Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics Letters 20 (2007) 177–182
Applied Mathematics Letters
www.elsevier.com/locate/aml

# An introduction to volatility models with indices

S. Peiris*, A. Thavaneswaran[1]

*School of Mathematics and Statistics, The University of Sydney, N.S.W. 2006, Australia*

Received 25 March 2005; received in revised form 15 March 2006; accepted 3 April 2006

**Abstract**

This paper considers a class of volatility models generated by autoregressive (AR) type models with indices. Some results associated with the autocorrelation function (acf) of this class are given and the spectral density is obtained in terms of the kurtosis of the error distribution and model parameters.


*Keywords:* Time series; Frequency; Spectrum; Autoregression; Correlation; Index; Moving average; Kurtosis; Moments; ARCH; GARCH

## 1. Introduction

It is well known that time series have their own frequency behaviour. This is a very common phenomenon in practice, especially in financial time series data (see, for instance, [10,8,9] for details) and the series cannot be identified using the existing standard time series techniques. In other words the acf, the pacf, and the spectrum are similar for many series and one may propose the same classical model for all of these cases. Obviously, this may produce poor forecast values leading to serious consequences in managerial decisions. One way of handling this problem in practice is to introduce a new class of time series models with an additional parameter (or an index) $\delta$ (>0). Therefore, this work considers a class of time series models satisfying

$$(I - \alpha B)^{\delta} X_t = e_t; \quad -1 < \alpha < 1; \delta > 0, \tag{1.1}$$

where $e_t$ is a white noise sequence and $B$ is the backshift operator such that $B^j X_t = X_{t-j}$; $j \geq 0$ with $B^0 X_t = X_t$.

This class of models covers the traditional $AR(1)$ family when $\delta = 1$. Peiris et al. [8,9] have discussed some useful properties of (1.1). It is clear that when $\alpha = 1$ and $0 < \delta < \frac{1}{2}$, (1.1) reduces to the well-known class of fractionally integrated white noise processes (see, for example, [6,2,4,7] for details). Therefore, (1.1) constitutes a new family of $AR(1)$ type models and can be applied to many standard time series in practice. This class of time series models generated by (1.1) is called '**Power Integrated AR(1)**' or '**PIAR(1)**'.

This paper attempts to generalize the class in (1.1) to incorporate ARCH and GARCH type models. With that view in mind, Section 2 reviews the class of the GARCH and gives some examples of calculating the kurtosis.

* Corresponding author.
*E-mail address:* shelton@maths.usyd.edu.au (S. Peiris).
[1] Visiting from the Department of Statistics, The University of Manitoba, Winnipeg, Manitoba, Canada R3T2N2.

0893-9659/$ - see front matter 
doi:10.1016/j.aml.2006.04.001

## 2. GARCH (p, q) process

Consider the general class of GARCH (p, q) models for a time series $y_t$ satisfying

$$y_t = \sqrt{h_t} Z_t, \tag{2.1}$$

$$h_t = \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j}, \tag{2.2}$$

where $\omega > 0$; $\alpha_i \geq 0$; $\beta_j \geq 0$ and $Z_t$ is a sequence of independent and identically distributed (iid) random variables with zero mean and unit variance.

It is well known that $u_t = y_t^2 - h_t$ is a martingale difference and (2.1) and (2.2) can be written as

$$\phi(B) y_t^2 = \omega + \beta(B) u_t, \tag{2.3}$$

where $\phi(B) = 1 - \sum_{i=1}^{r} \phi_i B^i, \phi_i = (\alpha_i + \beta_i), \beta(B) = 1 - \sum_{j=1}^{q} \beta_j B^j, r = \max(p, q)$ and $\mu' = E(y_t^2) = \frac{\omega}{1-\phi_1-\phi_2-\cdots-\phi_r}$.

Suppose that the following assumptions hold:

(A.1) All the zeros of the polynomial $\phi(B)$ lie outside of the unit circle.

(A.2) There exists a sequence of constants $\psi_i$ such that $\sum_{i=0}^{\infty} \psi_i^2 < \infty$, where the $\psi_i'$s are obtained from the relation $\psi(B)\, \phi(B) = \beta(B)$ satisfying $\psi(B) = 1 + \sum_{i=1}^{\infty} \psi_i B^i$.

Recall that the kurtosis, $K^{(X)}$, of any random variable $X$ is given by

$$K^{(X)} = \frac{E[(X - \mu)^4]}{[\mathrm{Var}(X)]^2},$$

where $\mu = E(X)$.

Below we give two examples of calculating the kurtosis for two specific cases:

**Example 2.1.** For the GARCH (1,1) model $(1 - \phi_1 B) y_t^2 = \omega + (1 - \beta_1 B) u_t$, $(\phi_1 = \alpha_1 + \beta_1)$ and $\psi_1 = \alpha_1$, $\psi_2 = \alpha_1(\alpha_1 + \beta_1)$, $\psi_3 = \alpha_1(\alpha_1 + \beta_1)^2, \ldots, \psi_j = \alpha_1(\alpha_1 + \beta_1)^{j-1}$. Clearly, $\sum_{j=1}^{\infty} \psi_j^2 = \alpha_1^2 + \alpha_1^2(\alpha_1 + \beta_1)^2 + \cdots = \frac{\alpha_1^2}{1-(\alpha_1+\beta_1)^2}$. The kurtosis $K^{(y)}$ of $\{y_t\}$ is

$$K^{(y)} = \frac{3}{1 - 2\sum_{j=1}^{\infty} \psi_j^2} = \frac{3}{1 - \frac{2\alpha_1^2}{1-(\alpha_1+\beta_1)^2}} = \frac{3[1 - (\alpha_1 + \beta_1)^2]}{1 - (\alpha_1 + \beta_1)^2 - 2\alpha_1^2},$$

and it turns out to be the same as the one given in [3]. Moreover, $\sigma_u^2 = \frac{\mu'^2 (K^{(y)} - 1)}{1 + \frac{\alpha_1^2}{1-(\alpha_1+\beta_1)^2}}$, where $\sigma_u^2 = \mathrm{Var}(u_t)$.

**Example 2.2.** For the ARCH (1) model of the form $y_t = \sqrt{h_t} Z_t$, $h_t = \omega + \alpha y_{t-1}^2$, $K^{(y)}$ can be obtained by setting $\beta_1 = 0$ in Example 2.1. The corresponding value is $K^{(y)} = \frac{3(1-\alpha^2)}{1-3\alpha^2}$ and $\sigma_u^2 = \mu'^2 (K^{(y)} - 1)(1 - \alpha^2)$.

Although ARCH and GARCH type models are very popular in volatility modelling in finance, we still have room to accommodate additional components in modelling in order to explain the volatility process without violating the principle of parsimony. With that view in mind, Section 3 introduces a class of volatility models with indices.

## 3. AR type volatility models with indices

Suppose that $\{y_t^2\}$ is an AR(p) process as in (2.3) with $h_t = \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2$, where $p$ is a large positive integer. In this case $\{y_t^2\}$ follows an AR(p) model of the form

$$y_t^2 = \omega + \sum_{i=1}^{p} \alpha_i y_{t-i}^2 + u_t, \tag{3.1}$$

where $u_t$ is as defined before.

Since the model (3.1) involves $p$ number of parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$, we consider that the corresponding analog of (1.1) for $\{y_t^2\}$ is given by

$$(I - \alpha B)^{\delta} y_t^2 = \omega + u_t, \tag{3.2}$$

where $|\alpha| < 1$, $\delta > 0$ and $u_t$ is a suitable martingale difference sequence.

Clearly when $\delta = 1$, (3.2) reduces to (3.1) with $p = 1$. However, when $\delta > 0$, Eq. (3.2) can be approximated by a $p$th order polynomial.

Let

$$(I - \alpha B)^{\delta} = \sum_{j=0}^{\infty} \pi_j \alpha^j B^j, \tag{3.3}$$

where $B^0 = I$, $\pi_0 = 1$ and

$$\pi_j = (-1)^j \binom{\delta}{j} = \frac{(-\delta)(-\delta+1)\cdots(-\delta+j-1)}{j!}; \quad j \geq 1.$$

**Notes:**

1. If $\delta$ is a positive integer, then $\pi_j = 0$ for $j \geq \delta + 1$. For any non-integer $\delta > 0$, it is known that

$$\pi_j = \frac{\Gamma(j-\delta)}{\Gamma(j+1)\Gamma(-\delta)}, \tag{3.4}$$

where $\Gamma(\cdot)$ is the gamma function.

2. The model in (3.2) can be thought of as a model incorporating all of the past volatilities in a parsimonious way using only one additional parameter $\delta$. This model can be used to forecast future volatilities using more of the available information than any other ARCH model. In practice we may use the approximation

$$(I - \alpha B)^{\delta} \approx \sum_{j=0}^{m} \pi_j \alpha^j B^j,$$

where $m$ is a suitably chosen large integer.

It is easy to see that the series $\sum_{j=0}^{\infty} \pi_j \alpha^j$ converges for all $\delta$ since $|\alpha| < 1$. Thus $y_t^2$ in (3.2) has a valid AR representation of the form

$$\sum_{j=0}^{\infty} \pi_j \alpha^j y_{t-j}^2 = \omega + u_t \tag{3.5}$$

with $\sum |\pi_j \alpha^j|^2 < \infty$. Now we state and prove the following theorem for a stationary solution of (3.2).

**Theorem 3.1.** *For all $\delta > 0$ and $|\alpha| < 1$, the infinite series*

$$y_t^2 - \mu' = \sum_{j=0}^{\infty} \Psi_j \alpha^j u_{t-j} \tag{3.6}$$

*converges absolutely with probability 1 provided $E(u_t^2) < C$, $C > 0$, where $\psi_j = \frac{\Gamma(j+\delta)}{\Gamma(j+1)\Gamma(\delta)}$ and $\mu' = E(y_t^2)$.*

**Proof.** Let $(I - \alpha B)^{-\delta} = \sum_{j=0}^{\infty} \Psi_j \alpha^j B^j$, where

$$\psi_j = (-1)^j \binom{-\delta}{j} = \frac{\Gamma(j+\delta)}{\Gamma(j+1)\,\Gamma(\delta)}; \quad j \geq 0.$$

Now $E\left(\sum_{j=0}^{\infty} |\Psi_j \alpha^j u_{t-j}|\right)^2 = \sum_{j=0}^{\infty} |\Psi_j \alpha^j|^2 E\{|u_{t-j}|^2\}$. $\square$

Since $\sum_{j=0}^{\infty} |\Psi_j \alpha^j|^2 < \infty$, the result follows. Thus (3.6) gives a stationary solution for the process in (3.2). For $\alpha = 1$, (3.6) converges for all $0 < \delta < 1/2$.

Let $\gamma_k = \mathrm{Cov}(y_t^2, y_{t-k}^2)$ be the autocovariance function at lag $k$ of $\{y_t^2\}$ satisfying the conditions of Theorem 3.1. It is clear from (3.5) that the $\{\gamma_k\}$ satisfy a Yule–Walker type of recursion

$$\sum_{j=0}^{\infty} \pi_j \, \alpha^j \, \gamma_{k-j} = 0; \quad k > 0 \tag{3.7}$$

and the corresponding autocorrelation function (acf), $\rho_k$, at lag $k$ is given by

$$\sum_{j=0}^{\infty} \pi_j \, \alpha^j \, \rho_{k-j} = 0; \quad k > 0. \tag{3.8}$$

It is interesting to note that $\rho_k = \alpha^k$ is a solution of (3.8), since $\sum_{j=0}^{\infty} \pi_j = 0$ for any $\delta > 0$. However, the general solution for $\rho_k$ may be expressed as

$$\rho_k = g(k, \alpha, \delta)\alpha^k,$$

where $g(.)$ is a suitably chosen function of $k$, $\alpha$, and $\delta$. To find this function $g$, we use the following approach:

The spectrum of $\{y_t^2\}$ in (3.2) is

$$\begin{aligned} f_{y_t^2}(\omega) &= |1 - \alpha \, \mathrm{e}^{-\mathrm{i}\omega}|^{-2\delta} \, \frac{\sigma_u^2}{2\pi}; \quad -\pi \le \omega \le \pi \\ &= (1 - 2\alpha \,\mathrm{Cos}\, \omega + \alpha^2)^{-\delta} \, \frac{\sigma_u^2}{2\pi}. \end{aligned} \tag{3.9}$$

In a neighbourhood of $\omega = 0$, $f_{y_t^2}^{g} \sim \frac{\sigma_u^2}{2\pi}(1-\alpha)^{-2\delta}$, where $g$ stands for generalized process in (3.2). Now the exact form of $\gamma_k$(or $\rho_k$) can be obtained from

$$\begin{aligned} \gamma_k &= \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}k\omega} \, f_{y_t^2}(\omega)\mathrm{d}\omega \\ &= \frac{\sigma_u^2}{\pi} \int_0^{\pi} \frac{\mathrm{Cos}(k\omega)}{(1 - 2\alpha \,\mathrm{Cos}\, \omega + \alpha^2)^{\delta}} \, \mathrm{d}\omega. \end{aligned} \tag{3.10}$$

In order to obtain the variance of the volatility process $y_t^2$, we evaluate the integral in (3.10) for $k = 0$. Section 4 reports this result.

## 4. Main results

This section is devoted to reporting some results associated with the AR type process given in (3.2). Since the variance plays a significant role in statistical modelling, the following theorem gives an expression for $\gamma_k$.

**Theorem 4.1.** *For the process given in* (3.2),

(a) *the variance*

$$\gamma_0 = \mathrm{Var}(y_t^2) = \sigma_u^2 F(\delta,\ \delta;\ 1;\ \alpha^2), \tag{4.1}$$

(b) *the autocovariance function of* $y_t^2$ *is*

$$\gamma_k = \frac{\sigma_u^2 \, \alpha^k \, \Gamma(k+\delta) \, F(\delta,\ k+\delta;\ k+1;\ \alpha^2)}{\Gamma(\delta) \, \Gamma(k+1)}; \quad k \ge 0, \tag{4.2}$$

*where* $F(\theta_1, \theta_2; \theta_3; \theta) = \sum_{j=0}^{\infty} \frac{\Gamma(\theta_1+j)\,\Gamma(\theta_2+j)\Gamma(\theta_3)\theta^j}{\Gamma(\theta_1)\Gamma(\theta_2)\Gamma(\theta_3+j)\Gamma(j+1)}$ *is the hypogeometric function (see* [5], *p.* 1039, *for details).*

**Proof.** We first evaluate (3.10) at $k = 0$ and then for any $k$. From [5] p. 384, we have

$$\gamma_0 = \frac{\sigma_u^2}{\pi} \int_0^\pi \frac{\mathrm{d}\omega}{(1 - 2\alpha\,\mathrm{Cos}\omega + \alpha^2)^\delta} = B\left(\frac{1}{2}, \frac{1}{2}\right) F(\delta, \delta; 1; \alpha^2), \tag{4.3}$$

where $B(x, y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$ is the Beta function and $B(\frac{1}{2}, \frac{1}{2}) = \pi$, and hence (a) follows. (b) follows by writing $y_t^2 - \mu' = \sum_{j=0}^{\infty} \psi_j\, \alpha^j\, u_{t-j}$ and using

$$\gamma_k = \sigma_u^2\, \alpha^k \sum_{j=0}^{\infty} \psi_j\, \psi_{j+k}\, \alpha^{2j} = \sigma_u^2 \alpha^k \sum_{j=0}^{\infty} \frac{\Gamma(j+\delta)\,\Gamma(j+k+\delta)(\alpha^2)^j}{\Gamma^2(\delta)\Gamma(j+1)\Gamma(j+k+1)},$$

where $\mu' = E(y_t^2) = \frac{\omega}{(1-\alpha)^\delta}$. □

From p. 556 of [1], we have

$$\sum_{j=0}^{\infty} \frac{\Gamma(\delta+j)\,\Gamma(k+\delta+j)(\alpha^2)^j}{\Gamma(k+1+j)\,\Gamma(j+1)} = \frac{\Gamma(\delta)\,\Gamma(k+\delta)\,F(\delta, k+\delta; k+1; \alpha^2)}{\Gamma(k+1)}$$

and hence (4.2) follows.

**Note:** Using the properties of $F()$, it is easy to see that for $\delta = 1$ one has $F(1, 1;\ 1;\ \alpha^2) = (1 - \alpha^2)^{-1}$ (compare with [5], p. 1040). That is, (4.1) turns out to be the variance of an ARCH(1) process:

$$\gamma_0 = \mathrm{Var}(y_t^2) = \frac{\sigma_u^2}{1 - \alpha^2} \quad \text{for } |\alpha| < 1.$$

It is known that for $\theta_3 - \theta_1 - \theta_2 > 0$, $F(\theta_1,\ \theta_2; \theta_3; 1) = \frac{\Gamma(\theta_3)\,\Gamma(\theta_3-\theta_1-\theta_2)}{\Gamma(\theta_3-\theta_1)\,\Gamma(\theta_3-\theta_2)}$ and hence part (a) of the Theorem 4.1 reduces to the $\mathrm{Var}(y_t^2)$ for a fractionally differenced (long memory) volatility process satisfying $(I - B)^\delta y_t^2 = u_t$ when $0 < \delta < \frac{1}{2}$. That is, when $\alpha = 1$ and $0 < \delta < \frac{1}{2}$, (4.1) gives $\gamma_0 = \frac{\sigma_u^2 \Gamma(1-2\delta_1)}{\Gamma^2(1-\delta)}$.

## 5. Conclusions

We have introduced a new class of models for modelling volatility and obtained the autocorrelation function of the underlying process. It is shown that the proposed class of models provides a novel way to incorporate additional components in volatility modelling. The new results in Eqs. (4.1) and (4.2) are particularly useful in theoretical developments of power integrated ARMA (PIARMA) and power integrated GARCH (PIGARCH) processes with indices and these will be discussed in a future paper. Moreover, the forecasting with this type of volatility models will be discussed in a forthcoming paper following [11].

## Acknowledgements

We thank the anonymous referee and the editor of the journal for their suggestions for improving the quality of this work. The second author thanks the School of Mathematics and Statistics at the University of Sydney for support during his visit.

## References

[1] M. Abramowitz, I. Segun, Handbook of Mathematical Functions, Dover, New York, 1968.
[2] J. Beran, Statistics for Long Memory Processes, Chapman and Hall, New York, 1994.
[3] T. Bolleslev, Generalised autoregressive conditional heteroscedasticity, J. Econometrics 31 (1986) 307–327.
[4] G. Chen, B. Abraham, S. Peiris, Lag window estimation of the degree of differencing in fractionally integrated time series models, J. Time Series Analysis 15 (1994) 473–487.
[5] I.S. Gradshteyn, I.M. Ryzhik, Tables of Integrals, Series, and Products, 4th ed., Academic Press, New York, 1980.
[6] J.R.M. Hosking, Fractional differencing, Biometrika 68 (1981) 154–176.
[7] W.K. Li, A.I. McLeod, Fractional time series modelling, Biometrika 73 (1986) 217–221.

[8] S. Peiris, Improving the quality of forecasting using generalized AR models: An application to statistical quality control, Stat. Methods 5 (2003) 156–171.
[9] S. Peiris, D. Allen, A. Thavaneswaran, An introduction to generalized moving average model and applications, J. Appl. Statist. Sci. 13 (2005) 251–267.
[10] M.B. Priestley, Spectral Analysis and Time Series, Academic Press, New York, 1981.
[11] A. Thavaneswaran, S. Appadoo, S. Peiris, Forecasting volatility, Statist. Probab. Lett. 75 (2005) 1–10.