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Explicit subsolutions and a Liouville theorem for fully nonlinear uniformly elliptic inequalities in halfspaces

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Abstract

We prove a Liouville type theorem for arbitrarily growing positive viscosity supersolutions of fully nonlinear uniformly elliptic equations in halfspaces. Precisely, let $\mathcal{M}^-_{\lambda,\Lambda}$ be the Pucci's inf-operator with ellipticity constants $\Lambda \geqslant \lambda > 0$. We prove that the inequality $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) + u^p \leqslant 0$ does not have any positive viscosity solution in a halfspace provided that $-1 \leqslant p \leqslant \frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}$, whereas positive solutions do exist if either $p < -1$ or $p > \frac{\frac{\Lambda}{\lambda}(n-1)+2}{\frac{\Lambda}{\lambda}(n-1)}$. The proof relies on the construction of explicit subsolutions of the homogeneous equation $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) = 0$ and on a nonlinear version in a halfspace of the classical Hadamard three-circles theorem for entire superharmonic functions.

Résumé

On démontre un théorème de type Liouville dans un demi-espace pour des sursolutions de viscosité avec croissance arbitraire d'équations complètement non linéaires uniformément elliptiques. Précisément, soit $\mathcal{M}^-_{\lambda,\Lambda}$ l'opérateur inf-extrémal de Pucci avec constantes d'ellipticité $\Lambda \geqslant \lambda > 0$. On montre que l'inégalité $\mathcal{M}^-_{\lambda,\Lambda}(D^2u)+u^p \leqslant 0$ n'a pas de solutions de viscosité positives dans un demi-espace si $-1 \leqslant p \leqslant \frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}$, tandis qu'il existe des solutions positives si $p < -1$ ou si $p > \frac{\frac{\Lambda}{\lambda}(n-1)+2}{\frac{\Lambda}{\lambda}(n-1)}$. Les demonstrations utilisent la construction de sous-solutions explicites de l'équation homogéne $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) = 0$ et une version non linéaire du théorème des trois-cercles de Hadamard pour les functions surharmoniques.

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1. Introduction

We focus on positive supersolutions of second order fully nonlinear uniformly elliptic equations of the form either

$$F\big(x, D^2u\big) = 0 \quad \text{in } \mathbb{R}^n_+, \tag{1}$$

or

$$F\big(x, D^2u\big) + u^p = 0 \quad \text{in } \mathbb{R}^n_+, \tag{2}$$

where $\mathbb{R}^n_+$ is the halfspace $\{x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}\colon\ x_n > 0\}$, with $n \geqslant 2$. Here $F : \mathbb{R}^n_+ \times \mathcal{S}_n \to \mathbb{R}$ is a continuous function of the space variable $x \in \mathbb{R}^n_+$ and of the Hessian matrix $D^2u \in \mathcal{S}_n$, the set of symmetric $n \times n$ matrices.

For Eq. (1) we first construct some explicit homogeneous subsolutions, vanishing on the boundary $\partial\mathbb{R}^n_+ \setminus \{0\}$, and then we use them to derive lower bounds and monotonicity properties for all nonnegative supersolutions. The result we obtain closely resembles the classical Hadamard three-spheres theorem for bounded from below superharmonic functions, and it will be applied in order to obtain a Liouville type theorem for positive supersolutions of (2).

Let us recall that the Liouville property for equations posed in halfspaces and having power-like zero order terms is one of the crucial steps for applying the blow-up method developed in [15], which yields L^∞ a priori estimates for solutions of boundary value problems in bounded domains. Liouville type properties have been largely studied mainly in case of semilinear equations, and our contribution is devoted to the extension to the fully nonlinear framework.

We assume that the operator F is uniformly elliptic with ellipticity constants $\Lambda \geqslant \lambda > 0$, that is F is assumed to satisfy

$$\lambda \operatorname{tr} P \leqslant F(x, M + P) - F(x, M) \leqslant \Lambda \operatorname{tr} P \tag{3}$$

for all $x \in \mathbb{R}^n_+$ and for every $M, P \in \mathcal{S}_n$, with $P \geqslant O$ (i.e. nonnegative definite).

We further assume that $F(x, O) = 0$, so that inequalities (3) amount to

$$\lambda \operatorname{tr} M^+ - \Lambda \operatorname{tr} M^- \leqslant F(x, M) \leqslant \Lambda \operatorname{tr} M^+ - \lambda \operatorname{tr} M^-$$

for all $x \in \mathbb{R}^n_+$ and $M \in \mathcal{S}_n$, where $M^+, M^- \geqslant O$ are the only nonnegative definite matrices decomposing M as $M = M^+ - M^-$ and satisfying $M^+M^- = 0$. Let us recall that the left and the right hand side of the above inequality represent the Pucci extremal operators (see e.g. [8]), that are the special uniformly elliptic operators given by

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \sum_{\mu_i > 0} \mu_i + \Lambda \sum_{\mu_i < 0} \mu_i = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AM),$$

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda \sum_{\mu_i > 0} \mu_i + \lambda \sum_{\mu_i < 0} \mu_i = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AM),$$

where $\mu_1, \dots, \mu_n$ stand for the eigenvalues of M and $\mathcal{A}_{\lambda,\Lambda}$ is the set of all symmetric matrices whose eigenvalues belong to the closed interval $[\lambda, \Lambda]$. Thus, the uniform ellipticity condition (3) is equivalent for the operator F to satisfy

$$\mathcal{M}^-_{\lambda,\Lambda}(M) \leqslant F(x, M) \leqslant \mathcal{M}^+_{\lambda,\Lambda}(M)$$

for every x and every M, and this implies that if u is a solution (or a supersolution) either of (1) or of (2), then u satisfies respectively either

$$\mathcal{M}^-_{\lambda,\Lambda}\big(D^2u\big) \leqslant 0 \quad \text{in } \mathbb{R}^n_+, \tag{4}$$

or

$$\mathcal{M}^-_{\lambda,\Lambda}\big(D^2u\big) + u^p \leqslant 0 \quad \text{in } \mathbb{R}^n_+. \tag{5}$$

In this respect, (4) and (5) are the inequalities naturally associated with all uniformly elliptic equations of the form either (1) or (2) respectively.

Our goal is to identify an explicit range of values for the exponent p for which (5) does not admit positive solutions. Note that weak solutions of inequality (5), because of nondivergence form of the principal part, have to be meant in the viscosity sense, and we refer to [8,10] for the viscosity solutions theory for Pucci and more general fully nonlinear operators.

As a consequence of our results, we obtain the following theorem.

Theorem 1.1. *Let $n \geqslant 2$ and $-1 \leqslant p \leqslant \frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}$. Then, there does not exist any positive viscosity solution of inequality (5).*

If $\Lambda = \lambda$, then (5) becomes, up to a scaling factor for the function u, the semilinear inequality

$$\Delta u + u^p \leqslant 0, \tag{6}$$

and Theorem 1.1 thus gives an extension of the well-known fact that inequality (6) does not have positive solutions in a halfspace for $-1 \leqslant p \leqslant \frac{n+1}{n-1}$ (see e.g. [1]). In other words, -1 and $\frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}$ work as critical exponents for the Liouville property for operator $\mathcal{M}^-_{\lambda,\Lambda}$ in a halfspace.

To show the existence of critical exponents for inequality (5) we can apply the same argument used in [18] for linear equations. Indeed, a straightforward computation shows that if $p > 1$ (or if $p < 1$) and u is a positive solution of inequality (5), then for any $q > p$ (or $q < p$, respectively) the function $v = (\frac{p-1}{q-1})^{1/(q-1)} u^{(p-1)/(q-1)}$ satisfies

$$\mathcal{M}^-_{\lambda,\Lambda}\big(D^2 v\big) + v^q \leqslant 0 \quad \text{in } \mathbb{R}^n_+.$$

Therefore one can define the exponents

$$p^* = \inf\big\{p > 1\colon \text{(5) has a positive solution}\big\},$$
$$p_* = \sup\big\{p < 1\colon \text{(5) has a positive solution}\big\},$$

and the Liouville property for inequality (5) certainly fails if either $p < p_*$ or $p > p^*$.

At this point let us recall that inequalities such as (6) and (5) have been extensively treated and subjected to different generalizations in past and recent works. For linear operators and inequalities posed in the whole space or in exterior domains, we just mention [14] for supersolutions of (6), [18] for uniformly elliptic nonconstant coefficient inequalities of the form

$$\operatorname{tr}\big(A(x) D^2 u\big) + u^p \leqslant 0 \tag{7}$$

and [5] for inequalities involving the Heisenberg–Laplace operator.

In the fully nonlinear case, inequalities posed in the whole space or in exterior domains have been considered for Pucci extremal operators in [11] for $p \geqslant 0$ and in [2] for $p < 0$, in [9] for Pucci extremal operators plus first order terms, in [1,6,13] for more general classes of fully nonlinear operators and zero order terms, and in [12] for fully nonlinear integrodifferential operators. We merely recall that when inequality (5) is considered in the whole space, then the critical exponents are

$$p_* = \begin{cases} -\infty & \text{if } \frac{\lambda}{\Lambda}(n-1) \geqslant 1, \\ \frac{\frac{\lambda}{\Lambda}(n-1)+1}{\frac{\lambda}{\Lambda}(n-1)-1} & \text{if } \frac{\lambda}{\Lambda}(n-1) < 1, \end{cases} \quad \text{and} \quad p^* = \frac{\frac{\Lambda}{\lambda}(n-1)+1}{\frac{\Lambda}{\lambda}(n-1)-1},$$

and the Liouville property holds if and only if $p_* \leqslant p \leqslant p^*$.

Inequality (6) posed in a halfspace or in more general cone-like domains has been studied in [4,16,17], and recently revised in [1]. In particular, the arguments used in [1] can be applied also to fully nonlinear principal parts, and this is, up to our knowledge, the only existing result for nondivergence form differential inequalities posed in conical domains, including the linear case of (7).

The results of [1] in particular relate the critical exponents p^*, p_* for (5) to the scaling exponents $\alpha^\pm$ of the homogeneous solutions of the homogeneous equation. Precisely, we recall that, in view of the results of [19] and their recent extensions in [3], the extremal homogeneous equation

$$\mathcal{M}^-_{\lambda,\Lambda}\big(D^2 \Phi\big) = 0 \tag{8}$$

is known to have in any axially symmetric cone $\mathcal{C}_\sigma = \{x \in \mathbb{R}^n\colon x_n > \sigma|x|\}$, with $-1 < \sigma < 1$, exactly two solutions, up to normalization, of the form

$$\Phi_{\alpha^\pm_\sigma}(x) = |x|^{-\alpha^\pm_\sigma} \phi_{\alpha^\pm_\sigma}\left(\frac{x_n}{|x|}\right)$$

with $\alpha_\sigma^- < 0 < \alpha_\sigma^+$ and $\phi_{\alpha_\sigma^\pm}$ C^2-functions defined on the interval $[\sigma, 1]$ satisfying $\phi_{\alpha_\sigma^\pm}(\sigma) = 0$ and $\phi_{\alpha_\sigma^\pm}(t) > 0$ for $\sigma < t \leqslant 1$.

By applying the proof of Theorem 5.1 in [1] with the functions $\Psi^\pm$ there replaced by $\Phi_{\alpha_\sigma^\pm}$, it follows that positive supersolutions in $\mathcal{C}_\sigma$ of (5) do not exist if and only if

$$1 + \frac{2}{\alpha_\sigma^-} \leqslant p \leqslant 1 + \frac{2}{\alpha_\sigma^+}.$$

Now, for the halfspace $\mathbb{R}^n_+ = \mathcal{C}_0$ it is clear that $\alpha_0^- = -1$ (and $\phi_{\alpha_\sigma^-}(t) = t$). Therefore, in this case, if we set $\alpha_0^+ = \alpha$, then the critical exponents are given by

$$p_* = -1 \quad \text{and} \quad p^* = 1 + \frac{2}{\alpha}. \tag{9}$$

On the other hand, the existence of the homogeneous solution Φ_α is obtained in [19] by means of an abstract existence and uniqueness result for nonlinear ODEs having singular monotone lower order terms, and in [3] by using a topological argument which leads to a fixed point theorem in Banach spaces. In both cases, the exponent α is not or not sharply estimated from above, so that no explicit lower bound for p^* can be deduced.

By the comparison principles of Phragmén–Lindelöf type given in [3,19], α can be estimated from above provided that an explicit subsolution of (8) vanishing on $\partial\mathbb{R}^n_+ \setminus \{0\}$ is known, as well as a homogeneous supersolution of (8) vanishing for $|x| \to \infty$ produces a lower bound for α. In [19], only a supersolution of (8) is exhibited, namely the function

$$\hat{\Phi} = \frac{x_n}{|x|^{\frac{\Lambda}{\lambda}(n-1)+1}}.$$

Note that the inequality $\mathcal{M}^-_{\lambda,\Lambda}(D^2\hat{\Phi}) \leqslant 0$ in $\mathbb{R}^n_+$ easily follows from the fact that $\mathcal{M}^-_{\lambda,\Lambda}$ is superadditive and $\hat{\Phi}$ is, up to a negative constant, the partial derivative with respect to x_n of a well-known radial solution for $\mathcal{M}^-_{\lambda,\Lambda}$ in $\mathbb{R}^n \setminus \{0\}$. The homogeneous supersolution $\hat{\Phi}$ gives the lower bound $\alpha \geqslant \frac{\Lambda}{\lambda}(n-1)$, which in turn implies, by (9),

$$p^* \leqslant \frac{\frac{\Lambda}{\lambda}(n-1)+2}{\frac{\Lambda}{\lambda}(n-1)}.$$

In other words, inequality (5) does admit positive solutions for $p > \frac{\frac{\Lambda}{\lambda}(n-1)+2}{\frac{\Lambda}{\lambda}(n-1)}$, $u(x) = \frac{x_n}{|x|^\beta}$ being an explicit supersolution of (5) for $\frac{p+1}{p-1} < \beta < \frac{\Lambda}{\lambda}(n-1)+1$.

Therefore, in order to obtain a nonexistence statement as in Theorem 1.1, we have to determine an explicit subsolution of (8) vanishing on $\partial\mathbb{R}^n_+ \setminus \{0\}$. This turns out to be a nontrivial task, since the standard separation of variables technique in polar representation hardly applies to operator $\mathcal{M}^-_{\lambda,\Lambda}$. To appreciate the strongly nonlinear character of $\mathcal{M}^-_{\lambda,\Lambda}$, note that, for $n = 2$, Eq. (8) reads as

$$\Delta v = \left(\sqrt{\frac{\Lambda}{\lambda}} - \sqrt{\frac{\lambda}{\Lambda}}\right)\sqrt{-\det D^2 v}.$$

We will prove that the function

$$\Phi(x) = \frac{x_n^{\frac{\Lambda}{\lambda}}}{|x|^{\frac{\Lambda}{\lambda}(n+1)-1}}$$

actually is a subsolution of (8). Hence, we obtain the upper bound

$$\alpha \leqslant \frac{\Lambda}{\lambda} n - 1,$$

and Theorem 1.1 can be deduced as a consequence of (9). Note that specific bounds for α are useful also when applying the extended comparison principles and the boundary singularity removability results given in [3,19], which require as assumptions growth conditions involving the exponent α. See also [7] for another application of homogeneous solutions of fully nonlinear elliptic equations in cones.

With the subsolution Φ at hand, we can bound from below not only the solution Φ_α, but all nonnegative supersolutions of (8), and we obtain a monotonicity property for supersolutions as in the classical three-circles Hadamard theorem for superharmonic functions (see [20]). This will be performed in Section 2. Furthermore, we apply this monotonicity property in Section 3, where we provide an alternative elementary proof of Theorem 1.1 in the superlinear case $1 \leqslant p \leqslant \frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}$. Indeed, Theorem 1.1 will be shown to follow easily from our nonlinear three-surfaces Hadamard theorem for $1 \leqslant p < \frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}$. In the limiting case $p = \frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}$ we will apply a bootstrap argument: first, if u satisfies (5), then u is a supersolution of (8), and then $u \geqslant c\Phi$ for some constant $c > 0$ and in a suitable subdomain of $\mathbb{R}^n_+$. Therefore, by (5) with $p = \frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}$, we will have that

$$-\mathcal{M}^-_{\lambda,\Lambda}\big(D^2u\big) \geqslant c\left(\frac{x_n^{\frac{\Lambda}{\lambda}}}{|x|^{\frac{\Lambda}{\lambda}(n+1)+1}}\right)^{\frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}}.$$

Again, we will construct an explicit solution of the opposite inequality, and the comparison principle will show that u is too large to satisfy (5).

2. Explicit subsolutions of $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) = 0$ and a Hadamard type theorem

In this section we first of all construct an explicit homogeneous subsolution of the homogeneous equation $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) = 0$ in the halfspace $\mathbb{R}^n_+$, vanishing on $\partial\mathbb{R}^n_+ \setminus \{0\}$. This will be then used to get information on solutions and supersolutions as well.

We will make use of the following algebraic result, whose proof is just a straightforward computation.

Lemma 2.1. *Let $v, w \in \mathbb{R}^n$ be unitary vectors and, given $a, b, c, d \in \mathbb{R}$, let us consider the symmetric matrix*

$$A = av \otimes v + bw \otimes w + c(v \otimes w + w \otimes v) + dI_n,$$

where $v \otimes w$ denotes the $n \times n$ matrix whose i, j-entry is $v_i w_j$. Then, the eigenvalues of A are:

- *d, with multiplicity (at least) $n-2$ and eigenspace given by $\langle v, w\rangle^\perp$;*
- *$d + \frac{a+b+2cv\cdot w \pm \sqrt{(a+b+2cv\cdot w)^2+4(1-(v\cdot w)^2)(c^2-ab)}}{2}$, which are simple (if different from d).*

In particular, if either $c^2 = ab$ or $(v \cdot w)^2 = 1$, then the eigenvalues are d, which has multiplicity $n-1$, and $d + a + b + 2cv \cdot w$, which is simple.

Remark 2.2. Let us explicitly remark that the radicand appearing in the expression of the eigenvalues above is nonnegative, since

$$\begin{aligned}
&(a+b+2cv\cdot w)^2 + 4\big(1-(v\cdot w)^2\big)\big(c^2-ab\big)\\
&\quad= (a-b)^2 + 4cv\cdot w(a+b) + 4c^2 + 4(v\cdot w)^2 ab\\
&\quad\geqslant \big(v\cdot w(a-b)\big)^2 + 4cv\cdot w(a+b) + 4c^2 + 4(v\cdot w)^2 ab\\
&\quad= \big(v\cdot w(a+b) + 2c\big)^2 \geqslant 0.
\end{aligned}$$

Theorem 2.3. *For any fixed $\Lambda \geqslant \lambda > 0$, the function*

$$\Phi(x) = \frac{x_n^{\frac{\Lambda}{\lambda}}}{|x|^{\frac{\Lambda}{\lambda}(n+1)-1}} \tag{10}$$

satisfies, in the classical sense,

$$\mathcal{M}^-_{\lambda,\Lambda}\big(D^2\Phi\big) \geqslant 0 \quad \text{in } \mathbb{R}^n_+. \tag{11}$$

Proof. Let us set $\rho = |x|$, and let us compute the Hessian matrix for functions of the form

$$\Phi(x) = \frac{x_n^\alpha}{\rho^\beta}$$

for any $\alpha, \beta > 0$. One has

$$D^2\Phi = \frac{x_n^\alpha}{\rho^{\beta+2}}\left[\beta(\beta+2)\frac{x}{\rho}\otimes\frac{x}{\rho} + \alpha(\alpha-1)\left(\frac{\rho}{x_n}\right)^2 e_n\otimes e_n - \alpha\beta\frac{\rho}{x_n}\left(\frac{x}{\rho}\otimes e_n + e_n\otimes\frac{x}{\rho}\right) - \beta I_n\right]$$

with $e_n = (0,1) \in \mathbb{R}^n$.

According to Lemma 2.1, the eigenvalues $\mu_1, \dots, \mu_n$ of $D^2\Phi(x)$ are

$$\begin{aligned}
\mu_1 &= \frac{x_n^\alpha}{\rho^{\beta+2}}\frac{\beta(\beta-2\alpha)+\alpha(\alpha-1)(\rho/x_n)^2+\sqrt{\mathcal{D}}}{2},\\
\mu_2 &= \frac{x_n^\alpha}{\rho^{\beta+2}}\frac{\beta(\beta-2\alpha)+\alpha(\alpha-1)(\rho/x_n)^2-\sqrt{\mathcal{D}}}{2},\\
\mu_i &= -\beta\frac{x_n^\alpha}{\rho^{\beta+2}}, \quad 3\leqslant i\leqslant n,
\end{aligned} \tag{12}$$

with

$$\mathcal{D} = \left(\beta(\beta-2\alpha+2)+\alpha(\alpha-1)\left(\frac{\rho}{x_n}\right)^2\right)^2 + 4\alpha\beta(\beta-2\alpha+2)\left(\frac{\rho}{x_n}\right)^2\left(1-\left(\frac{x_n}{\rho}\right)^2\right).$$

We notice that

$$\mathcal{D} = \left(\beta(\beta-2\alpha)+\alpha(\alpha-1)\left(\frac{\rho}{x_n}\right)^2\right)^2 + 4\beta(\beta-\alpha+1)\left(\beta-2\alpha+\alpha\left(\frac{\rho}{x_n}\right)^2\right);$$

therefore, for $\beta \geqslant \alpha$, one has $\mu_1 \geqslant 0$ and $\mu_i \leqslant 0$ for $2 \leqslant i \leqslant n$. Hence

$$\begin{aligned}
\mathcal{M}^-_{\lambda,\Lambda}(D^2\Phi) &= \lambda\mu_1 + \Lambda\sum_{i=2}^n \mu_i\\
&= \lambda\frac{x_n^\alpha}{\rho^{\beta+2}}\left[\beta\left(\frac{1}{2}\left(\frac{\Lambda}{\lambda}+1\right)(\beta-2\alpha) - \frac{\Lambda}{\lambda}(n-2)\right)\right.\\
&\quad \left. + \frac{\alpha}{2}(\alpha-1)\left(\frac{\Lambda}{\lambda}+1\right)\left(\frac{\rho}{x_n}\right)^2 - \frac{1}{2}\left(\frac{\Lambda}{\lambda}-1\right)\sqrt{\mathcal{D}}\right].
\end{aligned} \tag{13}$$

Furthermore, the radicand $\mathcal{D}$ can be easily estimated as follows:

$$\begin{aligned}
\mathcal{D} &\leqslant \left(\beta(\beta-2\alpha+2)+\alpha(\alpha-1)\left(\frac{\rho}{x_n}\right)^2\right)^2 + 4\alpha\beta(\beta-2\alpha+2)\left(\frac{\rho}{x_n}\right)^2\\
&= \beta^2(\beta-2\alpha+2)^2 + \alpha^2(\alpha-1)^2\left(\frac{\rho}{x_n}\right)^4 + 2\alpha\beta(\alpha+1)(\beta-2\alpha+2)\left(\frac{\rho}{x_n}\right)^2\\
&\leqslant \left(\beta(\beta-2\alpha+2)+\alpha(\alpha+1)\left(\frac{\rho}{x_n}\right)^2\right)^2.
\end{aligned}$$

Inserting the above inequality into (13) then yields

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2\Phi) \geqslant \lambda\frac{x_n^\alpha}{\rho^{\beta+2}}\left[\beta\left(\beta-2\alpha-\frac{\Lambda}{\lambda}(n-1)+1\right)+\alpha\left(\alpha-\frac{\Lambda}{\lambda}\right)\left(\frac{\rho}{x_n}\right)^2\right]. \tag{14}$$

The choices $\alpha = \frac{\Lambda}{\lambda} \geqslant 1$ and $\beta = \frac{\Lambda}{\lambda}(n-1)+2\alpha-1 = \frac{\Lambda}{\lambda}(n+1)-1 \geqslant \alpha$ then give (11). □

Remark 2.4. Let us point out that for $\Lambda = \lambda$ the function Φ coincides with the harmonic function $\frac{x_n}{|x|^n}$, and equality holds in (11). For $\Lambda > \lambda$, different choices for the exponents $\beta \geqslant \alpha > 0$ are possible to make $\Phi(x) = \frac{x_n^\alpha}{|x|^\beta}$ a solution of (11). Indeed, from the above proof it follows that Φ satisfies (11) if and only if the following inequality holds true

$$\begin{aligned}&\frac{\beta}{\frac{\Lambda}{\lambda}-1}\left(\left(\frac{\Lambda}{\lambda}+1\right)(\beta-2\alpha)-2\frac{\Lambda}{\lambda}(n-2)\right)t+\alpha(\alpha-1)\left(\frac{\Lambda}{\lambda}+1\right)\\&\quad\geqslant\sqrt{\beta(\beta+2)(\beta-2\alpha)(\beta-2\alpha+2)t^2+\alpha^2(\alpha-1)^2+2\alpha\beta(\alpha+1)(\beta-2\alpha+2)t}\\&\quad\text{for all } t\in[0,1]\quad\left(t=\left(\frac{x_n}{|x|}\right)^2\right).\end{aligned}\tag{15}$$

First, we note that testing (15) for $t = 0$ yields $\alpha > 1$. Then, we observe that (15) is satisfied also by $\beta = 2\alpha$, $\alpha = 2\frac{\Lambda}{\lambda}(n-1) - 1$. However, the smaller scaling exponent $\beta - \alpha = \frac{\Lambda}{\lambda}n - 1$ selected in Theorem 2.3 will produce better estimates.

Remark 2.5. As far as supersolutions for operator $\mathcal{M}^-_{\lambda,\Lambda}$ are concerned, it is easy to prove that the function, already found in [19],

$$\hat{\Phi}(x) = \frac{x_n}{|x|^{\frac{\Lambda}{\lambda}(n-1)+1}}$$

satisfies, in the classical sense,

$$\mathcal{M}^-_{\lambda,\Lambda}\big(D^2\hat{\Phi}\big) \leqslant 0 \quad \text{in } \mathbb{R}^n_+.$$

This can be checked either directly, by using formulas (12), or by observing that $\mathcal{M}^-_{\lambda,\Lambda}$ is superadditive and $\hat{\Phi}$ is, up to a negative constant, the derivative with respect to x_n of the well-known radial solution for $\mathcal{M}^-_{\lambda,\Lambda}$

$$\phi(x) = \begin{cases} -\log|x| & \text{if } \beta = 2,\\ |x|^{2-\beta} & \text{if } \beta > 2,\end{cases}\tag{16}$$

with $\beta = \frac{\Lambda}{\lambda}(n-1)+1$.

The subsolution Φ given in Theorem 2.3 can be used to estimate solutions and supersolutions by means of extended comparison principles of Phragmén–Lindelöf type, such as the ones given in [3,19]. We present here another form of comparison principle, namely a nonlinear three-surfaces version of the classical Hadamard three-circles theorem. Let us recall, see e.g. [20], that this classical result provides a decay estimate at infinity for entire nonnegative superharmonic functions. More precisely, by comparing a nonnegative function u superharmonic in $\mathbb{R}^n$ with the fundamental solution, one has that the function $m(r) = \inf_{B_r} u$ satisfies the concavity inequality

$$m(r) \geqslant \begin{cases} \frac{m(r_2)\log(r_1/r)+m(r_1)\log(r/r_2)}{\log(r_1/r_2)} & \text{if } n=2,\\ \frac{m(r_2)(r^{2-n}-r_1^{2-n})+m(r_1)(r_2^{2-n}-r^{2-n})}{(r_2^{2-n}-r_1^{2-n})} & \text{if } n>2,\end{cases}$$

for every fixed $r_1 > r_2 > 0$ and for all $r_2 \leqslant r \leqslant r_1$. This immediately yields that u is constant if $n = 2$ (Liouville theorem), and that $r \in (0, +\infty) \mapsto r^{n-2}m(r)$ is nondecreasing if $n \geqslant 3$.

The same argument can be used in the fully nonlinear framework, see [11], where it has been proved that if u is a bounded from below solution of $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leqslant 0$ in $\mathbb{R}^n$, then the infimum function $m(r)$ satisfies

$$m(r) \geqslant \begin{cases} \frac{m(r_2)\log(r_1/r)+m(r_1)\log(r/r_2)}{\log(r_1/r_2)} & \text{if } \beta=2,\\ \frac{m(r_2)(r^{2-\beta}-r_1^{2-\beta})+m(r_1)(r_2^{2-\beta}-r^{2-\beta})}{(r_2^{2-\beta}-r_1^{2-\beta})} & \text{if } \beta>2,\end{cases}\tag{17}$$

with $\beta = \frac{\Lambda}{\lambda}(n-1)+1$. This has been accomplished by comparing u in annular domains with the new "fundamental solution", that is the radial solution of $\mathcal{M}^-_{\lambda,\Lambda}(D^2\phi) = 0$ in $\mathbb{R}^n \setminus \{0\}$ given by (16).

In order to obtain analogous results in $\mathbb{R}^n_+$, we have to consider suitable subdomains (suggested by the subsolution Φ of Theorem 2.3) where the comparison principle can be applied. For $x \in \mathbb{R}^n_+$, let us define the positive function

$$d = d(x) = \left(\frac{|x|}{x_n}\right)^k |x|, \tag{18}$$

with

$$k = \frac{\Lambda - \lambda}{\Lambda n},$$

and observe that Φ can be written as

$$\Phi(x) = \frac{x_n}{d^{\frac{\Lambda}{\lambda} n}}.$$

Let us also introduce, for every $r > 0$, the sub-level sets

$$\mathcal{B}_r = \big\{ x \in \mathbb{R}^n_+ \,\big|\, d(x) < r \big\}. \tag{19}$$

We notice that, for $\Lambda = \lambda$, $d(x)$ reduces to $|x|$ and the set $\mathcal{B}_r$ is nothing but the upper halfball $B_r^+ = B_r \cap \mathbb{R}^n_+$. In the case $\Lambda > \lambda$, $\mathcal{B}_r$ is an open subset of B_r^+, being $d(x) \geqslant |x|$. It is rotationally symmetric around the x_n-axis and it satisfies

$$\partial \mathcal{B}_r = \big\{ x \in \mathbb{R}^n_+ \,\big|\, d = r \big\} \cup \big\{(0,0)\big\}, \qquad \partial \mathcal{B}_r \cap \partial B_r^+ = \big\{(0,0), (0,r)\big\}. \tag{20}$$

Let us consider now a lower semicontinuous function $u : \overline{\mathbb{R}^n_+} \to [0, +\infty]$ satisfying in the viscosity sense

$$u \geqslant 0, \qquad \mathcal{M}^-_{\lambda,\Lambda}\big(D^2 u\big) \leqslant 0 \quad \text{in } \mathbb{R}^n_+. \tag{21}$$

By the strong maximum principle, if u does not vanish identically then it is strictly positive in $\mathbb{R}^n_+$. Therefore, by translating upward the domain if necessary, we can assume that u is strictly positive on the closure $\overline{\mathbb{R}^n_+}$. For positive r let us define the function

$$\mu(r) = \inf_{x \in \mathcal{B}_r} \frac{u(x)}{x_n}. \tag{22}$$

Some immediate properties of $\mu(r)$ are summarized in the following lemma.

Lemma 2.6. *Let u be a positive lower semicontinuous function in $\overline{\mathbb{R}^n_+}$ satisfying (21), and let $\mu(r)$ be defined by (22). Then, for every $r > 0$, there exists a point $\hat{x} \in \partial \mathcal{B}_r \cap \mathbb{R}^n_+$ such that*

$$\mu(r) = \frac{u(\hat{x})}{\hat{x}_n}.$$

In particular, $\mu(r)$ is a positive and decreasing function of $r \in (0, +\infty)$.

Proof. The function $\frac{u(x)}{x_n}$ is positive and lower semicontinuous in $\overline{\mathbb{R}^n_+}$, so that the infimum $\mu(r)$ actually is a minimum on $\overline{\mathcal{B}_r}$, attained at some point belonging to $\overline{\mathcal{B}_r} \cap \mathbb{R}^n_+$. Let us consider the function

$$v_r(x) = u(x) - \mu(r) x_n,$$

which is nonnegative in $\overline{\mathcal{B}_r}$ and satisfies $\mathcal{M}^-_{\lambda,\Lambda}(D^2 v_r) \leqslant 0$ in $\mathcal{B}_r$. By the maximum principle the minimum of v_r on $\overline{\mathcal{B}_r}$ is attained on $\partial \mathcal{B}_r$. On the other hand, we have $\min_{\overline{\mathcal{B}_r}} v_r = 0$ and $v_r = u > 0$ for $x_n = 0$, so that from (20) the first part of the statement follows.

Observing further that, for every $R > r > 0$, one has $\partial \mathcal{B}_R \cap \partial \mathcal{B}_r \cap \mathbb{R}^n_+ = \emptyset$, from the above it follows that $v_R(x) > 0$ in $\overline{\mathcal{B}_r}$, that is

$$\frac{u(x)}{x_n} > \mu(R) \quad \forall x \in \overline{\mathcal{B}_r} \cap \mathbb{R}^n_+$$

and the claim is completely proved. □

We can now prove our nonlinear Hadamard type theorem.

Theorem 2.7. *Let $u:\overline{\mathbb{R}^n_+}\to[0,+\infty]$ be a lower semicontinuous function satisfying (21). Then the function $\mu(r)$ defined by (22) is a concave function of $r^{-\frac{\Lambda}{\lambda}n}$, i.e. for every fixed $R>r>0$ and for all $r\leqslant\rho\leqslant R$ one has*

$$\mu(\rho)\geqslant\frac{\mu(r)(\rho^{-\frac{\Lambda}{\lambda}n}-R^{-\frac{\Lambda}{\lambda}n})+\mu(R)(r^{-\frac{\Lambda}{\lambda}n}-\rho^{-\frac{\Lambda}{\lambda}n})}{r^{-\frac{\Lambda}{\lambda}n}-R^{-\frac{\Lambda}{\lambda}n}}. \tag{23}$$

Consequently, we have that

$$r\in(0,+\infty)\mapsto\mu(r)r^{\frac{\Lambda}{\lambda}n}\quad\text{is nondecreasing.} \tag{24}$$

Proof. We fix $R>r>0$ and we apply the comparison principle in the domain $\mathcal{B}_R\setminus\mathcal{B}_r$, where we consider the function

$$\Phi(x)=x_n\big(c_1d(x)^{-\frac{\Lambda}{\lambda}n}+c_2\big),$$

with constants $c_1\geqslant0$ and $c_2\in\mathbb{R}$ to be appropriately fixed. Notice that Φ has a continuous extension in $\overline{\mathcal{B}_R\setminus\mathcal{B}_r}$ vanishing at the origin. By Theorem 2.3, we have in particular

$$\mathcal{M}^-_{\lambda,\Lambda}\big(D^2\Phi\big)\geqslant0\quad\text{in }\mathcal{B}_R\setminus\mathcal{B}_r.$$

Let us now fix the constants $c_1\geqslant0$ and $c_2\in\mathbb{R}$ in such a way that $\Phi\leqslant u$ on $\partial(\mathcal{B}_R\setminus\mathcal{B}_r)$. We impose

$$\begin{cases}c_1r^{-\frac{\Lambda}{\lambda}n}+c_2=\mu(r),\\ c_1R^{-\frac{\Lambda}{\lambda}n}+c_2=\mu(R),\end{cases}$$

which yields

$$\begin{cases}c_1=\dfrac{\mu(r)-\mu(R)}{r^{-\frac{\Lambda}{\lambda}n}-R^{-\frac{\Lambda}{\lambda}n}}\geqslant0,\\ c_2=\dfrac{\mu(R)r^{-\frac{\Lambda}{\lambda}n}-\mu(r)R^{-\frac{\Lambda}{\lambda}n}}{r^{-\frac{\Lambda}{\lambda}n}-R^{-\frac{\Lambda}{\lambda}n}}.\end{cases}$$

With this choice of c_1 and c_2 we can apply the comparison principle to the subsolution Φ and to the supersolution u in the domain $\mathcal{B}_R\setminus\mathcal{B}_r$, which gives $\Phi\leqslant u$, that is

$$\frac{u(x)}{x_n}\geqslant\frac{\mu(r)(d(x)^{-\frac{\Lambda}{\lambda}n}-R^{-\frac{\Lambda}{\lambda}n})+\mu(R)(r^{-\frac{\Lambda}{\lambda}n}-d(x)^{-\frac{\Lambda}{\lambda}n})}{r^{-\frac{\Lambda}{\lambda}n}-R^{-\frac{\Lambda}{\lambda}n}}.$$

By Lemma 2.6, for every $r\leqslant\rho\leqslant R$ there exists a point $\hat{x}$ such that $d(\hat{x})=\rho$ and $\mu(\rho)=\frac{u(\hat{x})}{\hat{x}_n}$; by applying the above inequality for $x=\hat{x}$, we then obtain (23).

By observing further that (23) implies

$$\mu(\rho)\geqslant\frac{\mu(r)(\rho^{-\frac{\Lambda}{\lambda}n}-R^{-\frac{\Lambda}{\lambda}n})}{r^{-\frac{\Lambda}{\lambda}n}-R^{-\frac{\Lambda}{\lambda}n}}$$

and by letting $R\to+\infty$, we finally get the monotonicity property (24). □

As a consequence of Theorem 2.7 we can obtain more specific bounds on the scaling exponent of the positive singular homogeneous solution, which is already known to exist. By a singular homogeneous function we mean a positively homogeneous function with negative homogeneity exponent. We recall that, by the results of [19] and their extensions in [3], it is known that there exists a unique positive exponent α, and a unique C^2-function $\phi_\alpha:[0,\frac{\pi}{2}]\to[0,+\infty)$, with $\phi_\alpha(0)=0$, $\phi_\alpha(\theta)>0$ for $0<\theta\leqslant\frac{\pi}{2}$, such that

$$\Phi_\alpha(x)=|x|^{-\alpha}\phi_\alpha\left(\arcsin\left(\frac{x_n}{|x|}\right)\right) \tag{25}$$

is the unique (up to normalization) singular homogeneous and continuous in $\overline{\mathbb{R}^n_+}\setminus\{0\}$ solution of

$$\Phi_\alpha>0,\qquad\mathcal{M}^-_{\lambda,\Lambda}\big(D^2\Phi_\alpha\big)=0\quad\text{in }\mathbb{R}^n_+,\qquad\Phi_\alpha=0\quad\text{on }\partial\mathbb{R}^n_+\setminus\{0\}.$$

Moreover, as observed in [19], a comparison argument applied to Φ_α, and the supersolution $\hat{\Phi}$ given in Remark 2.5, yields

$$\alpha \geqslant \frac{\Lambda}{\lambda}(n-1). \tag{26}$$

On the other hand, by Theorem 2.7 we immediately obtain the following upper bound.

Corollary 2.8. *The scaling exponent of the solution Φ_α in (25) satisfies*

$$\alpha \leqslant \frac{\Lambda}{\lambda}n - 1.$$

Proof. It is enough to observe that the function ϕ_α can be normalized in order to satisfy

$$\phi_\alpha(\theta) \leqslant \sin\theta \quad \text{for all } \theta \in \left[0, \frac{\pi}{2}\right].$$

Therefore, the infimum function $\mu(r)$ for Φ_α satisfies $\mu(r) \leqslant \frac{1}{r^{\alpha+1}}$, and (24) is violated for $\alpha > \frac{\Lambda}{\lambda}n - 1$. □

Remark 2.9. We cannot prove that the exponent appearing in the growth condition (24) is sharp, since it is derived by a comparison argument with a subsolution, not a solution. In order to obtain the optimal condition, we can repeat the proof of Theorem 2.7 with the subsolution Φ replaced by the solution Φ_α. In this case, we consider, for positive r, the function

$$m(r) = \inf_{B_r^+} \frac{u(x)}{x_n}.$$

We further observe that, by Hopf's Lemma, the function ϕ_α satisfies $\phi_\alpha'(0) > 0$, so that, up to a normalization, one has

$$c\sin\theta \leqslant \phi_\alpha(\theta) \leqslant \sin\theta \quad \text{for all } \theta \in \left[0, \frac{\pi}{2}\right],$$

with $c = \inf_{(0,\frac{\pi}{2})} \frac{\phi_\alpha(\theta)}{\sin\theta} > 0$ depending only on Λ, λ and n. As in the proof of Theorem 2.7, the comparison principle applied in the upper annular domain $B_R^+ \setminus B_r^+$ then yields that $m(r)$ satisfies

$$m(\rho) \geqslant \frac{m(r)(c\rho^{-(\alpha+1)} - R^{-(\alpha+1)}) + m(R)(r^{-(\alpha+1)} - c\rho^{-(\alpha+1)})}{r^{-(\alpha+1)} - R^{-(\alpha+1)}}$$

for any fixed $R > r > 0$ and all $r \leqslant \rho \leqslant R$. Hence,

$$\rho^{\alpha+1} m(\rho) \geqslant cr^{\alpha+1} m(r) \quad \text{for all } \rho \geqslant r.$$

Remark 2.10. All the statements we have given in this section for operator $\mathcal{M}^-_{\lambda,\Lambda}$ correspond to analogous results for operator $\mathcal{M}^+_{\lambda,\Lambda}$. In particular, the function

$$\hat{\Psi}(x) = \frac{x_n}{|x|^{\frac{\lambda}{\Lambda}(n-1)+1}}$$

satisfies, in the classical sense,

$$\mathcal{M}^+_{\lambda,\Lambda}\big(D^2\hat{\Psi}\big) \geqslant 0 \quad \text{in } \mathbb{R}^n_+.$$

Note that $\hat{\Psi}$ is, up to a negative constant, the partial derivative with respect to x_n of the radial solution for $\mathcal{M}^+_{\lambda,\Lambda}$

$$\psi(x) = \begin{cases} -|x|^{2-\beta} & \text{if } \beta < 2, \\ -\log|x| & \text{if } \beta = 2, \\ |x|^{2-\beta} & \text{if } \beta > 2, \end{cases}$$

with $\beta = \frac{\lambda}{\Lambda}(n-1)+1$. Therefore, the same proof of Theorem 2.7, which is in this case even simpler, yields that, if u is a supersolution for $\mathcal{M}^+_{\lambda,\Lambda}$, then the function

$$m(r) = \inf_{B_r^+} \frac{u(x)}{x_n}$$

is a concave function of $r^{-(\frac{\lambda}{\Lambda}(n-1)+1)}$, i.e. for every fixed $R > r > 0$ and for all $r \leqslant \rho \leqslant R$ one has

$$m(\rho) \geqslant m(r)\frac{\rho^{-(\frac{\lambda}{\Lambda}(n-1)+1)} - R^{-(\frac{\lambda}{\Lambda}(n-1)+1)}}{r^{-(\frac{\lambda}{\Lambda}(n-1)+1)} - R^{-(\frac{\lambda}{\Lambda}(n-1)+1)}} + m(R)\frac{r^{-(\frac{\lambda}{\Lambda}(n-1)+1)} - \rho^{-(\frac{\lambda}{\Lambda}(n-1)+1)}}{r^{-(\frac{\lambda}{\Lambda}(n-1)+1)} - R^{-(\frac{\lambda}{\Lambda}(n-1)+1)}}.$$

Hence,

$$r \in (0,+\infty) \mapsto m(r) r^{\frac{\lambda}{\Lambda}(n-1)+1} \quad \text{is nondecreasing.}$$

As far as supersolutions are concerned, the same proof of Theorem 2.3 carried out for operator $\mathcal{M}^+_{\lambda,\Lambda}$ shows that, under the assumption $\frac{\lambda}{\Lambda}n \geqslant 1$, the function

$$\Psi(x) = \frac{x_n^{\frac{\lambda}{\Lambda}}}{|x|^{\frac{\lambda}{\Lambda}(n+1)-1}}$$

satisfies, in the classical sense,

$$\mathcal{M}^+_{\lambda,\Lambda}\big(D^2\Psi\big) \leqslant 0 \quad \text{in } \mathbb{R}^n_+.$$

It then follows that the positive singular homogeneous solution Ψ_α for operator $\mathcal{M}^+_{\lambda,\Lambda}$ has a positive scaling exponent $\alpha = \alpha(\mathcal{M}^+_{\lambda,\Lambda})$ satisfying

$$\frac{\lambda}{\Lambda}n - 1 \leqslant \alpha \leqslant \frac{\lambda}{\Lambda}(n-1).$$

This improves the lower bound $\alpha \geqslant \frac{\lambda}{\Lambda}(n-1)-1$ proved in [19] by comparing Ψ_α with the radial (super)solution ψ in the case $\frac{\lambda}{\Lambda}(n-1) > 1$.

3. Explicit subsolutions and a Liouville type theorem

In this section we give an elementary proof, purely based on the comparison principle, of the following Liouville type theorem for inequalities with superlinear zero order terms.

Theorem 3.1. *Let $n \geqslant 2$ and $1 \leqslant p \leqslant \frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}$. Then, $u \equiv 0$ is the only nonnegative viscosity solution of inequality*

$$\mathcal{M}^-_{\lambda,\Lambda}\big(D^2u\big) + u^p \leqslant 0 \quad \text{in } \mathbb{R}^n_+. \tag{27}$$

To prove the above result in the limiting case $p = \frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}$, we will compare the supersolution u with an explicit subsolution of the equation

$$-\mathcal{M}^-_{\lambda,\Lambda}\big(D^2v\big) = \left(\frac{x_n}{d^{\frac{\Lambda}{\lambda}n}}\right)^{\frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}},$$

where $d = d(x)$ is as in (18). Such a subsolution is constructed in the following preliminary result.

Lemma 3.2. *There exist positive constants $a, b > 0$ and $d_0 \geqslant 1$, depending only on λ, Λ and n, such that the function*

$$\Gamma(x) = \frac{x_n}{d^{\frac{\Lambda}{\lambda}n}}\left(a\ln d + b\left(\frac{x_n}{|x|}\right)^2\right) \tag{28}$$

satisfies, in the classical sense,

$$-\mathcal{M}^-_{\lambda,\Lambda}(D^2\Gamma) \leqslant \left(\frac{x_n}{d^{\frac{\Lambda}{\lambda}n}}\right)^{\frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}} \quad \text{in } \mathbb{R}^n_+ \setminus \overline{\mathcal{B}_{d_0}}. \tag{29}$$

Proof. Let us consider the two functions

$$\Gamma_1(x) = \frac{x_n}{d^{\frac{\Lambda}{\lambda}n}} \ln d$$

and

$$\Gamma_2(x) = \frac{x_n}{d^{\frac{\Lambda}{\lambda}n}}\left(\frac{x_n}{\rho}\right)^2 = \frac{x_n^{\frac{\Lambda}{\lambda}+2}}{\rho^{\frac{\Lambda}{\lambda}(n+1)+1}},$$

with $\rho = |x|$. If $a, b > 0$ and Γ is given by (28), then, being $\mathcal{M}^-_{\lambda,\Lambda}$ superadditive and positively homogeneous, we have that

$$-\mathcal{M}^-_{\lambda,\Lambda}(D^2\Gamma) \leqslant -a\mathcal{M}^-_{\lambda,\Lambda}(D^2\Gamma_1) - b\mathcal{M}^-_{\lambda,\Lambda}(D^2\Gamma_2). \tag{30}$$

Therefore, in order to prove (29), we estimate separately the two terms appearing in the right hand side of (30).

As far as Γ_1 is concerned, definition (18) of d and a direct computation show that

$$\begin{aligned} D^2\Gamma_1(x) = (k+1)\frac{x_n}{d^{\frac{\Lambda}{\lambda}n+2}}\left(\frac{\rho}{x_n}\right)^{2k}\Bigg\{&\left[\frac{\Lambda}{\lambda}n\left(\frac{\Lambda}{\lambda}(n+1)+1\right)\ln d - 2\frac{\Lambda}{\lambda}(n+1)\right]\frac{x}{\rho}\otimes\frac{x}{\rho} \\ &+ \frac{k}{k+1}\left(\frac{\rho}{x_n}\right)^2\left(\left(\frac{\Lambda}{\lambda}\right)^2 n\ln d - 2\frac{\Lambda}{\lambda}+1\right)e_n\otimes e_n \\ &- \frac{\rho}{x_n}\left(\left(\frac{\Lambda}{\lambda}\right)^2 n\ln d - 2\frac{\Lambda}{\lambda}+1\right)\left(\frac{x}{\rho}\otimes e_n + e_n\otimes\frac{x}{\rho}\right) \\ &- \left(\frac{\Lambda}{\lambda}n\ln d - 1\right)I_n\Bigg\} \end{aligned}$$

with $k = \frac{\Lambda-\lambda}{\Lambda n}$. According to Lemma 2.1, the eigenvalues $\mu_1, \dots, \mu_n$ of $D^2\Gamma_1(x)$ are

$$\begin{aligned} \mu_1 = \frac{k+1}{2}\frac{x_n}{d^{\frac{\Lambda}{\lambda}n+2}}\left(\frac{\rho}{x_n}\right)^{2k}\Bigg[&\frac{\Lambda}{\lambda}n\left(\frac{\Lambda}{\lambda}(n-1)-1\right)\ln d - 2\frac{\Lambda}{\lambda}(n-1) \\ &+ \frac{k}{k+1}\left(\frac{\rho}{x_n}\right)^2\left(\left(\frac{\Lambda}{\lambda}\right)^2 n\ln d - 2\frac{\Lambda}{\lambda}+1\right) + \sqrt{\mathcal{D}}\Bigg], \end{aligned}$$

$$\begin{aligned} \mu_2 = \frac{k+1}{2}\frac{x_n}{d^{\frac{\Lambda}{\lambda}n+2}}\left(\frac{\rho}{x_n}\right)^{2k}\Bigg[&\frac{\Lambda}{\lambda}n\left(\frac{\Lambda}{\lambda}(n-1)-1\right)\ln d - 2\frac{\Lambda}{\lambda}(n-1) \\ &+ \frac{k}{k+1}\left(\frac{\rho}{x_n}\right)^2\left(\left(\frac{\Lambda}{\lambda}\right)^2 n\ln d - 2\frac{\Lambda}{\lambda}+1\right) - \sqrt{\mathcal{D}}\Bigg], \end{aligned}$$

$$\mu_i = -(k+1)\frac{x_n}{d^{\frac{\Lambda}{\lambda}n+2}}\left(\frac{\rho}{x_n}\right)^{2k}\left(\frac{\Lambda}{\lambda}n\ln d - 1\right), \quad 3 \leqslant i \leqslant n,$$

where

$$\begin{aligned} \mathcal{D} = &\left[\left(\frac{\Lambda}{\lambda}(n-1)+1\right)\left(\frac{\Lambda}{\lambda}n\ln d - 2\right) + \frac{k}{k+1}\left(\frac{\rho}{x_n}\right)^2\left(\left(\frac{\Lambda}{\lambda}\right)^2 n\ln d - 2\frac{\Lambda}{\lambda}+1\right)\right]^2 \\ &+ 4\frac{\frac{\Lambda}{\lambda}(n-1)+1}{\frac{\Lambda}{\lambda}(n+1)-1}\left(1-\left(\frac{x_n}{\rho}\right)^2\right)\left(\frac{\rho}{x_n}\right)^2\left(\left(\frac{\Lambda}{\lambda}\right)^2 n\ln d - 2\frac{\Lambda}{\lambda}+1\right)\left(\frac{\Lambda}{\lambda}n\ln d - 1\right). \end{aligned} \tag{31}$$

For $d \geqslant d_0$, with d_0 depending only on Λ, λ and n, it is easy to see that $\mu_1 \geqslant 0$ and $\mu_i \leqslant 0$ for $2 \leqslant i \leqslant n$. Therefore, one has

$$\begin{aligned}\mathcal{M}^-_{\lambda,\Lambda}\big(D^2\Gamma_1(x)\big) &= \lambda\mu_1 + \Lambda\sum_{i=2}^{n}\mu_i \\ &= -\frac{\lambda}{2}(k+1)\frac{x_n}{d^{\frac{\Lambda}{\lambda}n+2}}\left(\frac{\rho}{x_n}\right)^{2k}\left[\frac{\Lambda}{\lambda}\left(\frac{\Lambda}{\lambda}(n-1)+1\right)\left(-n\left(\frac{\Lambda}{\lambda}-1\right)\ln d+2\right)\right. \\ &\quad \left. -\frac{k}{k+1}\left(\frac{\Lambda}{\lambda}+1\right)\left(\frac{\rho}{x_n}\right)^2\left(\left(\frac{\Lambda}{\lambda}\right)^2 n\ln d - 2\frac{\Lambda}{\lambda}+1\right)+\left(\frac{\Lambda}{\lambda}-1\right)\sqrt{\mathcal{D}}\right]. \end{aligned} \tag{32}$$

Moreover, from (31) it follows that

$$\begin{aligned}\mathcal{D} &\leqslant \left[\left(\frac{\Lambda}{\lambda}(n-1)+1\right)\left(\frac{\Lambda}{\lambda}n\ln d-2\right)+\frac{k}{k+1}\left(\frac{\rho}{x_n}\right)^2\left(\left(\frac{\Lambda}{\lambda}\right)^2 n\ln d-2\frac{\Lambda}{\lambda}+1\right)\right]^2 \\ &\quad +4\frac{\frac{\Lambda}{\lambda}(n-1)+1}{\frac{\Lambda}{\lambda}(n+1)-1}\left(\frac{\rho}{x_n}\right)^2\left(\left(\frac{\Lambda}{\lambda}\right)^2 n\ln d-2\frac{\Lambda}{\lambda}+1\right)\left(\frac{\Lambda}{\lambda}n\ln d-1\right) \\ &\leqslant \left[\left(\frac{\Lambda}{\lambda}(n-1)+1\right)\left(\frac{\Lambda}{\lambda}n\ln d-\frac{2\frac{\Lambda}{\lambda}}{\frac{\Lambda}{\lambda}+1}\right)+\frac{\frac{\Lambda}{\lambda}+1}{\frac{\Lambda}{\lambda}(n+1)-1}\left(\frac{\rho}{x_n}\right)^2\left(\left(\frac{\Lambda}{\lambda}\right)^2 n\ln d-2\frac{\Lambda}{\lambda}+1\right)\right]^2. \end{aligned}$$

The above estimate plugged into (32) gives

$$\begin{aligned}\mathcal{M}^-_{\lambda,\Lambda}\big(D^2\Gamma_1(x)\big) &\geqslant -\frac{2\lambda}{\frac{\Lambda}{\lambda}+1}\frac{(\frac{\Lambda}{\lambda}(n-1)+1)(\frac{\Lambda}{\lambda}(n+1)-1)}{n}\frac{x_n}{d^{\frac{\Lambda}{\lambda}n+2}}\left(\frac{\rho}{x_n}\right)^{2k} \\ &= -\frac{c_1}{d^{\frac{\Lambda}{\lambda}n+1}}\left(\frac{x_n}{\rho}\right)^{1-k} \end{aligned} \tag{33}$$

with $c_1 = \frac{2\lambda}{\frac{\Lambda}{\lambda}+1}\frac{(\frac{\Lambda}{\lambda}(n-1)+1)(\frac{\Lambda}{\lambda}(n+1)-1)}{n}$.

Let us now turn to estimate $\mathcal{M}^-_{\lambda,\Lambda}(D^2\Gamma_2(x))$. By applying inequality (14) with $\alpha = \frac{\Lambda}{\lambda}+2$ and $\beta = \frac{\Lambda}{\lambda}(n+1)+1 > \alpha$, we obtain

$$\begin{aligned}\mathcal{M}^-_{\lambda,\Lambda}\big(D^2\Gamma_2(x)\big) &\geqslant -2\lambda\frac{x_n^{\frac{\Lambda}{\lambda}+2}}{\rho^{\frac{\Lambda}{\lambda}(n+1)+3}}\left(\frac{\Lambda}{\lambda}(n+1)+1-\left(\frac{\Lambda}{\lambda}+2\right)\left(\frac{\rho}{x_n}\right)^2\right) \\ &= -\frac{1}{d^{\frac{\Lambda}{\lambda}n+1}}\left(\frac{x_n}{\rho}\right)^{1-k}\left(c_2\left(\frac{x_n}{\rho}\right)^2-c_3\right), \end{aligned} \tag{34}$$

where $c_2 = 2(\Lambda(n+1)+\lambda)$ and $c_3 = 2(\Lambda+2\lambda)$.

Inequalities (33) and (34) combined with (30) with $a = \frac{c_3}{c_1c_2}$ and $b = \frac{1}{c_2}$ then imply

$$-\mathcal{M}^-_{\lambda,\Lambda}\big(D^2\Gamma\big) \leqslant \frac{1}{d^{\frac{\Lambda}{\lambda}n+1}}\left(\frac{x_n}{\rho}\right)^{3-k}.$$

We finally observe that, since $\frac{\Lambda}{\lambda}(n-1) \geqslant 1$, one has $3-k \geqslant (1+k)\frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}$. Hence

$$-\mathcal{M}^-_{\lambda,\Lambda}\big(D^2\Gamma\big) \leqslant \frac{1}{d^{\frac{\Lambda}{\lambda}n+1}}\left(\frac{x_n}{\rho}\right)^{(1+k)\frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}} = \left(\frac{x_n}{d^{\frac{\Lambda}{\lambda}n}}\right)^{\frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}}. \qquad \square$$

Remark 3.3. Let us observe that in the linear planar case, that is for $\Lambda = \lambda$ and $n = 2$, inequality (29) becomes equality.

Proof of Theorem 3.1. For a contradiction, let us assume that there exists a nontrivial solution u of (27). As in the previous section, by using the strong maximum principle and by translating upward the domain if necessary, we can assume without loss that u is strictly positive in $\overline{\mathbb{R}^n_+}$.

Let us re-scale inequality (27), that is, for every $r > 0$ let us set

$$u_r(x) = u(rx).$$

Then, u_r satisfies

$$u_r > 0 \quad \text{in } \overline{\mathbb{R}^n_+}, \qquad \mathcal{M}^-_{\lambda,\Lambda}\big(D^2 u_r\big) + r^2 u_r^p \leqslant 0 \quad \text{in } \mathbb{R}^n_+. \tag{35}$$

We now test inequality (35) with a suitable cut-off function, chosen constant on the ball $B_{1/2}((0,1))$ centered at $(0,1)$ and having radius $1/2$, and negative outside $B_{3/4}((0,1))$. Precisely, let us select a smooth, concave, nonincreasing function $\zeta : [0,+\infty) \to \mathbb{R}$ satisfying

$$\zeta(t) = \begin{cases} 1 & \text{for } 0 \leqslant t \leqslant 1/2, \\ > 0 & \text{for } 1/2 < t < 3/4, \\ \leqslant 0 & \text{for } t \geqslant 3/4, \end{cases}$$

and let us consider the radial function

$$z(x) = \Big(\inf_{B_{1/2}((0,1))} u_r \Big) \zeta\big(|x - (0,1)|\big).$$

Note that $u_r \geqslant z$ in $\overline{B_{1/2}((0,1))}$, $u_r = z$ at some point in $\partial B_{1/2}((0,1))$, and $u_r > z$ outside $B_{3/4}((0,1))$. Therefore, the infimum of $u - z$ is nonpositive and it is achieved at some point $x^* \in B_{3/4}((0,1)) \setminus B_{1/2}((0,1))$. By definition of viscosity solution of inequality (35), it then follows

$$u_r(x^*)^p \leqslant \frac{C}{r^2} \inf_{B_{1/2}((0,1))} u_r, \tag{36}$$

where

$$\begin{aligned} C &= \sup_{B_{3/4}((0,1))} \big(-\mathcal{M}^-_{\lambda,\Lambda}\big(D^2\zeta\big)\big) \\ &= \sup_{B_{3/4}((0,1))} (-\Lambda\Delta\zeta) = -\Lambda \inf_{[1/2,3/4]} \left(\zeta''(t) + (n-1)\frac{\zeta'(t)}{t} \right) \end{aligned}$$

is a positive constant depending only on Λ and n. Here and throughout in the sequel, we will use c and C to denote positive constants, which may change from line to line, not depending on r.

By the nonlinear Hadamard three-spheres theorem for positive supersolutions of the equation $\mathcal{M}^-_{\lambda,\Lambda}(D^2 u) = 0$ (see (17) and Theorem 3.1 in [11]), we also have

$$\inf_{B_{1/2}((0,1))} u_r \leqslant C \inf_{B_{3/4}((0,1))} u_r,$$

and from (36) it then follows

$$\Big(\inf_{B_{3/4}((0,1))} u_r \Big)^p \leqslant u_r\big(x^*\big)^p \leqslant \frac{C}{r^2} \inf_{B_{3/4}((0,1))} u_r.$$

Now, the contradiction is evident if $p = 1$. For $p > 1$, we re-scale back from u_r to u and we further observe that

$$\inf_{B_{3/4}((0,1))} u_r = \inf_{B_{\frac{3}{4}r}((0,r))} u \geqslant \frac{r}{4} \inf_{B_{\frac{3}{4}r}((0,r))} \frac{u}{x_n} \geqslant \frac{r}{4} \inf_{\mathcal{B}_{2r}} \frac{u}{x_n} = \frac{r}{4}\mu(2r),$$

where μ is defined in (22). Hence, we obtain

$$\mu(r) \leqslant \frac{C}{r^{\frac{p+1}{p-1}}}. \tag{37}$$

If $\frac{p+1}{p-1} > \frac{\Lambda}{\lambda}n$, that is if $p < \frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}$, then (37) contradicts the monotonicity property (24). Thus, only the case $p = \frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}$ remains to be considered. In this case, (37) gives the upper bound

$$r^{\frac{\Lambda}{\lambda}n}\mu(r) \leqslant C \quad \text{for all } r > 0. \tag{38}$$

On the other hand, by (24) we also have

$$r^{\frac{\Lambda}{\lambda}n}\mu(r) \geqslant d_0^{\frac{\Lambda}{\lambda}n}\mu(d_0) = c > 0 \quad \text{for all } r \geqslant d_0,$$

which implies

$$u(x) \geqslant c\frac{x_n}{d(x)^{\frac{\Lambda}{\lambda}n}} \quad \text{for } x \in \mathbb{R}^n_+ \setminus \mathcal{B}_{d_0},$$

where $d_0 > 0$ is given by Lemma 3.2. By inequality (27) with $p = \frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}$ it then follows that u satisfies

$$-\mathcal{M}^-_{\lambda,\Lambda}\big(D^2u\big) \geqslant c\left(\frac{x_n}{d(x)^{\frac{\Lambda}{\lambda}n}}\right)^{\frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1}} \quad \text{in } \mathbb{R}^n_+ \setminus \mathcal{B}_{d_0}. \tag{39}$$

By Lemma 3.2, the opposite inequality is satisfied by $\gamma\Gamma(x)$, where Γ is given by (28) and $0 < \gamma \leqslant c$. If γ is further assumed to satisfy $\gamma \leqslant \mu(d_0)\frac{d_0^{\frac{\Lambda}{\lambda}n}}{a\ln d_o + b}$, then we have

$$\gamma\Gamma(x) \leqslant u(x) \quad \text{on } \partial\mathcal{B}_{d_0}.$$

Moreover, for any fixed $\epsilon > 0$, let $R > 0$ be large enough so that

$$\gamma\Gamma(x) \leqslant \epsilon \quad \text{for } x \in \mathbb{R}^n_+ \setminus \mathcal{B}_R.$$

The comparison principle applied to $\gamma\Gamma$ and $u + \epsilon$ in $\mathcal{B}_R \setminus \overline{\mathcal{B}_{d_0}}$ then gives $\gamma\Gamma(x) \leqslant u(x) + \epsilon$ in $\overline{\mathcal{B}_R} \setminus \mathcal{B}_{d_0}$ for all R sufficiently large. If we let first $R \to \infty$ and then $\epsilon \to 0$, we obtain

$$u(x) \geqslant \gamma\Gamma(x) \quad \text{for } x \in \mathbb{R}^n_+ \setminus \mathcal{B}_{d_0},$$

which yields, by (28),

$$u(x) \geqslant c\frac{x_n}{d(x)^{\frac{\Lambda}{\lambda}n}}\ln d(x) \quad \text{for } x \in \mathbb{R}^n_+ \setminus \mathcal{B}_{d_0}.$$

By Lemma 2.6, this implies that $r^{\frac{\Lambda}{\lambda}n}\mu(r) \geqslant c\ln r$ for all $r \geqslant d_0$, and this contradicts (38). □

Remark 3.4. Theorem 3.1 of course still holds if u is a supersolution in the exterior domain $\mathbb{R}^n_+ \setminus \overline{B_r}$ for any $r > 0$. In fact, in this case inequality (5) is satisfied in the translated halfspace $\{x \in \mathbb{R}^n\colon x_n > r\}$.

Remark 3.5. By the characterization (9) of the critical exponents p_* and p^*, and by the lower bound (26), we know that the Liouville property does not hold for inequality (27) if either $p < -1$ or $p > \frac{\frac{\Lambda}{\lambda}(n-1)+2}{\frac{\Lambda}{\lambda}(n-1)}$. This can be checked also directly, by finding some explicit supersolution u. Indeed, it is immediate to verify that, if $p < -1$, then $u(x) = x_n^\delta$, with $1 > \delta > \frac{2}{1-p}$, is a supersolution in the halfspace $\{x_n \geqslant (\Lambda\delta(1-\delta))^{\frac{1}{\delta(p-1)+2}}\}$. On the other hand, for $p > \frac{\frac{\Lambda}{\lambda}(n-1)+2}{\frac{\Lambda}{\lambda}(n-1)}$ we can consider the function

$$u(x) = \frac{x_n}{|x|^\beta}$$

with $\frac{\Lambda}{\lambda}(n-1) + 1 > \beta > \frac{p+1}{p-1}$, which satisfies, by formula (13) with $\alpha = 1$,

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) = \frac{\beta\lambda}{2}\frac{x_n}{|x|^{\beta+2}}\left[\left(\frac{\Lambda}{\lambda}+1\right)\beta - 2\frac{\Lambda}{\lambda}(n-1) - 2 - \left(\frac{\Lambda}{\lambda}-1\right)\sqrt{\beta^2+4\left(\left(\frac{|x|}{x_n}\right)^2-1\right)}\right]$$
$$\leqslant -\beta\lambda\frac{x_n}{|x|^{\beta+2}}\left(\frac{\Lambda}{\lambda}(n-1)+1-\beta\right) \leqslant -u^p$$

for $x \in \mathbb{R}^n_+ \setminus \overline{B_r}$, with $r = (\lambda\beta(\frac{\Lambda}{\lambda}(n-1)+1-\beta))^{-\frac{1}{\beta(p-1)-p-1}}$.

Let us observe that for $\Lambda > \lambda$, one has $\frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1} < \frac{\frac{\Lambda}{\lambda}(n-1)+2}{\frac{\Lambda}{\lambda}(n-1)}$, so that in this case the existence or nonexistence of solutions for (27) is somehow indeterminate for $\frac{\frac{\Lambda}{\lambda}n+1}{\frac{\Lambda}{\lambda}n-1} < p \leqslant \frac{\frac{\Lambda}{\lambda}(n-1)+2}{\frac{\Lambda}{\lambda}(n-1)}$.

Analogously, by (9) and Remark 2.10, it follows that the inequality

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + u^p \leqslant 0 \quad \text{in } \mathbb{R}^n_+$$

does not have any positive solution for $-1 \leqslant p \leqslant \frac{\frac{\lambda}{\Lambda}(n-1)+2}{\frac{\lambda}{\Lambda}(n-1)}$. Positive solutions do exist if either $p < -1$ or $p > \frac{\frac{\lambda}{\Lambda}n+1}{\frac{\lambda}{\Lambda}n-1}$, provided that $\frac{\lambda}{\Lambda}n > 1$. Note that if $\frac{\lambda}{\Lambda}n \leqslant 1$, no upper bound for $p^*(\mathcal{M}^+_{\lambda,\Lambda})$ is given.

Remark 3.6. By applying the proof of Theorem 5.1 given in [1], with the functions Ψ^+ and Ψ^- there replaced respectively by Φ_α given in (25) and by x_n, and by using Corollary 2.8, a more general result than Theorem 3.1 can be obtained. Precisely, it can be proved the following statement:

Let $r_0 \geqslant 0$, $\gamma > -2$ and $f : (0,+\infty) \to (0,+\infty)$ be continuous with

$$\liminf_{t\to 0} t^{-\frac{\frac{\Lambda}{\lambda}n+1+\gamma}{\frac{\Lambda}{\lambda}n-1}} f(t) > 0 \quad \text{and} \quad \liminf_{t\to+\infty} t^{1+\gamma} f(t) > 0.$$

Then, there does not exist any positive solution of

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) + |x|^\gamma f(u) \leqslant 0 \quad \text{in } \mathbb{R}^n_+ \setminus \overline{B_{r_0}}.$$

Note that the proof relies on Alexandrov–Bakelmann–Pucci estimate and a weakened form of weak Harnack inequality.

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