VELDKAMP $Q^{(1)}$ has to be called a BALL point with exces *one*.) For the solution we have $A_0 = 0$ and therefore $Z_0 = Z_3 = Z_4 = 0$. From $\Delta_1 = 0$ it follows that there are four coefficients λ_i, not all zero, such that

$$(17) \qquad \Sigma\lambda_i a_i = \Sigma\lambda_i b_i = \Sigma\lambda_i(1-\cos\varphi_i) = \Sigma\lambda_i \sin\varphi_i = 0$$

and therefore *all* solutions of (10) satisfy

$$\tfrac{1}{2}Z_0\,\Sigma\lambda_i(a_i{}^2+b_i{}^2)+Z_3\,\Sigma\lambda_i a_i{}^1+Z_4\,\Sigma\lambda_i b_i{}^1 = 0.$$

Hence for each solution different from $Z_0 = Z_3 = Z_4 = 0$:

$$(18) \qquad \tfrac{1}{2}\,\Sigma\lambda_i(a_i{}^2+b_i{}^2)+x_0\,\Sigma\lambda_i a_i{}^1+y_0\,\Sigma\lambda_i b_i{}^1 = 0$$

but that means that the (three) non-singular Burmester points are on a straight line. We have proved the theorem: *if for five distinct positions of a plane one Burmester point is such that the corresponding centre is at infinity, then the remaining Burmester points are collinear.* The well-known theorem of MÜLLER, valid for instantaneous kinematics, is therefore seen to be a special case of a more general result.

If $\Delta_2 = 0$ we have obviously a dual statement for the inverse motion.

In the problem as considered until now twelve parameters appear:

$$a_i, b_i, \varphi_i \qquad (i = 1, 2, 3, 4).$$

In the set (10) all equations are of a similar type, which makes it suitable to be solved by a systematic procedure. But if we want to, we are able to reduce our equations by a special choice of OXY, which is still arbitrary. For instance we may take the relative rotation centre P_{01} as the origin and moreover the X-axis through the centre P_{02} (if we suppose these centres to be finite points).

Then we have $a_1 = b_1 = b_2 = 0$ and there are only nine parameters, which is the minimum to describe five positions of the plane V^1.

Until now a position of V has been given by a, b and φ. In view of the shape of equations (10) it may be convenient to make use of polar coordinates. Putting

$$a_i = \varrho_i \cos \psi_i, \ b_i = \varrho_i \sin \psi_i \tag{19}$$

the system (10) reads

$$\begin{cases} \tfrac{1}{2}\varrho_i^2 Z_0 - \varrho_i \cos \psi_i Z_1 - \varrho_i \sin \psi_i Z_0 + \varrho_i \cos (\psi_i - \varphi_i) Z_3 \\ \qquad + \varrho_i \sin (\psi_i - \varphi_i) Z_4 + (1 - \cos \varphi_i) Z_5 + \sin \varphi_i Z_6 = 0 \end{cases} \tag{20}$$

and there is a strong resemblance to the mathematical apparatus used by FREUDENSTEIN and SANDOR, who define the displacements by complex numbers.

5. Numerical examples for the method developed here to determine Burmester pairs will be given elsewhere. But we want to consider the special case in which three of the five positions of V^1 are parallel, *viz.* D_0, D_1 and D_2.

Then $\varphi_1 = \varphi_2 = 0$ and the two first equations (10) read

$$\tfrac{1}{2}(a_i^2 + b_i^2) Z_0 - a_i(Z_1 - Z_3) - b_i(Z_2 - Z_4) = 0 \qquad (i = 1, 2) \tag{21}$$

from which it follows

$$Z_1 - Z_3 = d_1 Z_0, \ Z_2 - Z_4 = d_2 Z_0. \tag{22}$$

If we eliminate Z_3 and Z_4 from (22) and (8b) the result is

$$Z_0(Z_6 + d_2 Z_1 - d_1 Z_2) = 0.$$

Therefore two of our solutions satisfy $Z_0 = 0$, hence they do not correspond to finite Burmester centres. The other two are, in general, finite points. We give the following example: $D_1 = (p, 0, 0)$, $D_2 = (0, p, 0)$, $D_3 = (0, 0, \pi/2)$, $D_4 = (q, 0, \pi/2)$.

If we apply our method we find for the non-singular solutions:

$$Z_0 = 4, Z_1 = 2p + q \pm w, \ Z_2 = q \mp w, \ Z_3 = q \pm w, \ Z_4 = -\,2p + q \mp w$$

where w stands for $(2p^2 - q^2)^{1/2}$. Therefore there are two real Burmester centres if $2p^2 - q^2 > 0$ and no such centres if $2p^2 - q^2 < 0$. We have given this example because there has been some confusion about this particular problem.

6. Our analytical method may be applied if two of the five positions of the moving plane are infinitesimally separated. Many cases may be distinguished. D_0 and D_1 may coincide, D_3 and D_4 may coincide also, being distinct from D_0 and D_1, D_5 being distinct from the others, and so on. Of course if k positions coincide, this position must be given up to the kth order. We consider here only the extreme case that all five positions coincide, so that we deal with instantaneous kinematics.

It is well-known that even in this case the Burmester points are found in the classical method by the insection of two cubic curves.

In order to define an instantaneous position up to the fifth order we consider the displacement $D(a, b, \varphi)$ as a function of a parameter which for the sake of simplicity we identify with the rotation angle φ. We take $\varphi=0$ in the position under consideration and suppose that $a_m=d^m a(0)/d\varphi^m$ and $b_m=d^m b(0)/d\varphi^m$ $(m=0, 1, 2, 3, 4)$ are given.

Furthermore we take the origin O at the pole and OX along the pole-tangent; hence $a_0=b_0=a_1=b_1=a_2=0$. We meet again the equations (10) and the respective coefficients are found as the m-derivatives of the general equation.

The derivatives of $\frac{1}{2}(a^2+b^2)$ are $a(da/d\varphi)+b(db/d\varphi)$, $a(d^2a)/d\varphi^2)+$ $+(da/d\varphi)^2+b(d^2b)/(d\varphi^2)+(db/d\varphi)^2$, and so on, and therefore if $\varphi=0$ they are $0, 0, 0, 3b_2{}^2$. If we do the same thing for the coefficients a, b, a^1, b^1, $1-\cos\varphi$ and $\sin\varphi$ we obtain a set of linear equations for Z_j the matrix of which is

$$\begin{Vmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -b_2 & 0 & b_2 & 1 & 0 \\ 0 & -a_3 & -b_3 & a_3+3b_2 & b_3 & 0 & -1 \\ 3b_2{}^2 & -a_4 & -b_4 & a_4+4b_3 & -4a_3-6b_2+b_4 & -1 & 0 \end{Vmatrix}. \tag{23}$$

To determine the Burmester pairs we have to solve (23) and (8) and again we have to calculate the intersections of two conics. The case on hand is somewhat simpler than the general one, for we have always $Z_6=0$, with an obvious geometrical meaning. We remark that the determinant Δ_1, defined in (12) reads here $-a_3b_4+a_4b_3-a_3b_2$ and that $\Delta_1=0$ is indeed the condition in instantaneous kinematics that one of the Burmester centres is at infinity [1]).

7. We finish this paper with two general remarks. It has given us a straightforward method to determine the Burmester pairs if five positions of the moving plane are given, but it does not contain a discussion about the number of real solutions of the problem. Such a discussion would not be very simple because a large variety of special cases must be considered. Speaking geometrically we deal with a problem in the six-dimensional space S_6 of the homogeneous point coordinates $Z_j (j=0, 1, \ldots, 6)$.

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1) VELDKAMP, Curvature theory in plane kinematics, T. H. Delft. Groningen, 1963), p. 31.

In this space four five-dimensional linear spaces L_i are given by the equations (10) and moreover two quadratic five-dimensional varieties F_a and F_b by the equations (8a) and (8b). In the general case the spaces L_i determine a plane which meets F_a and F_b in two conics the points of intersection of which give us the solutions. But the rank of the matrix of the linear equations may be less than four so that the intersection of L_i has a dimension larger than two. Add to this that the spaces L_i are not general spaces: they belong to a certain set of linear spaces in S_6 for the seven coefficients of their equations are functions of the three parameters a, b and φ. On the other hand the problem is simplified by the fact that F_a and F_b are independent of the data of the given positions, but we must keep in mind that linear spaces lie on them so that the intersection of L_i and F_a or F_b may be not a mere conic but a plane.

Our second remark deals with a comparison between the method developed here and the classical way to determine the Burmester pairs. The latter may be described in terms of the mathematical apparatus build up in this paper. We have four linear equations L_i and the two quadratic ones F_a and F_b. In the classical method they are combined as follows: (L_1, L_2, L_3, F_a, F_b) and (L_2, L_3, L_4, F_a, F_b), which both give rise to a cubic equation for two non-homogeneous unknowns. In this paper we choose (L_1, L_2, L_3, L_4, F_a) and (L_1, L_2, L_3, L_4, F_b), so that the Burmester problem is reduced to the intersection of two conics.

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