The Hopf algebroids of functions on étale groupoids and their principal Morita equivalence

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Abstract

We show that the bimodules associated to the maps between étale groupoids admit a natural cocommutative coalgebra structure which is preserved under composition. Moreover, we obtain a Hopf algebroid structure on the Connes convolution algebra of an étale groupoid, which is invariant under Morita-equivalence. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The Hopf algebroids were first introduced as the cogroupoid objects in the category of commutative algebras [14]. After that, various non-commutative generalizations have been studied [2,9,10], with aim to obtain, among others, a “quantisation” of Poisson groupoids [8,16]. In this article, we will show that the Connes convolution algebra of functions with compact support on an étale groupoid [3,4,15] provides an example of a Hopf algebroid which is in general non-commutative. This Hopf algebroid enjoys some interesting properties. First, it is cocommutative and the base algebra is a commutative subalgebra of the total algebra, but does not in general lie in the center of the total algebra. Second, the antipode preserves its coalgebra structure, while the counit is compatible with the multiplicative structure in a stronger sense. And finally, it has a property which originates in the fact that an étale groupoid is a principal bundle over its base space, and relates the multiplication, the comultiplication and the antipode in
a new way. The Hopf algebroids which satisfy those properties will be referred to as étale Hopf algebroids.

In the rest of this article we study the bimodules over étale Hopf algebroids, in connection with the notion of Morita equivalence of Hopf algebroids. Our motivation comes again from topology. There are two kinds of maps between smooth groupoids: functors and more general Hilsum–Skandalis maps [5,7] (see also [12]), which are isomorphism classes of principal bibundles. The category of Hilsum–Skandalis maps is in many ways more suitable since it makes Morita (or essentially) equivalent smooth groupoids isomorphic. For example, the quotient map from a foliated manifold to its “space” of leaves (i.e. the holonomy groupoid of the foliation) is a perfectly well-defined Hilsum–Skandalis map but not a functor in general. Analogously, one considers the isomorphism classes of bimodules as generalized maps between algebras, and indeed it was shown in [12] that a Hilsum–Skandalis map between separated étale groupoids gives rise to a bimodule over the corresponding Connes convolution algebras in a natural functorial fashion.

The bimodule associated to a Hilsum–Skandalis map consists of the compactly supported functions on the underlying principal bibundle. We show that there is also a cocommutative coalgebra structure on this bimodule which is compatible with the étale Hopf algebroid structure on the corresponding convolution algebras. This compatibility follows in part from the fact that the original bibundle is principal, hence, we shall use the name “principal” bimodules for the bimodules over the étale Hopf algebroids which are equipped such a compatible coalgebra structure.

Next we show that the tensor product $M \otimes_B N$ of a principal $A$–$B$-bimodule $M$ and principal $B$–$C$-bimodule $N$ is again a principal bimodule in a natural way, so that the principal bimodules may be considered as generalized maps between étale Hopf algebroids. In particular, they may be used to define an equivalence relation between étale Hopf algebroids in the style of Morita which we call principal Morita equivalence. We then conclude that Morita equivalent separated étale groupoids give rise to principally Morita equivalent étale Hopf algebroids. For the importance of this type of results see [1,6,13]. Finally, we show that the equivalence of étale Hopf algebroids is stable in the sense that an étale Hopf algebroid $A$ is principally Morita equivalent to the étale Hopf algebroid of $p \times p$ matrices with coefficients in $A$. In fact, we show that there is a unique (up to isomorphism) principal $\mathcal{M}_p$–$\mathcal{M}_q$-bimodule for any $p,q \in \mathbb{Z}^+ \cup \{\infty\}$, where $\mathcal{M}_p$ stands for the étale Hopf algebroid of $p \times p$ matrices over the base field.

In this paper, we are working with separated smooth étale groupoids only. The smoothness assumption is in fact not essential, since all our results clearly hold true for separated $C^0$ étale groupoids as well. However, we should remark that there are many interesting examples of étale groupoids which are not separated. For instance, the étale holonomy groupoid of a foliation, which is in fact determined up to Morita equivalence, may not be separated, and in fact not even Morita equivalent to a separated one. Nevertheless, the class of separated étale groupoids is quite rich: for example, it includes actions of discrete groups on manifolds, orbifolds and étale holonomy groupoids of analytic or Riemannian foliations.
2. Étale Hopf algebroids

In this section, we shall show that the Connes convolution algebra of compactly supported functions on a separated étale groupoid [3,4] has a natural structure of a Hopf algebroid [2,8,9,14,16]. In fact, it has a structure of an étale Hopf algebroid defined below.

Notation. Throughout this paper we shall assume that our algebras are associative, perhaps without a unit, and over a fixed base field \( \mathbb{F} \). In our examples the base field will be either \( \mathbb{R} \) or \( \mathbb{C} \). An algebra \( A \) has local units in a commutative subalgebra \( A_0 \) of \( A \) if for any \( a_1, a_2, \ldots, a_k \in A \) there exists \( a_0 \in A_0 \) such that \( a_ia_0 = a_0a_i = a_i \) for any \( 1 \leq i \leq k \). Let \( A \) be an algebra with local units in a commutative subalgebra \( A_0 \), and let \( B \) be an algebra with local units in a commutative subalgebra \( B_0 \). Then an \( A-B \)-bimodule \( M \) is called locally \( A_0-B_0 \)-unital if for any \( m_1, m_2, \ldots, m_k \in M \) there exist \( a_0 \in A_0 \) and \( b_0 \in B_0 \) such that \( a_0m_i = m_ib_0 = m_i \) for any \( 1 \leq i \leq k \). If \( C \) is another algebra with local units in a commutative subalgebra \( C_0 \), \( M \) a locally \( A_0-B_0 \)-unital \( A-B \)-bimodule and \( N \) a locally \( B_0-C_0 \)-unital \( B-C \)-bimodule, then \( M \otimes_B N \) is a locally \( A_0-C_0 \)-unital \( A-C \)-bimodule. Moreover, \( B \) itself can be viewed as a locally \( B_0-B_0 \)-unital \( B-B \)-bimodule, and there are canonical isomorphisms of bimodules \( M \otimes_B B \cong M \) and \( B \otimes_B N \cong N \). Note also that for any \( m_1, m_2, \ldots, m_k \in M \) and \( n_1, n_2, \ldots, n_l \in N \) one can find \( b_0 \in B_0 \) such that \( m_i b_0 = m_i \) and \( b_0 n_j = n_j \), for any \( 1 \leq i \leq k \) and \( 1 \leq j \leq l \).

Let \( A \) be an algebra with local units in a commutative subalgebra \( A_0 \), and assume that \((A, \varepsilon)\) is a cocommutative coalgebra structure on \( A \) over the right \( A_0 \)-action. In particular, \( \varepsilon : A \rightarrow A_0 \) and \( \Delta : A \rightarrow A \otimes_{A_0} A \) are homomorphisms of right \( A_0 \)-modules. Here we used the notation \( A \otimes_{A_0} A \) for the tensor product of right \( A_0 \)-modules, to emphasize that the right action of \( A_0 \) on the first and on the second factor is relevant for the tensor product. This makes sense since \( A_0 \) is commutative. We prefer this notation because we want to see \( A \otimes_{A_0} A \) as a right \( A_0 \)-module with respect to the action \((a \otimes a')a_0 = a_0a' \otimes a = a \otimes a_0a' \). We shall regard \( A \otimes_{A_0} A \) as a left \( A \)-module with respect to the left action on the first factor of \( A \otimes_{A_0} A \). Note that there is also a left \( A \)-action on the second factor of \( A \otimes_{A_0} A \) which commutes with the action on the first factor.

Assume also that for any \( a_0 \in A_0 \) and \( a \in A \) we have \( \Delta(a_0a) = a_0\Delta(a) \), or in other words, that \( \Delta \) is a homomorphism of left \( A_0 \)-modules. Denote by \( \sigma : A \otimes_{A_0} A \rightarrow A \otimes_{A_0} A \) the flip, and write \( \Delta(a) = \sum a_i' \otimes a_i'' \). Cocommutativity yields \( \Delta(a_0a) = \sigma(\Delta(a_0a)) = \sigma(a_0\sigma\Delta(a)) \) and hence

\[
\sum a_0a_i' \otimes a_i'' = a_0\Delta(a) = \sigma(a_0\sigma\Delta(a)) = \sum a_i' \otimes a_0a_i''.
\]

Thus, \( \Delta A \) is an \( A_0-A_0 \)-bimodule, and the left \( A_0 \)-module structure of \( \Delta A \) coincide with the left \( A_0 \)-action on the second factor of \( \Delta A \subset A \otimes_{A_0} A \). This implies that for any \( a' \otimes a'' \in A \otimes_{A_0} A \) and any \( \sum a_i' \otimes a_i'' \in \Delta A \), the product

\[
(a' \otimes a'') \left( \sum a_i' \otimes a_i'' \right) = \sum a' a_i' \otimes a'' a_i'' \in A \otimes_{A_0} A
\]
is well defined. We may consider this product as a map \((\mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{A}) \otimes_{\mathcal{A}_0} \mathcal{A} \rightarrow \mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{A}\). In this context, we shall denote by \(\tilde{\Delta} : \mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{A} \rightarrow \mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{A}\) the map given by \(\tilde{\Delta}(a \otimes a') = a\Delta(a')\). Observe that this map is a homomorphism of \(\mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{A}\)-\(\mathcal{A}_0\)-bimodules.

**Definition 2.1.** étale Hopf algebroid \(A\) (total algebra) together with a commutative subalgebra \(\mathcal{A}_0\) (base algebra) in which \(A\) has local units, equipped with a cocommutative coalgebra structure \((\Delta, \varepsilon)\) over the right \(\mathcal{A}_0\)-action and with a linear involution \(\mathcal{S} : A \rightarrow A\) (antipod) such that

1. \(\varepsilon\) restricted to \(\mathcal{A}_0\) is the identity,
2. \(\varepsilon(a'a) = \varepsilon(a)\varepsilon(a')\) for any \(a, a' \in \mathcal{A}\),
3. \(\Delta\) restricted to \(\mathcal{A}_0\) is the canonical embedding \(\mathcal{A}_0 \cong \mathcal{A}_0 \otimes_{\mathcal{A}_0} \mathcal{A}_0 \subset \mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{A}\),
4. \(\Delta\) is a homomorphism of left \(\mathcal{A}_0\)-modules and \(\Delta(a'a) = \Delta(a')\Delta(a)\) for any \(a, a' \in \mathcal{A}\),
5. \(\mathcal{S}\) restricted to \(\mathcal{A}_0\) is the identity,
6. \(\mathcal{S}(a'a) = \mathcal{S}(a)\mathcal{S}(a')\) for any \(a, a' \in \mathcal{A}\),
7. if \(\Delta(a) = \sum_i d_i' \otimes d_{ii}'\), then \(\Delta(\mathcal{S}(a)) = \sum_i \mathcal{S}(d_i') \otimes \mathcal{S}(d_{ii}')\), for any \(a \in \mathcal{A}\), and
8. \((id \otimes \mathcal{S}) \circ \tilde{\Delta} \circ (id \otimes S) \circ \tilde{\Delta} = \mu\).

**Remark.** (1) Observe that for an étale Hopf algebroid \(A = (\mathcal{A}, \mathcal{A}_0, \mathcal{A}, \varepsilon, \mathcal{S})\) the homomorphism \(\tilde{\Delta}\) is in fact an isomorphism, with the inverse \((id \otimes \mathcal{S}) \circ \tilde{\Delta} \circ (id \otimes S)\). Note also that \((S \otimes S) \circ \Delta\) does not equal \(\Delta \circ S\) because these two maps have different targets. Indeed, we have \(\mathcal{S}_0 \otimes \mathcal{A} : \mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{A} \rightarrow \mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{A}\), where \(\mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{A}\) denotes the tensor product with respect to the left actions of \(\mathcal{A}_0\) on both factors. However, the definition implies that there is a well-defined linear isomorphism \(\gamma : \mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{A} \rightarrow (S \otimes S)\mathcal{A}\) given by \(\gamma(\sum_i d_i' \otimes d_{ii}') = \sum_i d_i' \otimes d_{ii}' \in \mathcal{A} \otimes_{\mathcal{A}_0} \mathcal{A}\). To see this, observe that the property (7) in the above definition implies that \(\gamma(\sum_i d_i' \otimes d_{ii}')\) lies in \((S \otimes S)\mathcal{A}\), while the fact that the two left actions of \(\mathcal{A}_0\) on \(\mathcal{A}\) coincide (and hence the two right actions of \(\mathcal{A}_0\) on \((S \otimes S)\mathcal{A}\) coincide) implies that \(\gamma\) is well-defined. The inverse of \(\gamma\) exists by an analogous argument. Therefore, we have

\[\gamma \circ \Delta \circ S = (S \otimes S) \circ \Delta.\]

(2) Let us make some comments about our definition with respect to the various definitions of Hopf algebroids in the literature \([2, 8–10, 16]\) (see the introduction of \([8]\) for a brief comparison of those definitions). First of all, in our case the base algebra \(\mathcal{A}_0\) is a commutative subalgebra of the total algebra \(\mathcal{A}\) and may not be in the center of \(\mathcal{A}\). A part of the structure of a Hopf algebroid are the source map and the target map from \(\mathcal{A}_0\) to \(\mathcal{A}\), but in our special case they are identities. In general, the source and the target map induce a left action, respectively a right action, of the base algebra on the total algebra, but in our case both those actions coincide with our right action of \(\mathcal{A}_0\) on \(\mathcal{A}\). With this it is easy to check that étale Hopf algebroids satisfy the axioms of bialgebroids \([8]\) (see also \([16]\)), but to see that they in fact satisfy the axioms of Hopf algebroids \([8]\) we have to prove that \(S\) is indeed the antipod in the standard sense, i.e. that

\[\mu \circ (S \otimes id) \circ \gamma \circ \Delta = \varepsilon\]
and
\[ \mu \circ (id \otimes S) \circ \Delta = \varepsilon \circ S. \]

Here \( \mu : A \otimes_{A_0} A \to A \) denotes the multiplication. To this end, denote by \( \lambda : A \to A \otimes_{A_0} A \) the map given by \( \lambda(a) = \eta \otimes a \), where \( \eta \) is any element of \( A_0 \) satisfying \( \eta a = a \). Clearly, we have \( \lambda \circ \lambda = \lambda \); therefore,

\[
\begin{align*}
\mu \circ (id \otimes S) \circ \Delta &= \mu \circ (id \otimes S) \circ \lambda \\
&= \mu \circ \lambda^{-1} \circ (id \otimes S) \circ \lambda \\
&= \mu \circ (id \otimes \varepsilon) \circ (id \otimes S) \circ \lambda \\
&= \varepsilon \circ S.
\end{align*}
\]

If we compose both sides of this equation with \( S \) we get

\[
\begin{align*}
\varepsilon &= \mu \circ (id \otimes S) \circ \Delta \circ S \\
&= \mu \circ (id \otimes S) \circ (S \otimes S) \circ \gamma \circ \Delta &= \mu \circ (S \otimes id) \circ \gamma \circ \Delta \quad \text{(involution).}
\end{align*}
\]

(3) Two étale Hopf algebroids \((A, A_0, \varepsilon, \gamma, S)\) and \((A', A'_0, \varepsilon', \gamma', S')\) are isomorphic if there exists an algebra isomorphism \( f : A \to A' \) such that \( f(A_0) = A'_0 \), \( f \otimes f \circ \Delta = \Delta' \circ f \), \( f \circ \varepsilon = \varepsilon' \circ f \) and \( f \circ S = S' \circ f \).

Recall that a groupoid is a small category in which every morphism is invertible. A separated smooth étale groupoid is a groupoid \( G \) such that the set of objects of \( G \) (denoted by \( G_0 \)) and the set of morphisms of \( G \) (also denoted by \( G \) or by \( G_1 \)) are smooth manifolds (finite-dimensional, Hausdorff, without boundary) and such that all the structure maps of \( G \) are smooth local diffeomorphisms (see also [1] or [12]). We shall denote the structure maps of \( G \) as follows: \( s, t : G \to G_0 \) will stand for the source (the domain) map, respectively, the target (the codomain) map, \( inv : G \to G \) for the inverse map, \( com : G \times_{G_0} G \to G \) for the composition map and \( uni : G_0 \to G \) for the unit map. We shall in fact identify \( G_0 \) with \( uni(G_0) \subset G \). In this paper, we will work with separated smooth étale groupoids only, therefore the separated smooth étale groupoids will be referred to simply as étale groupoids.

Let \( G \) be an étale groupoid and let \( C_c^\infty(G) \) be the Connes algebra of (smooth) complex (or real) functions with compact support on \( G \) [3,4] (see also [1]). The product is given by the convolution

\[ (ad')(g'') = \sum_{gg' = g''} a(g) d'(g') \]

for any \( a, d' \in C_c^\infty(G) \) and \( g'' \in G \). The sum is over all possible decompositions of \( g'' \) in \( G \). For the base algebra \( C_c^\infty(G)_0 \) we take the subalgebra of \( C_c^\infty(G) \) of those functions which have support in \( G_0 \subset G \). Note that this subalgebra is commutative and may be identified with the commutative algebra \( C_c^\infty(G_0) \), and that \( C_c^\infty(G) \) has local units in \( C_c^\infty(G_0) \). Define the counit \( \varepsilon : C_c^\infty(G) \to C_c^\infty(G_0) \) by

\[ \varepsilon(a)(x) = \sum_{s(g) = x} a(g) \]
for any $a \in C_c^\infty(G)$ and $x \in G_0$. Here the sum is over all the elements $g$ of $G$ which satisfy $s(g) = x$. The antipode is defined by

$$S(a)(g) = a(g^{-1}).$$

Finally, define the comultiplication $\Delta : C_c^\infty(G) \to C_c^\infty(G) \otimes_{C_c^\infty(G_0)} C_c^\infty(G)$ as follows: let $d : G \to G \times G_0$ be the diagonal open embedding, i.e. $d(g) = (g,g)$. Here $G \times G_0$ denotes the pullback with respect to the source map on both factors. This map gives us an inclusion $C_c^\infty(G) \to C_c^\infty(G \times G_0 G)$. Now we define $\Delta$ to be the composition of this inclusion with the inverse of the isomorphism $\Omega : C_c^\infty(G) \otimes_{C_c^\infty(G_0)} C_c^\infty(G) \to C_c^\infty(G \times G_0 G)$, which is given by $\Omega(a \otimes a')(g,g') = a(g)a'(g')$. For the proof that this is indeed an isomorphism see [12]. The map $\Delta$ can be described as follows: if $a \in C_c^\infty(G)$ has the support in an open subset $U$ of $G$ which is so small that $s|_U$ is injective, then

$$\Delta(a) = a \otimes \xi = \xi \otimes a,$$

where $\xi$ is any smooth function with compact support in $U$ which constantly equals 1 on the support of $a$. Note that the functions $a \in C_c^\infty(G)$ which satisfy the condition above generate the linear space $C_c^\infty(G)$.

**Remark.** For any local diffeomorphism $\phi : N \to M$ between two smooth manifolds, denote by $\phi_+ : C_c^\infty(N) \to C_c^\infty(M)$ the linear map given by

$$\phi_+(a)(x) = \sum_{\phi(y) = x} a(y).$$

This association is functorial. Now observe that $S = inv_+, \varepsilon = s_+$ and $\Delta = \Omega^{-1} \circ d_+$. Furthermore, the convolution multiplication $\mu : C_c^\infty(G) \otimes_{C_c^\infty(G_0)} C_c^\infty(G) \to C_c^\infty(G)$ is exactly $\text{com}_+ \circ \Omega'$, where $\Omega' : C_c^\infty(G) \otimes_{C_c^\infty(G_0)} C_c^\infty(G) \to C_c^\infty(G \times G_0 G)$ is the isomorphism given by the same formula as $\Omega$ [12]. Note also that $t_+ = \varepsilon \circ S$ and that $\text{uni}_+$ is the embedding of $C_c^\infty(G_0)$ into $C_c^\infty(G)$.

**Proposition 2.2.** The convolution algebra $C_c^\infty(G)$ is an étale Hopf algebroid with respect to the coalgebra structure $(\Delta, \varepsilon)$ and the antipod $S$ defined above.

**Proof.** It is easy to see that $(\Delta, \varepsilon)$ is a cocommutative coalgebra structure over the left $C_c^\infty(G_0)$-action on $C_c^\infty(G)$ and that the properties (1)–(7) in Definition 2.1 are satisfied. Observe that we have the diffeomorphism $\tilde{d} : G \times G_0 G \to G \times G_0 G$ given by $\tilde{d}(g',g) = (g'g,g)$ and that $\Omega^{-1} \circ \tilde{d}_+ \circ \Omega' = \tilde{A}$. The property (8) now follows from the equation $(id \times inv) \circ \tilde{d} \circ (id \times inv) \circ \tilde{A} = id$. □

**Proposition 2.3.** If $(A,A_0,\Delta,\varepsilon,S)$ and $(B,B_0,A',\varepsilon',S')$ are étale Hopf algebroids, then, $(A \otimes B, A_0 \otimes B_0, \sigma \circ (\Delta \otimes A'), \varepsilon \otimes \varepsilon', S \otimes S')$ is also an étale Hopf algebroid.

**Remark.** Here $\sigma$ denotes the flip isomorphism $\sigma = \sigma_{23} : (A \otimes B_0) \otimes (B \otimes B_0) \to (A \otimes B) \otimes (B \otimes B_0)$. (A \otimes B).
Example 2.4. (1) Let \( \mathcal{A}_p \) be the algebra of \( p \times p \) matrices with coefficients in \( \mathbb{F} \). Then there is an étale Hopf algebroid structure on \( \mathcal{A}_p \) given as follows: the base algebra is the algebra of diagonal matrices, the antipode is given by the transposition, and the coalgebra structure is given by \( A(e_{ij}) = e_{ij} \otimes e_{ij} \) and \( \Delta(e_{ij}) = e_{ii} \otimes e_{jj} \). Here \( e_{ij} \) denotes the matrix of the standard basis of \( \mathcal{A}_p \) for which the coefficient \((e_{ij})_{kl}\) equals 1 if \( i = k \) and \( j = l \), and equals 0 otherwise. Note that \( p \) can be any positive integer or \( \infty \), where \( \mathcal{A}_\infty \) are the infinite matrices with only finitely many non-zero entries.

If \( G \) is the discrete groupoid with \( p \) objects and with exactly one morphism between each two objects, then \( C^\infty_e(G) \) is isomorphic to the étale Hopf algebroid \( \mathcal{A}_p \) over the base field \( \mathbb{C} \) (or \( \mathbb{R} \)).

Let \( A \) be an étale Hopf algebroid. Then, \( \mathcal{A}_p(A) = A \otimes \mathcal{A}_p \) of \( p \times p \) matrices with coefficients in \( A \) is also an étale Hopf algebroid, by Proposition 2.3.

(2) Let \( A \) be an étale Hopf algebroid such that \( A_0 = A \). Then \( \varepsilon \) and \( S \) are the identities and \( A \) is the standard isomorphism \( A \to A \otimes^A A \). Conversely, any commutative algebra \( A \) with local units may be considered as an étale Hopf algebroid with \( A_0 = A \). The étale Hopf algebroid \( C^\infty_e(G) \) is of this type if \( G \) is a smooth manifold viewed as an étale groupoid.

(3) Let \( A \) be an étale Hopf algebroid with a unit \( 1 \) such that \( A_0 = \mathbb{F}1 \). Then, \( A \) is a Hopf algebra over \( \mathbb{F} \). The étale Hopf algebroid \( C^\infty_e(G) \) is of this type if \( G \) is a discrete group viewed as an étale groupoid.

3. Principal bimodules

Let \( A \) and \( B \) be étale Hopf algebroids and \( M \) a locally \( A_0 \)-unital \( A \)-\( B \)-bimodule. Consider a cocommutative coalgebra structure \( (A, \varepsilon) \) on \( M \) over the right \( B_0 \)-action. Thus, \( \varepsilon : M \to B_0 \) and \( \Delta : M \to M \otimes^{B_0} M \) are homomorphisms of right \( B_0 \)-modules.

We shall regard \( M \otimes^{B_0} M \) as a left \( A \)-module with respect to the action on the first factor. Observe that for any \( m' \otimes m'' \in M \otimes^{B_0} M \) and \( \sum_k b_k' \otimes b_k'' \in AB \) the product

\[
(m' \otimes m'') \sum_k b_k' \otimes b_k'' = \sum_k m' b_k' \otimes m'' b_k'' \in M \otimes^{B_0} M
\]

is well-defined. This product is well defined as a map \((M \otimes^{B_0} M) \otimes B_0 \to AB \otimes^{B_0} M\). In particular, \( M \otimes^{B_0} M \) is a locally \( A_0 \)-unital \( A \)-\( B \)-bimodule, where the right \( B \)-action is \((m' \otimes m'') \cdot b = (m' \otimes m'') A(b)\).

Now assume that \( A \) is a homomorphism of left \( A_0 \)-modules. By the same argument as in Section 2 we see that \( \Delta M \) is an \( A_0 \)-\( B_0 \)-bimodule and that the left \( A_0 \)-module structure of \( \Delta M \) coincide with the left \( A_0 \)-action on the second factor of \( \Delta M \subset M \otimes^{B_0} M \). Therefore, the product

\[
(a' \otimes a'') \sum_i m_i' \otimes m_i'' = \sum_i a' m_i' \otimes a'' m_i''
\]

is well defined as a map \((A \otimes^{A_0} A) \otimes_{A_0} \Delta M \to M \otimes^{B_0} M\). Denote by \( \tilde{\Delta} : A \otimes_{A_0} \Delta M \to M \otimes^{A_0} M \) the homomorphism of \( A \)-\( B_0 \)-bimodules given by \( \tilde{\Delta}(a \otimes m) = a\Delta(m) \).
Remark. (1) Observe that for a principal $A-B$-bimodule $M$ equipped with a cocommutative coalgebra structure $(\Delta, \varepsilon)$ over the right $B_0$-action such that

1. $\varepsilon$ is surjective,
2. $\varepsilon(am) = \varepsilon(a)m$ and $\varepsilon(mb) = \varepsilon(m)b$ for any $a \in A$, $m \in M$ and $b \in B$,
3. $\Delta$ is a homomorphism of left $A_0$-modules, $\Delta(am) = \Delta(a)\Delta(m)$ and $\Delta(mb) = \Delta(m)\Delta(b)$ for any $a \in A$, $m \in M$ and $b \in B$, and
4. $\tilde{\Delta}$ is an isomorphism.

Definition 3.1. Let $A$ and $B$ be étale Hopf algebroids. A principal $A-B$-bimodule is a locally $A_0-B_0$-unital $A-B$-bimodule $M$ equipped with a cocommutative coalgebra structure $(\Delta, \varepsilon)$ over the right $B_0$-action such that

1. $\varepsilon$ is surjective,
2. $\varepsilon(am) = \varepsilon(a)m$ and $\varepsilon(mb) = \varepsilon(m)b$ for any $a \in A$, $m \in M$ and $b \in B$,
3. $\Delta$ is a homomorphism of left $A_0$-modules, $\Delta(am) = \Delta(a)\Delta(m)$ and $\Delta(mb) = \Delta(m)\Delta(b)$ for any $a \in A$, $m \in M$ and $b \in B$, and
4. $\tilde{\Delta}$ is an isomorphism.

Remark. (1) Observe that for a principal $A-B$-bimodule $M = (M, \Delta, \varepsilon)$ the space $\Delta M$ is a right $B$-submodule of $M \otimes_{B_0} M$. Moreover, there is also a left $A$-action on $\Delta M$ given by $a \cdot \sum_i m_i' \otimes m_i'' = \Delta(a) \sum_i m_i' \otimes m_i''$, making $\Delta M$ into a locally $A_0-B_0$-unital $A-B$-bimodule. Note that this left $A$-action is in general different from the $A$-module structure on $M \otimes_{B_0} M$, so we used the dot to denote the actions which are given by the comultiplication. Further observe that $\tilde{\Delta}$ is an isomorphism of $A-B$-bimodules.

(2) Any étale Hopf algebroid $A$ is in particular a principal $A-A$-bimodule.

(3) Two principal $A-B$-bimodules $(M, \Delta, \varepsilon)$ and $(M', \Delta', \varepsilon')$ are isomorphic if there exists an isomorphism $\phi : M \to M'$ of $A-B$-bimodules such that $\varepsilon = \varepsilon' \circ \phi$ and $(\phi \otimes \phi) \circ \Delta = \Delta' \circ \phi$.

Now, let $A$, $B$ and $C$ be étale Hopf algebroids, let $M$ be a principal $A-B$-bimodule and let $N$ be a principal $B-C$-bimodule. Then there is an induced coalgebra structure on $M \otimes_B N$ which makes it into a principal $A-C$-bimodule, given as follows: the comultiplication on $M \otimes_B N$ is given by

$$\Delta(m \otimes n) = \sum_{i,j} (m_{i}^j \otimes n_{j}^i),$$

where $\Delta(m) = \sum_i m_i' \otimes m_i''$ and $\Delta(n) = \sum_j n_j' \otimes n_j''$, while the counit on $M \otimes_B N$ is defined by

$$\varepsilon(m \otimes n) = \varepsilon(\varepsilon(m)n),$$

for any $m \otimes n \in M \otimes_B N$. Let us show that this indeed gives a well-defined coalgebra structure on $M \otimes_B N$. First, the counit is well-defined since

$$\varepsilon(mb \otimes n) = \varepsilon(\varepsilon(mb)n) = \varepsilon(\varepsilon(m)b)n \quad (\text{property (2) in Definition 3.1})$$
$$\varepsilon(mb \otimes bn) = \varepsilon(\varepsilon(m)bn) \quad (\text{property (2) in Definition 3.1})$$
$$\varepsilon(m \otimes bn)$$

for any $b \in B$. To show that the comultiplication on $M \otimes_B N$ is well defined, observe first that it equals the composition

$$M \otimes_B N \xrightarrow{\Delta \otimes A}(M \otimes_{B_0} M) \otimes_B \Delta N \xrightarrow{\varepsilon}(M \otimes_B N) \otimes C \circ (M \otimes_B N),$$
where $\sigma = \sigma_{23}$ is the flip map,

$$
\sigma \left( (m' \otimes m'') \otimes \sum_j n_j \otimes n_j'' \right) = \sum_j (m' \otimes n_j') \otimes (m'' \otimes n_j'').
$$

Now the flip is well defined since for any $b \in B$ with $A(b) = \sum_k b'_k \otimes b''_k$ we have

$$
\sigma \left( (m' \otimes m'') \cdot b \otimes \sum_j (n_j \otimes n_j'') \right) = \sum_{j,k} \sigma((m'b'_k \otimes m''b''_k) \otimes (n_j \otimes n_j''))
= \sum_{j,k} (m'b'_k \otimes n_j') \otimes (m''b''_k \otimes n_j'')
= \sum_{j,k} (m' \otimes b'_k n_j') \otimes (m'' \otimes b''_k n_j'')
= \sum_{j,k} \sigma((m' \otimes m'') \otimes (b'_k n_j' \otimes b''_k n_j''))
= \sigma \left( (m' \otimes m'') \otimes b \cdot \sum_j (n_j \otimes n_j'') \right).
$$

By using what we already know about principal bimodules it is straightforward to check that we obtain a cocommutative coalgebra structure over the right $C_0$-action on $M \otimes_B N$ which satisfies the conditions (1), (2) and (3) in Definition 3.1. Let us now show that the coalgebra structure on $M \otimes_B N$ satisfies the condition (4) as well, i.e. that $\hat{A}$ is an isomorphism. To this end, we shall describe explicitly the inverse of $\hat{A}$.

It is given as the composition of four maps:

$$
(M \otimes_B N) \otimes C_0 (M \otimes_B N) \xrightarrow{\sigma_{312}} M \otimes_B (M \otimes_B (N \otimes C_0 N)) \xrightarrow{id \otimes id \otimes \hat{A}_N^{-1}} M \otimes_B (M \otimes_B (B \otimes_B N)) \xrightarrow{\nu} (M \otimes_B^{B_0} M) \otimes_B N \xrightarrow{\hat{A}_M^{-1} \otimes id} A \otimes_{A_0} M \otimes_B N.
$$

Here we denoted the comultiplications on $M$ and on $N$ with $\Lambda_M$, respectively $\Lambda_N$, to avoid confusion. In $M \otimes_B (M \otimes_B (N \otimes C_0 N))$ the first tensor product is with respect to the action of $B$ on the third factor of $M \otimes_B (N \otimes C_0 N)$. So here we are using both left $B$-actions on $N \otimes C_0 N$, together with the fact that they commute with each other. Since $\hat{A}_N$ is an isomorphism, the left $B$-action on the second factor of $N \otimes C_0 N$ corresponds to a left $B$-action on $B \otimes_B N$, which will be denoted by $\triangleright$. This action is relevant for the first tensor product in $M \otimes_B (M \otimes_B (B \otimes_B N))$. The map $\sigma_{312}$ is the permutation $\sigma_{312}((m \otimes n) \otimes (m' \otimes n')) = m' \otimes (m \otimes (n \otimes n'))$, and is clearly well defined. Finally, the map $\nu$ is given by

$$
\nu(m' \otimes (m \otimes (b \otimes n))) = (mb \otimes m') \otimes n.
$$
As the rest is clear, we should check here that this is a sound definition with respect to the first tensor product. First we have \( v(m' b' \otimes (m \otimes (b \otimes n))) = (mb \otimes m'b') \otimes n. \) On the other hand,

\[
m \otimes b' \triangleright (b \otimes n) = (id \otimes \tilde{\Delta}_N^{-1})(mb \otimes (\eta \otimes b')\Delta_N(n)),
\]

where \( \eta \) is an element of \( B_0 \) with \( \eta \Delta_N(n) = \Delta_N(n) \). We can choose \( \eta \) so that \( b\eta = b \). Denote the comultiplication on \( B \) by \( \Delta_B \). Since \( \hat{\Delta}_B \) is an isomorphism we can write \( \eta \otimes b' = \sum_l b_l \Delta_B(b'_l) \). Now,

\[
mb \otimes (\eta \otimes b')\Delta_N(n) = \sum_l mb \otimes b_l \Delta_B(b'_l)\Delta_N(n) = \sum_l mb \otimes b_l \Delta_N(b'_ln),
\]

and hence,

\[
m \otimes b' \triangleright (b \otimes n) = (id \otimes \tilde{\Delta}_N^{-1}) \left( \sum_l mb \otimes b_l \Delta_N(b'_l)n \right) = \sum_l mb \otimes (b_l \otimes b'_ln).
\]

Therefore,

\[
v(m' \otimes (m \otimes b' \triangleright (b \otimes n))) = v \left( m' \otimes \sum_l mb \otimes (b_l \otimes b'_ln) \right)
= \sum_l (mbb_l \otimes m') \otimes b'_ln
= \sum_l (mbb_l \otimes m')\Delta_B(b'_l) \otimes n
= (mb \eta \otimes m'b') \otimes n
= v(m'b' \otimes (m \otimes (b \otimes n)))
\]

and this shows that \( v \) is well defined. It is then straightforward to check that the composition of the four maps described above gives the inverse of \( \hat{\Delta} : A \otimes_{A_0} (M \otimes_B N) \to (M \otimes_B N) \otimes^{C_0} (M \otimes_B N) \). We proved:

**Proposition 3.2.** Let \( A, B \) and \( C \) be étale Hopf algebroids, let \( M \) be a principal \( A-B \)-bimodule and let \( N \) be a principal \( B-C \)-bimodule. Then, \( M \otimes_B N \) is a principal \( A-C \)-bimodule with respect to the coalgebra structure defined above.

**Proposition 3.3.** Let \( A, A', B \) and \( B' \) be étale Hopf algebroids, let \( M \) be a principal \( A-B \)-bimodule and let \( M' \) be a principal \( A'-B' \)-bimodule. Then \( M \otimes M' \) is a principal \( A \otimes A' \otimes B \otimes B' \)-bimodule in the natural way. If \( C \) and \( C' \) are another two étale Hopf algebroids, \( N \) a principal \( B-C \)-bimodule and \( N' \) a principal \( B'-C' \)-bimodule, then the flip isomorphism

\[
(M \otimes M') \otimes_{B \otimes B'} (N \otimes N') \cong (M \otimes_B N) \otimes (M' \otimes_{B'} N')
\]

is an isomorphism of principal \( A \otimes A'-C \otimes C' \)-bimodules.

**Remark.** The coalgebra structure on \( M \otimes M' \) is analogous to the one on \( A \otimes A' \).
4. Principal Morita equivalence

Definition 4.1. The category $\mathcal{eHa}$ of étale Hopf algebroids and principal bimodules is given by

1. objects of $\mathcal{eHa}$ are étale Hopf algebroids,
2. morphisms in $\mathcal{eHa}$ from an étale Hopf algebroid $B$ to an étale Hopf algebroid $A$ are the isomorphism classes of principal $A$–$B$-bimodules, and
3. the composition in $\mathcal{eHa}$ is induced by the tensor product of principal bimodules (Proposition 3.2).

Two étale Hopf algebroids are \textit{principally Morita equivalent} if they are isomorphic in the category $\mathcal{eHa}$.

Remark. We will denote an isomorphism class of a principal bimodule $M$ again by $M$, and the composition in $\mathcal{eHa}$ by $\circ$, i.e. $M \circ N = M \otimes_B N$ for $M \in \mathcal{eHa}(B,A)$ and $N \in \mathcal{eHa}(C,B)$. It is easy to verify that $\mathcal{eHa}$ is indeed a category. For instance, the unit morphism of an étale Hopf algebroid $A$ is the (isomorphism class of the) principal $A$–$A$-bimodule $A$. Alternatively one could define a two-category of étale Hopf algebroids, principal bimodules and isomorphisms of principal bimodules in the analogous way. Proposition 3.3 says, in other words, that $\mathcal{eHa}$ is a monoidal category.

Example 4.2. Let $A$ and $B$ be étale Hopf algebroids and assume that $A_0 = A$. Let $M$ be a principal $A$–$B$-bimodule. The principality implies that the comultiplication on $M$ is an isomorphism, and therefore $\varepsilon : M \to B_0$ is an isomorphism as well. Conversely, if we have a (left) action of $A_0 = A$ on $B_0$, then $B_0$ is a principal $A$–$B$-bimodule: the right action of $B$ on $B_0$ is given the product in $B$ composed with $\varepsilon : B \to B_0$. It follows that two commutative algebras with local units, viewed as étale Hopf algebroids (see Example 2.4(2)), are principally Morita equivalent if and only if they are isomorphic.

Proposition 4.3. Let $A$ be an étale Hopf algebroid. Then all the étale Hopf algebroids $\mathcal{M}_p(A)$ are principally Morita equivalent, for $p \in \mathbb{Z}^+ \cup \{\infty\}$.

Proof. The vector space $\mathcal{M}_{p \times q}$ of $p \times q$-matrices is naturally a $\mathcal{M}_p$–$\mathcal{M}_q$-bimodule. But in fact it is a principal $\mathcal{M}_p$–$\mathcal{M}_q$-bimodule, with $A$ and $\varepsilon$ defined analogously as in Example 2.4(1).

Now let $M$ be any principal $\mathcal{M}_p$–$\mathcal{M}_q$-bimodule. Observe that $M$ is a direct sum of subspaces $M_{ij} = e_{i,j}Me_{i,j}$, and that $M_{ij} = e_{jk}M_{kk}e_{ij}$. Take any $m \in M_{ij}$. Since $M$ is principal, we can write $m \otimes m = \sum_{kl} e_{kl}A(m_{kl})$. But since $m \otimes m = e_{i,m} \otimes m = m \otimes e_{i,m}$, it follows that $m \otimes m = \sum_{kl} e_{i,l}e_{i,j}A(m_{kl}) = A(m_{ij})$. Hence, $m_{ij} = m\varepsilon(m)$ and therefore,

$$m \otimes m = A(m)\varepsilon(m).$$

Now, if $m \neq 0$ we have $\varepsilon(m) \neq 0$, which yields that $M_{ij}$ is of dimension one. It is now easy to see that $M$ is isomorphic to $\mathcal{M}_{p \times q}$. In particular, it follows that all the étale Hopf algebroids $\mathcal{M}_p$ are isomorphic in $\mathcal{eHa}$. The same is then true for $\mathcal{M}_p(A) = A \otimes \mathcal{M}_p$ by Proposition 3.3. \(\square\)
We shall now describe our main example of principal bimodules. Let $G$ and $H$ be two étale groupoids and let $E = (E, p, w)$ be a principal $G$–$H$-bibundle [11,12]. Recall that a $G$–$H$-bibundle is a (smooth) manifold $E$ equipped with a (smooth) left action of $G$ with respect to $p : E \to G_0$ and a (smooth) right action of $H$ with respect to $w : E \to H_0$ such that the actions commute with each other. Such a bibundle is principal if $w$ is a surjective local diffeomorphism and the map $\tilde{d} : G \times_{G_0} E \to E \times_{H_0} E$, given by $\tilde{d}(g, e) = (ge, e)$, is a diffeomorphism. We have shown in [12] that $C_c^\infty(E)$ is a $C_c^\infty(G)$–$C_c^\infty(H)$-bimodule, with the actions given by

\[(am)(e) = \sum_{t(g) = p(e)} a(g)m(g^{-1} e)\]

and

\[(mb)(e) = \sum_{t(h) = w(e)} m(eh)b(h^{-1})\]

for any $a \in C_c^\infty(G)$, $m \in C_c^\infty(E)$, $b \in C_c^\infty(H)$ and $e \in E$. Furthermore, there is a coalgebra structure on $C_c^\infty(E)$ over the $C_c^\infty(H_0)$-action, analogous to the one on the algebra $C_c^\infty(G)$ described in Section 2: The counit is $\varepsilon = w_+$, i.e,

\[\varepsilon(m)(y) = \sum_{w(e) = y} m(e)\].

The comultiplication is induced by the diagonal map $d : E \to E \times_{H_0} E$, $d(g) = (g, g)$, or more precisely, $\Delta = \Omega^{-1} \circ \tilde{d}_+ \circ \Omega'$, where $\Omega : C_c^\infty(E) \otimes_{C_c^\infty(H_0)} C_c^\infty(E) \to C_c^\infty(E \times_{H_0} E)$ is the isomorphism $\Omega(m \otimes m')(e, e') = m(e)m'(e')$ [12]. If $m$ has support in an open subset $U$ of $E$ such that $w|_U$ is injective one has

\[\Delta(m) = m \otimes \xi = \xi \otimes m\]

where $\xi \in C_c^\infty(E)$ is any function with support in $U$ which constantly equals 1 on the support of $m$.

**Proposition 4.4.** The bimodule $C_c^\infty(E)$ is a principal $C_c^\infty(G)$–$C_c^\infty(H)$-bimodule with respect to the coalgebra structure $(\Delta, \varepsilon)$ defined above.

**Proof.** The map $\tilde{d}$ is an isomorphism because it equals $\Omega^{-1} \circ \tilde{d}_+ \circ \Omega'$, where $\Omega' : C_c^\infty(G) \otimes_{C_c^\infty(G_0)} C_c^\infty(G) \to C_c^\infty(G \times_{G_0} G)$ is the isomorphism given by $\Omega'(a \otimes m)(g, e) = a(g)m(e)$ [12].

Let $G$, $H$, and $K$ be étale groupoids, let $(E, w, p)$ be a principal $G$–$H$-bibundle and let $(E', p', w')$ be a principal $H$–$K$-bibundle. Then there is an isomorphism of $C_c^\infty(G)$–$C_c^\infty(K)$-bimodules (see [12]) $\Omega_{E, E'} : C_c^\infty(E) \otimes_{C_c^\infty(H)} C_c^\infty(E') \to C_c^\infty(E \otimes_K E')$.

\[\Omega_{E, E'}(m \otimes n)(e \otimes e') = \sum_{t(h) = p'(e')} m(eh)n(h^{-1} e').\]

Here $E \otimes_K E'$ denotes the quotient of $E \times_{H_0} E'$ with respect to the identifications of $(eh, e')$ with $(e, he')$, for any $e, e' \in E$ and $h \in H$ with $w(e) = t(h)$ and $s(h) = p'(e')$. 
The equivalence class of \((e, e')\) is denoted by \(e \otimes e' \in E \otimes_H E'\). The space \(E \otimes_H E'\) is a principal \(G\)-\(K\)-bibundle in the natural way: the left \(G\) action is given by \(g(e \otimes e') = ge \otimes e'\) and is with respect to the map \(p''(e \otimes e') = p(e)\), while the right \(K\)-action is given by \((e \otimes e')k = e \otimes e'k\) and is with respect to the map \(w''(e \otimes e') = w'(e')\). This operation between principal bibundles induces the composition between the associated Hilsum–Skandalis maps \([11, 12]\).

**Theorem 4.5.** Let \(G, H,\) and \(K\) be separated étale groupoids, let \(E\) be a principal \(G\)-\(H\)-bibundle and let \(E'\) be a principal \(H\)-\(K\)-bibundle. Then \(\Omega_{E,E'}\) is an isomorphism between the principal \(C^\infty_c(G)\)-\(C^\infty_c(K)\)-bimodules \(C^\infty_c(E) \odot C^\infty_c(E')\) and \(C^\infty_c(E \otimes_H E')\).

**Proof.** Let \(m \otimes n \in C^\infty_c(E) \odot C^\infty_c(E')\). Take any \(z \in K_0\) and fix \(e' \in E'\) with \(w'(e') = z\). Now we have

\[
\varepsilon(\Omega_{E,E'}(m \otimes n))(z) = \sum_{w(e) = p'(e')} \Omega_{E,E'}(m \otimes n)(e \otimes e')
\]

\[
= \sum_{w(e) = p'(e')} \sum_{n(h) = p'(e')} m(eh)n(h^{-1}e')
\]

\[
= \sum_{n(h) = p'(e')} \left( \sum_{w(e) = p'(e')} m(eh) \right) n(h^{-1}e')
\]

\[
= \sum_{n(h) = p'(e')} \varepsilon(m(s(h)))n(h^{-1}e')
\]

\[
= \varepsilon(\varepsilon(m)n)(z)
\]

\[
= \varepsilon(m \otimes n)(z).
\]

We may assume without loss of generality that the support of \(m\) is in an open subset \(U\) of \(E\) for which \(w|_U\) is injective, and that the support of \(n\) is in an open subset \(U'\) of \(E'\) for which \(w'|_{U'}\) is injective. Therefore, we have \(\Delta(m) = m \otimes \xi\) and \(\Delta(n) = n \otimes \xi'\), where the support of \(\xi \in C^\infty_c(E)\) is in \(U\) and \(\xi\) constantly equals 1 on the support of \(m\) while the support of \(\xi' \in C^\infty_c(E')\) is in \(U'\) and \(\xi'\) constantly equals 1 on the support of \(m'\). Now observe that \(U'' = \{e \otimes e' \mid e \in U, e' \in U'\}\) is an open subset of \(E \otimes_H E'\) such that \(w''|_{U''}\) is injective. Furthermore, the supports of \(\Omega_{E,E'}(m \otimes n)\) and \(\Omega_{E,E'}(\xi \otimes \xi')\) are in \(U''\) and \(\Omega_{E,E'}(\xi \otimes \xi')\) constantly equals 1 on the support of \(\Omega_{E,E'}(m \otimes n)\). Therefore, we have

\[
\Delta(\Omega_{E,E'}(m \otimes n)) = \Omega_{E,E'}(m \otimes n) \otimes \Omega_{E,E'}(\xi \otimes \xi')
\]

\[
= (\Omega_{E,E'} \otimes \Omega_{E,E'})(m \otimes n) \otimes (\xi \otimes \xi')
\]

\[
= (\Omega_{E,E'} \otimes \Omega_{E,E'})\Delta(m \otimes n).
\]

\(\Box\)
Corollary 4.6. By associating the principal bimodule $C^\infty_c(E)$ to a principal bibundle $E$ we get a functor from the category of Hilsum–Skandalis maps between separated étale groupoids to the category $eHa$. In particular, the étale Hopf algebroids associated to Morita-equivalent separated étale groupoids are principally Morita equivalent.

References