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On a question by Grunenfelder, Omladič and Radjavi

Mikhail Chebotar *, Victor Lomonosov

Department of Mathematical Sciences, Kent State University, Kent, OH 44242, USA

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Abstract

Let $n \geqslant 3$ be an odd number. If S is a Lie subalgebra of $\mathrm{gl}_n(\mathbb{C})$ which acts transitively on $\mathbb{C}^n$, then either $S = \mathrm{gl}_n(\mathbb{C})$ or $S = \mathrm{sl}_n(\mathbb{C})$. This answers a question by Grunenfelder, Omladič and Radjavi.

Keywords: Transitive action; 2-Transitive action; Lie algebra

A set S of operators *acts transitively* on a vector space V if for all pairs of vectors $x, y \in V$ with $x \neq 0$ there is an element $A \in S$ such that $Ax = y$. For a given positive integer k the set S *acts k-transitively* on V if for any linearly independent k vectors $\{x_1, x_2, \ldots, x_k\}$ and any k vectors $\{y_1, y_2, \ldots, y_k\}$ of V there is $A \in S$ such that $Ax_i = y_i$, $i = 1, \ldots, k$.

A classical theorem due to Burnside asserts that the only associative subalgebra of the full matrix algebra $\mathrm{gl}_n(\mathbb{C})$ which acts transitively on $\mathbb{C}^n$ is $\mathrm{gl}_n(\mathbb{C})$ itself (see [6,7]). Recently, Grunenfelder et al. [4,5] initiated the study of nonassociative subalgebras of $\mathrm{gl}_n(\mathbb{C})$ which act k-transitively on $\mathbb{C}^n$. In particular, they proved the following theorem:

Theorem 1 [5, Theorem 2.4]. *If $n \geqslant 3$ and S is a Lie subalgebra of* $\mathrm{gl}_n(\mathbb{C})$ *which acts* 2*-transitively on* $\mathbb{C}^n$, *then either* $S = \mathrm{gl}_n(\mathbb{C})$ *or* $S = \mathrm{sl}_n(\mathbb{C})$.

Grunenfelder et al. [5, p. 92] noticed that in even dimensions $n = 2m > 2$ there are Lie algebras (to be precise, the symplectic Lie algebras $\mathrm{sp}_n(\mathbb{C})$ in their example) which act transitively, but not 2-transitively. On the other hand they proved the following result.

* Corresponding author. Tel.: +1 330 672 9080.
E-mail addresses: chebotar@math.kent.edu (M. Chebotar), lomonoso@math.kent.edu (V. Lomonosov).

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Proposition 2 [5, Proposition 3.1]. *If S is a Lie subalgebra of* $\mathrm{gl}_3(\mathbb{C})$ *which acts transitively on* $\mathbb{C}^3$, *then either* $S = \mathrm{gl}_3(\mathbb{C})$ *or* $S = \mathrm{sl}_3(\mathbb{C})$.

In other words every Lie subalgebra of $\mathrm{gl}_3(\mathbb{C})$ which acts transitively acts 2-transitively.

This observation was the reason for their question whether in odd dimensions transitive action of Lie subalgebras always implies 2-transitive action [5, p. 92]. We answer this question positively, namely we prove the following result.

Theorem 3. *Let $n \geqslant 3$ be an odd number. If S is a Lie subalgebra of* $\mathrm{gl}_n(\mathbb{C})$ *which acts transitively on* $\mathbb{C}^n$, *then either* $S = \mathrm{gl}_n(\mathbb{C})$ *or* $S = \mathrm{sl}_n(\mathbb{C})$.

Throughout the paper V, V_i denotes finite-dimensional complex vector spaces and I denotes the identity matrix. Following the classical works by Dynkin [2,3] let us characterize a maximal Lie subalgebra h of $\mathrm{sl}_n(\mathbb{C})$. Then only three following cases may occur:

1. h is reducible, i.e. it has an invariant subspace;
2. h is either skew-symmetric Lie algebra $\mathrm{o}(V)$ or symplectic Lie algebra $\mathrm{sp}(V)$;
3. $h = \mathrm{sl}(V_1) \otimes \mathrm{sl}(V_2)$ in $\mathrm{sl}(V_1 \otimes V_2)$, provided that $\dim(V_2) \geqslant \dim(V_1) \geqslant 2$.

The last case needs more explanation. Recall that a representation φ of a Lie algebra g on a vector space V assigns to each $X \in g$ an operator $\varphi(X)$ on V. Let φ_1 and φ_2 be two representations. The tensor product $\varphi_1 \otimes \varphi_2$ on the tensor product $V_1 \otimes V_2$ is defined as

$$\varphi_1 \otimes \varphi_2(X)(v_1 \otimes v_2) = \varphi_1(X)(v_1) \otimes v_2 + v_1 \otimes \varphi_2(X)(v_2).$$

In other words, $\varphi_1 \otimes \varphi_2(X)$ is not the tensor product of operators $\varphi_1(X)$ and $\varphi_2(X)$. As it is mentioned in [8, p. 13], it would be better to call it the infinitesimal tensor product or tensor sum.

Lemma 4. *Let V_1 and V_2 be finite dimensional complex vector spaces with* $\dim(V_1) = k \geqslant 2$ *and* $\dim(V_2) = l \geqslant 2$. *Then* $\mathrm{gl}(V_1) \otimes \mathrm{gl}(V_2)$ *does not act transitively on* $V_1 \otimes V_2$.

Proof. We may represent each vector x from $V_1 \otimes V_2$ as a rectangular $k \times l$ matrix with entries x_{ij}. For $A \in \mathrm{gl}(V_1)$ and $B \in \mathrm{gl}(V_2)$ the action of $A \otimes B$ on x is a sum of the action of A on rows and the action of B on columns of the matrix x.

Let $x = e_{11}$, that is, $x_{11} = 1$ and other entries equal 0. Then for any $A \in \mathrm{gl}(V_1)$ and $B \in \mathrm{gl}(V_2)$ put $y = A \otimes B(x)$. It is obvious that the lower rightmost entry of the matrix y, $y_{kl} = 0$, that yields $\mathrm{gl}(V_1) \otimes \mathrm{gl}(V_2)(x) \neq V_1 \otimes V_2$ and so $\mathrm{gl}(V_1) \otimes \mathrm{gl}(V_2)$ does not act transitively on $V_1 \otimes V_2$. □

Remark 5. Let us underline one more time that $\mathrm{gl}(V_1) \otimes \mathrm{gl}(V_2)$ is not a tensor product of matrix subspaces. In the recent paper by Davidson et al. [1] there are several interesting results on the tensor products of matrix subspaces. In particular, the tensor product of matrix subspaces $\mathrm{sl}(V_1) \otimes \mathrm{gl}(V_2)$ is $[\dim(V_1) - 1]$-transitive according to [1, Example 3.10].

We are ready to prove Theorem 3.

Proof of Theorem 3. Suppose that S is a Lie subalgebra of $\mathrm{gl}_n(\mathbb{C})$ which acts transitively on $\mathbb{C}^n$, and different from $\mathrm{gl}_n(\mathbb{C})$ and $\mathrm{sl}_n(\mathbb{C})$.

Let S_0 be a Lie subalgebra of $S + \mathbb{C} \cdot I$ consisting of matrices with zero trace. It is clear that $\dim(S_0) \leqslant \dim(S)$, so S_0 is also a proper Lie subalgebra of $\mathrm{sl}_n(\mathbb{C})$.

Embed S_0 in the maximal Lie subalgebra h of $\mathrm{sl}_n(\mathbb{C})$. According to Dynkin's results mentioned above we have three possible cases. The first case when h is reducible cannot happen because $h + \mathbb{C} \cdot I$ then has an invariant subspace and so S does not act transitively.

Consider now the third case when $h = \mathrm{sl}(V_1) \otimes \mathrm{sl}(V_2)$. Clearly the identity operator belongs to $\mathrm{gl}(V_1) \otimes \mathrm{gl}(V_2)$ so the inclusion

$$S + \mathbb{C} \cdot I \subseteq \mathrm{gl}(V_1) \otimes \mathrm{gl}(V_2)$$

holds. By Lemma 4 $\mathrm{gl}(V_1) \otimes \mathrm{gl}(V_2)$ does not act transitively and so S does not act transitively.

Therefore, the only possible situation is the second case, when h is isomorphic either to $\mathrm{o}(V)$ or to $\mathrm{sp}(V)$.

Since n is odd we do not need to consider the case of $\mathrm{sp}(V)$. We have now that $S \subseteq \mathrm{o}_n(\mathbb{C}) + \mathbb{C} \cdot I$, and so the Lie subalgebra $L = \mathrm{o}_n(\mathbb{C}) + \mathbb{C} \cdot I$ acts transitively on V. However, it is not true. Indeed, let $u = (1, i, 0, \ldots, 0)$ and $v = (-i, 1, 0, \ldots, 0)$ with $i^2 = -1$. Then it is easy to see that for all $A \in L$ we have the inner product $Au \cdot v = 0$ and so $Lu \neq V$, a contradiction. Therefore S does not act transitively. □

Remark 6. The proof of Theorem 3 also shows the reason why for $n = 2m$ there are transitive Lie subalgebras different from $\mathrm{gl}_n(\mathbb{C})$ and $\mathrm{sl}_n(\mathbb{C})$. In this case there is a maximal simple Lie subalgebra $\mathrm{sp}_n(\mathbb{C})$ of $\mathrm{sl}_n(\mathbb{C})$, which does not appear in the odd case.

Remark 7. Theorem 3 essentially means that if S is a Lie subalgebra of $\mathrm{gl}_n(\mathbb{C})$ (with odd n) which acts transitively on $\mathbb{C}^n$, then it acts 2-transitively on $\mathbb{C}^n$. One may wonder whether this remains true if we replace $\mathbb{C}^n$ with $\mathbb{R}^n$. The answer is: No.

Let K be the algebra of 3 by 3 skew-symmetric real matrices and $S = K + \mathbb{R} \cdot I$. It is straightforward to check that S acts transitively on $\mathbb{R}^3$. If we take $T = e_{11} - e_{22}$, then T is of rank 2 and the trace of TA equals zero for all $A \in S$. According to [5, Proposition 1.1] we get that S does not act 2-transitively on $\mathbb{R}^3$.

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