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NOTE

TWO-FOLD TRIPLE SYSTEMS WITHOUT REPEATED BLOCKS

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Direct, easy, and self-contained proofs are presented of the existence of triple systems with $\lambda = 2$ having no repeated blocks.

1. Introduction

A λ-fold triple system $S_\lambda(2, 3, v)$ is a way of selecting unordered triples from a v-set so that any given pair of elements lies in precisely λ triples. An $S_\lambda(2, 3, v)$ is *simple* if it contains no repeated triple.

An $S_2(2, 3, v)$ can exist only if $v \equiv 0$ or 1 (mod 3). Bose [1] showed in 1939 that these conditions are sufficient. More recently, Van Buggenhaut [6] has proven that there is a *simple* $S_2(2, 3, v)$ in every such case (except, obviously, $v = 3$). However, his construction assumes knowledge of Hanani's inductive construction [3] of 3-wise balanced designs with block sizes {4, 6} of all orders, and of Doyen's construction [2] of disjoint Steiner triple systems of all orders $v \equiv 3$ (mod 6). He also uses pairwise balanced designs with block sizes {3, 5} and other designs, and constructs eight specific cases. Street [5] has produced an interesting construction in terms of other triple systems, but again it is recursive.

Our aim here is to give direct constructions of simple $S_2(2, 3, v)$ for all possible v, based only on the idea of a Latin square. (No use of the properties of Latin squares is made, other than those defined herein.)

The longest part of the paper is the construction of what we call 'skew transversal squares' for orders congruent to 3 and 4 modulo 6; and these are only used or needed for $S_2(2, 3, v)$ with $v \equiv 10$ or 13 (mod 18).

2. Some Latin squares

A Latin square of order n is an $n \times n$ array whose rows and columns are all permutations of an n-set. Unless otherwise stated, our Latin squares are based on

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the consecutive positive integers. (If, for example, we have an $n \times n$ square and speak of a calculation 'reduced (mod n)', the base set will be the residues $\{1, 2, \ldots, n\}$.)

A *transversal square* is a Latin square with diagonal $(1, 2, \ldots, n)$. A *skew* transversal square $L = (l_{ij})$ is one for which $l_{ij} = l_{ji}$ never occurs unless $i = j$. Clearly a transversal square of side 2, or a skew transversal square of side 2 or 3, is impossible.

We define a Latin square $L_n = (l_{ij})$ of odd order by $l_{ij} \equiv 2i - j \pmod n$. If n is even, $n \neq 2$, we construct L_n from L_{n-1} by replacing the $(1,2)$, $(2,3), \ldots,$ $(n-1, 1)$ entries by n and appending last row $(n-2, n-1, 1, \ldots, n-3, n)$ and last column $(n-1, 1, 2, \ldots, n-2, n)$. Then L_n is a transversal square; the necessary conditions are sufficient. When n is odd, L_n will be skew unless $2i - j \equiv 2j - i \pmod n$, and so will L_{n+1}. So we have skew transversal squares of every possible order n except $n = 3m$ or $3m + 1$, m odd.

Cases 4 and 9 are covered by examples in Fig. 1. The square of order 4 is actually the one constructed above (the replacement process removes all cases where L_3 is non-skew); the order 9 square is from [4].

In the remaining odd cases, $n = 3m$, we can write $n = pq$, where p is a power of $3, q > 3$, and a skew transversal square S of order q is already known. We also need symmetric transversal squares R and T of orders p and q, but these are easily constructed: R has (i, j) entry $(\frac{1}{2}p + 1)(i + j) \pmod p$, and similarly for T. (This is just the back-circulant matrix with diagonal $(1, 2, 3, \ldots)$.) We write T^k to mean T with every entry increased by k and then reduced modulo q to the usual range $\{1, 2, \ldots, q\}$. The required skew transversal square is a $p \times p$ array of $q \times q$ blocks; the ith diagonal block is $S + (i-1)q$, and if $i \neq j$ the (i, j) block is $T^k + (r_{ij} - 1)q$ where $k \in \{0, 1, 2\}$ and $k \equiv i - j \pmod 3$. This is obviously a skew transversal square.

The remaining even cases $n = 3m + 1$ are easily solved from the preceding construction. The diagonals of the $(1, 2)$, $(2, 3), \ldots, (p - 1)$ blocks are replaced by the symbol $3m + 1$, and the deleted entries are transported to a new last row and column just as in the construction of L_n for even n.

Observe that this construction is not recursive: although the square S is needed, the order q is always such that S was directly constructed in the earlier part of this section.

1	4	2	3
3	2	4	1
4	1	3	2
2	3	1	4

1	8	7	6	9	3	5	4	2
4	2	9	8	3	7	1	5	6
5	6	3	2	7	9	4	1	8
3	5	6	4	8	2	9	7	1
2	9	4	7	5	1	8	6	3
7	1	8	5	2	6	3	9	4
9	4	2	1	6	8	7	3	5
6	3	5	9	1	4	2	8	7
8	7	1	3	4	5	6	2	9

Fig. 1.

3. The two-fold triple systems

(i) *Case* $v \equiv 1 \pmod 3$

We use the $3n$ symbols x^i, where $1 \leqslant x \leqslant n$, $0 \leqslant i \leqslant 2$, and a symbol ∞. Suppose $L = (l_{ij})$ is a skew transversal square of side n. Then, if superscripts are reduced modulo 3 when necessary, the triples

$$\begin{array}{ll} \{x^0, x^1, x^2\}: & 1 \leqslant x \leqslant n; \\ \{x^i, x^{i+1}, \infty\}: & 1 \leqslant x \leqslant n; \quad 0 \leqslant i \leqslant 2; \\ \{x^i, y^i, l_{xy}^{i+1}\}: & 1 \leqslant x \leqslant n, 1 \leqslant y \leqslant n, x \neq y; \quad 0 \leqslant i \leqslant 2 \end{array} \tag{1}$$

form a simple $S_2(2, 3, 3n+1)$. To handle the case $n = 2$, we exhibit the triples: 123, 145, 167, 246, 257, 347, 356, 124, 137, 156, 235, 267, 346, 357. For $n = 3$, we could also exhibit the triples. However, writing $n = 3$ and

$$L = (l_{ij}) = L_3 = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

we have the following interesting formula:

$$\begin{array}{ll} \{x^0, x^1, x^2\}: & 1 \leqslant x \leqslant n; \\ \{x^i, x^{i+1}, \infty\}: & 1 \leqslant x \leqslant n; \quad 0 \leqslant i \leqslant 2; \\ \{x^i, y^i, l_{xy}^{i+1}\}: & 1 \leqslant x < y \leqslant n; \quad 0 \leqslant i \leqslant 2; \\ \{x^i, y^i, l_{xy}^{i+2}\}: & 1 \leqslant y < x \leqslant n; \quad 0 \leqslant i \leqslant 2. \end{array} \tag{2}$$

This could be used whenever there is a *symmetric* transversal square of order n; but such objects can obviously only exist when n is odd, so we would still need formula (1) for the cases $n \equiv 1 \pmod 6$.

(ii) *Case* $v \equiv 0 \pmod 6$

For $v = 6n$, we use symbols x^i for $1 \leqslant x \leqslant n$ and $0 \leqslant i \leqslant 5$, and any transversal square (l_{ij}) of side n. The triples are:

$$\begin{array}{l} \{x^1, x^3, x^5\}: \quad 1 \leqslant x \leqslant n; \\ \left.\begin{array}{l} \{x^i, x^{i+1}, x^{i+3}\} \\ \{x^i, x^{i+3}, x^{i+4}\} \\ \{x^i, x^{i+4}, x^{i+5}\} \end{array}\right\} \quad 1 \leqslant x \leqslant n; \quad i = 0, 2, 4; \\ \left.\begin{array}{l} \{x^i, y^i, l_{xy}^{i+2}\} \quad \{x^i, y^i, l_{yx}^{i+3}\} \\ \{x^i, y^{i+1}, l_{xy}^{i+3}\}, \{x^i, y^{i+1}, l_{yx}^{i+2}\} \\ \{x^{i+1}, y^i, l_{xy}^{i+3}\}, \{x^{i+1}, y^i, l_{yx}^{i+2}\} \\ \{x^{i+1}, y^{i+1}, l_{xy}^{i+2}\}, \{x^{i+1}, y^{i+1}, l_{yx}^{i+3}\} \end{array}\right\} \quad 1 \leqslant x < y \leqslant n; \quad i = 0, 2, 4. \end{array} \tag{3}$$

The case $n = 2$, $v = 12$, must be treated separately. If $T(a, b, c, d)$ denotes the set of four possible triples with elements in $\{a, b, c, d\}$, then

$$T(j, j+3, j+6, j+9): \quad j=0,1,2;$$
$$\{3x, 3y+1, 3z+2\}: \quad x, y, z \in \{0,1,2,3\},\ x+y+z \text{ odd} \tag{4}$$

form an $S_2(2, 3, 12)$ on $\{0, 1, \ldots, 11\}$.

(iii) *Case* $v \equiv 3 \pmod 6$

We use the symbols x^i for $1 \leqslant x \leqslant n$ and $i = 0, 1, 2$, where n is odd, and a symmetric transversal square (l_{ij}) of order n. The triples are

$$\{x^1, x^2, x^3\}, \{x^1, x^2, (x+1)^3\}: \quad 1 \leqslant x \leqslant n;$$
$$\left.\begin{array}{l} \{x^1, y^1, l_{xy}^2\}, \{x^1, y^1, (l_{xy}+1)^3\} \\ \{x^2, y^2, l_{xy}^3\}, \{x^2, y^2, l_{xy}^1\} \\ \{x^3, y^3, l_{xy}^1\}, \{x^3, y^3, (l_{xy}-1)^2\} \end{array}\right\} \quad 1 \leqslant x < y \leqslant n \tag{5}$$

where additions and subtractions are carried out modulo n.

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