

Note

# Partition identities and the coin exchange problem<sup>☆</sup>

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Received 15 June 2007

Available online 14 January 2008

Communicated by George Andrews

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## Abstract

The number of partitions of  $n$  into parts divisible by  $a$  or  $b$  equals the number of partitions of  $n$  in which each part and each difference of two parts is expressible as a non-negative integer combination of  $a$  and  $b$ . This generalizes identities of MacMahon and Andrews. The analogous identities for three or more integers (in place of  $a, b$ ) hold in certain cases.

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*Keywords:* Partition identity; Coin problem; Frobenius problem

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## 1. Introduction

A *partition* of  $n$  is an unordered multiset of positive integers (called *parts*) whose sum is  $n$ . For positive integers  $a_1, \dots, a_m$  we denote the set of non-negative integer combinations

$$S = S(a_1, \dots, a_m) := \left\{ \sum_{i=1}^m x_i a_i : x_1, \dots, x_m \in \mathbb{N}_0 \right\},$$

where  $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ .

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<sup>☆</sup> Funded in part by an NSERC Discovery Grant and by Microsoft Research.  
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**Theorem 1.** For positive integers  $n, a_1$  and  $a_2$ , the following are all equinumerous:

- (i) partitions of  $n$  in which each part and each difference between two parts lies in  $S(a_1, a_2)$ ;
- (ii) partitions of  $n$  in which each part appears with multiplicity lying in  $S(a_1, a_2)$ ;
- (iii) partitions of  $n$  in which each part is divisible by  $a_1$  or  $a_2$ .

For example, when  $(n, a_1, a_2) = (13, 3, 4)$ , the three sets of partitions are:

- (i)  $\{(13), (10, 3), (7, 3, 3)\}$ ;
- (ii)  $\{(3, 3, 3, 1, 1, 1, 1), (2, 2, 2, 1, \dots, 1), (1, \dots, 1)\}$ ;
- (iii)  $\{(9, 4), (6, 4, 3), (4, 3, 3, 3)\}$ .

We also establish the following partial extension to three or more integers  $a_1, \dots, a_m$ . Let  $\sqcap$  and  $\sqcup$  denote greatest common divisor and least common multiple respectively.

**Theorem 2.** For any positive integers  $n$  and  $a_1, \dots, a_m$ , the following are equinumerous:

- (i) partitions of  $n$  in which each part and each difference between two parts lies in  $S(a_1, \dots, a_m)$ ;
- (ii) partitions of  $n$  in which each part appears with multiplicity lying in  $S(a_1, \dots, a_m)$ .

If  $a_1, \dots, a_m$  can be ordered such that

$$\forall i = 2, \dots, m, \exists j < i \text{ such that } (a_1 \sqcap \dots \sqcap a_{i-1}) \sqcup a_i = a_j \sqcup a_i, \tag{*}$$

then in addition the following are equinumerous with (i) and (ii):

- (iii) partitions of  $n$  in which each part is divisible by some  $a_i$ .

Note that (\*) holds automatically when  $m = 2$ , so Theorem 1 is a special case of Theorem 2.

## 2. Remarks

To avoid uninteresting cases,  $a_1, \dots, a_m$  should be coprime, and none should be a multiple of another. (Indeed, if the greatest common divisor is  $g > 1$  then Theorem 2 reduces easily to the case  $(n', a'_1, \dots, a'_m) = g^{-1}(n, a_1, \dots, a_m)$ , while if  $a_j$  is a multiple of  $a_i$  then the statements of the theorem are unchanged by removing  $a_j$  from  $a_1, \dots, a_m$ .)

The set  $S$  is sometimes interpreted as describing sums of money that can be formed using coins of given denominations. When  $a_1, \dots, a_m$  are coprime, the complement  $S^C := \mathbb{N}_0 \setminus S$  is finite; see e.g. [10]. The case  $m = 2$  was studied by Sylvester [11], who proved for  $a_1, a_2$  coprime that  $|S^C| = \frac{1}{2}(a_1 - 1)(a_2 - 1)$  and  $\max S^C = (a_1 - 1)(a_2 - 1) - 1$ . The case  $m \geq 3$  was proposed by Frobenius, and is much less well understood in general. An exception is when  $a_1, \dots, a_m$  satisfy a certain condition which is implied by our condition (\*); see [9]. For more information see [10].

When  $m = 2$  we have for example  $S(2, 3)^C = \{1\}$ ;  $S(3, 4)^C = \{1, 2, 5\}$ ;  $S(2, 5)^C = \{1, 3\}$ ;  $S(3, 5)^C = \{1, 2, 4, 7\}$ ;  $S(4, 5)^C = \{1, 2, 3, 6, 7, 11\}$ . Larger sets  $\{a_1, \dots, a_m\}$  satisfying condition (\*) include  $\{4, 6, 9\}$ ;  $\{6, 8, 9\}$ ;  $\{6, 9, 10\}$ ;  $\{p^{m-1}, p^{m-2}q, \dots, q^{m-1}\}$  for  $p, q$  coprime;

$\{\pi/p_1, \dots, \pi/p_m\}$  for  $p_1, \dots, p_m$  pairwise coprime and  $\pi := \prod_i p_i$ . We have for instance  $S(4, 6, 9)^C = \{1, 2, 3, 5, 7, 11\}$ .

In the case  $\{a_1, a_2\} = \{2, 3\}$ , the equality between (i) and (iii) in Theorem 1 gives the following partition identity due to MacMahon [8, pp. 299–300] (see also [3, p. 14, Examples 9–10]).

*The number of partitions of  $n$  into parts not congruent to  $\pm 1$  modulo 6 equals the number of partitions of  $n$  with no consecutive integers and no ones as parts.*

The generalization to  $\{a_1, a_2\} = \{2, 2r + 1\}$ ,  $r \in \mathbb{N}_0$ , was proved (in a form similar to that above) by Andrews [2]. The other cases of Theorems 1 and 2 appear to be new. Other recent work related to MacMahon’s identity appears in [1,4,7]. Somewhat similar identities are proved in [5]. For more information on partitions and partition identities see e.g. [3].

Finally we note that the second assertion in Theorem 2 cannot hold for arbitrary  $a_1, \dots, a_m$  with  $m \geq 3$ . For example, it does not hold for  $\{a_1, a_2, a_3\} = \{2, 3, 5\}$ : we have  $S(2, 3, 5) = S(2, 3)$ , but allowing multiples of 5 in addition to multiples of 2 and 3 clearly increases the number of partitions of type (iii) for some  $n$ .

**3. Proofs**

As remarked above, Theorem 1 is the  $m = 2$  case of Theorem 2. We will prove the two assertions of Theorem 2 separately. The proofs are simpler when  $m = 2$ , and the reader may find it helpful to bear this case in mind throughout.

**Proof of Theorem 2 (first equality).** Fix  $a_1, \dots, a_m$ , and let  $F_n$  and  $M_n$  be the sets of partitions in (i) and (ii) respectively. We will show that  $|F_n| = |M_n|$ .

For a partition  $\lambda = (\lambda_1, \dots, \lambda_r)$  (where  $n = \sum_i \lambda_i$  and  $\lambda_1 \geq \dots \geq \lambda_r$ ), the *conjugate* partition  $\lambda' = (\lambda'_1, \dots, \lambda'_{r'})$  is defined as usual by  $r' = \lambda_1$  and  $\lambda'_i = \max\{j : \lambda_j \geq i\}$ . Since the set  $S$  is closed under addition, the condition that  $\lambda$  has all parts and differences between parts in  $S$  is equivalent to the condition that each *adjacent* pair in the sequence  $\lambda_1, \lambda_2, \dots, \lambda_r, 0$  differs by an element of  $S$ . On the other hand, it is readily seen that the latter condition is equivalent to the condition that  $\lambda'$  has all multiplicities in  $S$  (indeed this holds for any set  $S$ ). Hence conjugation is a bijection between  $F_n$  and  $M_n$ . □

Our proof of the second assertion in Theorem 2 relies on the two simple lemmas below. Given integers  $a_1, \dots, a_m$  we write

$$\ell_i := (a_1 \sqcap \dots \sqcap a_{i-1}) \sqcup a_i.$$

**Lemma 3.** *If  $a_1, \dots, a_m$  satisfy condition (\*) then we have the formal power series identity*

$$\sum_{k \in S(a_1, \dots, a_m)} q^k = \frac{\prod_{i=2}^m (1 - q^{\ell_i})}{\prod_{i=1}^m (1 - q^{a_i})}.$$

In the case when  $m = 2$  and  $a_1, a_2$  are coprime, the above expression has the appealing form  $(1 - q^{a_1 a_2})(1 - q^{a_1})^{-1}(1 - q^{a_2})^{-1}$ , as noted in [12]. Expressions for the left side for  $m = 3$  and arbitrary  $a_1, a_2, a_3$ , are derived in [6,12].

**Proof of Lemma 3.** We use induction on  $m$ . When  $m = 1$  we have

$$\sum_{k \in S(a_1)} q^k = 1 + q^{a_1} + q^{2a_1} + \dots = \frac{1}{1 - q^{a_1}}$$

as required.

For  $m \geq 2$ , clearly any  $k \in S(a_1, \dots, a_m)$  can be expressed as

$$k = xa_m + y, \quad \text{where } x \in \mathbb{N}_0 \text{ and } y \in S(a_1, \dots, a_{m-1}). \tag{1}$$

We claim that under condition (\*), each such  $k$  has a *unique* such representation subject to the additional constraint

$$x < \ell_m/a_m. \tag{2}$$

Once this is proved we obtain

$$\sum_{k \in S(a_1, \dots, a_m)} q^k = (1 + q^{a_m} + q^{2a_m} + \dots + q^{\ell_m - a_m}) \sum_{k \in S(a_1, \dots, a_{m-1})} q^k.$$

By the inductive hypothesis this equals

$$\frac{1 - q^{\ell_m}}{1 - q^{a_m}} \times \frac{\prod_{i=2}^{m-1} (1 - q^{\ell_i})}{\prod_{i=1}^{m-1} (1 - q^{a_i})},$$

which is the required expression.

To check the above claim, let  $j = j(m)$  be as in condition (\*), and write  $d = a_1 \sqcap \dots \sqcap a_{m-1}$ , so that  $\ell_m = d \sqcup a_m = a_j \sqcup a_m$ . Now note that any representation  $k = xa_m + y$  as in (1) that violates (2) may be re-expressed as  $k = (x - \ell_m/a_m)a_m + (y + \ell_m)$ , where  $x - \ell_m/a_m \in \mathbb{N}_0$ , and  $y + \ell_m \in S(a_1, \dots, a_{m-1})$  (since  $\ell_m$  is a multiple of  $a_j$ ). By repeatedly applying this we can reduce  $x$  until (2) is satisfied, as required. To check uniqueness, note that all elements of  $S(a_1, \dots, a_{m-1})$  are divisible by  $d$ , while the  $\ell_m/a_m$  quantities  $0, a_m, 2a_m, \dots, \ell_m - a_m$  are all distinct modulo  $d$  (since  $\ell_m = d \sqcup a_m$ ). Hence we see that no two distinct expressions  $xa_m + y$  satisfying (1), (2) can be equal.  $\square$

Let  $\mathbf{1}[\cdot]$  denote an indicator function and let  $|$  denote “divides.”

**Lemma 4.** *If  $a_1, \dots, a_m$  satisfy condition (\*) then for any positive integer  $k$ ,*

$$\mathbf{1}[a_i | k \text{ for some } i] = \sum_{i=1}^m \mathbf{1}[a_i | k] - \sum_{i=2}^m \mathbf{1}[\ell_i | k].$$

When  $m = 2$  and  $a_1, a_2$  are coprime, the lemma is the familiar inclusion/exclusion formula  $\mathbf{1}[a_1 | k \text{ or } a_2 | k] = \mathbf{1}[a_1 | k] + \mathbf{1}[a_2 | k] - \mathbf{1}[a_1 a_2 | k]$ .

**Proof of Lemma 4.** We use induction on  $m$ . The case  $m = 1$  is trivial. For  $m \geq 2$  we have

$$\begin{aligned} \mathbf{1}[a_i | k \text{ for some } i] &= \mathbf{1}[a_m | k] + \mathbf{1}[a_i | k \text{ for some } i < m] \\ &\quad - \mathbf{1}[a_m | k, \text{ and } a_i | k \text{ for some } i < m]. \end{aligned}$$

We claim that the last condition “ $a_m | k$ , and  $a_i | k$  for some  $i < m$ ” is equivalent to  $\ell_m | k$ . Once this is established, the result follows by substituting the inductive hypothesis and the claim into the above equation.

Turning to the proof of the claim, if the given condition holds then  $a_m|k$  and  $d|k$ , where  $d = a_1 \sqcap \dots \sqcap a_{m-1}$ . So  $k$  is divisible by  $a_m \sqcup d = \ell_m$ . For the converse, recall from (\*) that  $\ell_m = a_m \sqcup a_j$  for some  $j < m$ , so  $\ell_m|k$  implies  $a_m|k$  and  $a_j|k$ .  $\square$

**Proof of Theorem 2 (second equality).** Suppose (\*) holds, and let  $M_n$  and  $D_n$  denote the sets of partitions in (ii) and (iii) respectively. We will show  $|M_n| = |D_n|$ .

Using Lemma 3, the generating function for  $|M_n|$  is

$$G(q) := \sum_{n=0}^{\infty} |M_n| q^n = \prod_{t=1}^{\infty} \left[ \sum_{k \in S} q^{kt} \right] = \prod_{t=1}^{\infty} \frac{\prod_{i=2}^m (1 - q^{\ell_i t})}{\prod_{i=1}^m (1 - q^{a_i t})}.$$

When the product over  $t$  is expanded, the factor  $(1 - q^{\ell_i t})$  contributes a factor  $(1 - q^k)$  in the numerator for each  $k$  that is a non-negative multiple of  $\ell_i$ ; similarly for the factors in the denominator. Thus

$$\begin{aligned} G(q) &= \prod_{k=1}^{\infty} (1 - q^k)^{-\sum_{i=1}^m \mathbf{1}_{[a_i|k]} + \sum_{i=2}^m \mathbf{1}_{[\ell_i|k]}} \\ &= \prod_{k=1}^{\infty} (1 - q^k)^{-\mathbf{1}_{[a_i|k \text{ for some } i]}} = \prod_{\substack{k \geq 1: \\ a_i|k \text{ for some } i}} \frac{1}{1 - q^k}. \end{aligned}$$

(In the second equality we have used Lemma 4.) But the last expression is the generating function for  $|D_n|$ .  $\square$

**4. Questions**

Can Theorems 1 and 2 be given simple bijective proofs? Dan Romik has found an affirmative answer for Theorem 1 (personal communication). Is condition (\*) necessary and sufficient for the identity between (i) and (iii) in Theorem 2? For those  $a_1, \dots, a_m$  not satisfying this identity, are the partitions of type (i) or type (iii) equinumerous with partitions in some other natural classes? Can condition (\*) be expressed in a more natural form?

**Acknowledgments**

I thank Dan Romik, George Andrews and the referees for valuable comments and encouragement.

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