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Sufficient conditions for super k-restricted edge connectivity in graphs of diameter 2[☆]

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Abstract

For a connected graph $G = (V, E)$, an edge set $S \subseteq E$ is a k-restricted edge cut if $G - S$ is disconnected and every component of $G - S$ has at least k vertices. The k-restricted edge connectivity of G, denoted by $\lambda_k(G)$, is defined as the cardinality of a minimum k-restricted edge cut. Let $\xi_k(G) = \min\{|[X, \overline{X}]| : |X| = k, G[X] \text{ is connected}\}$. G is λ_k-optimal if $\lambda_k(G) = \xi_k(G)$. Moreover, G is super-λ_k if every minimum k-restricted edge cut of G isolates one connected subgraph of order k. In this paper, we prove that if $|N_G(u) \cap N_G(v)| \geq 2k - 1$ for all pairs u, v of nonadjacent vertices, then G is λ_k-optimal; and if $|N_G(u) \cap N_G(v)| \geq 2k$ for all pairs u, v of nonadjacent vertices, then G is either super-λ_k or in a special class of graphs. In addition, for k-isoperimetric edge connectivity, which is closely related with the concept of k-restricted edge connectivity, we show similar results.

Keywords: Restricted edge connectivity; Isoperimetric edge connectivity; λ_k-optimality; Diameter

1. Terminology and introduction

For graph-theoretical terminology and notation not defined here we follow [4]. We consider finite, undirected and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. For any vertex v in G, we define the neighbour set of v in G to be the set of all vertices adjacent to v; this set is denoted by $N(v) = N_G(v)$. If G' is a subgraph of G and v is a vertex of G', we define $N_{G'}(v) = N_G(v) \cap V(G')$. The degree $d_G(v)$ of a vertex $v \in V(G)$ equals the number of vertices in $N_G(v)$. Let $\delta(G)$ denote the minimum degree in G. For $U \subseteq V(G)$ let $G[U]$ be the subgraph induced by U. For subsets U and U' of $V(G)$, we denote by $[U, U']$ the set of edges with one end in U and the other in U'. If vertices u and v are connected in G, the distance between u and v in G, denoted by $d_G(u, v)$, is the length of a shortest path from u to v in G; if there is no path connecting u and v we define $d_G(u, v)$ to be infinite. The diameter of G, denoted by $D(G)$, is the maximum distance between two vertices of G. Let G_1 and G_2 be two graphs. The union $G_1 \cup G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

The edge connectivity $\lambda(G)$ of a graph G is the minimum cardinality of an edge cut of G. It is well known that $\lambda(G) \leq \delta(G)$. A graph G is λ-optimal if $\lambda(G) = \delta(G)$. Furthermore, G is super-λ if every minimum edge cut consists

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of edges adjacent to a vertex of minimum degree. As a more refined index than the edge connectivity, restricted edge connectivity was proposed by Esfahanian and Hakimi [6]. A set of edges S in a connected graph G is called a restricted edge cut if $G - S$ is disconnected and contains no isolated vertex. If such an edge cut exists, then the restricted edge connectivity of G, denoted by $\lambda'(G)$, is defined to be the minimum number of edges over all restricted edge cuts of G. A graph is called λ'-connected if it contains restricted edge cuts. Esfahanian and Hakimi [6] showed that each connected graph G of order $\nu(G) \geq 4$ except a star $K_{1,\nu-1}$ is λ'-connected and satisfies $\lambda(G) \leq \lambda'(G) \leq \xi(G)$, where $\xi(G) = \min\{d_G(u) + d_G(v) - 2 : uv \in E(G)\}$ is the minimum edge degree of G. A graph G is λ'-optimal if $\lambda'(G) = \xi(G)$. Moreover, G is super-λ' if every minimum restricted edge cut of G isolates one edge, that is, every minimum restricted edge cut of G is a set of edges adjacent to a certain edge with minimum edge degree in G. There has been much research on λ-optimal graphs, super-λ graphs, λ'-optimal graphs and super-λ' graphs (cf. e.g. [1–3,6,7,9–12,16–19]).

Generally, for a connected graph G, an edge set $S \subseteq E(G)$ is called a k-restricted edge cut of G if $G - S$ is disconnected and every component of $G - S$ has at least k vertices. The k-restricted edge connectivity of G, denoted by $\lambda_k(G)$, is defined as the cardinality of a minimum k-restricted edge cut. A minimum k-restricted edge cut is called a λ_k-cut. By definition, if S is a λ_k-cut, then $|S| = \lambda_k(G)$. It should be pointed out that not all connected graphs have k-restricted edge cuts. A connected graph G is called λ_k-connected if $\lambda_k(G)$ exists. It is easy to see that if G is λ_k-connected for $k \geq 2$, then G is also λ_{k-1}-connected and $\lambda_{k-1}(G) \leq \lambda_k(G)$. Sufficient conditions for graphs to be λ_k-connected were given by several authors [5,6,15,20]. In view of recent studies on k-restricted edge connectivity, it seems that the larger $\lambda_k(G)$ is, the more reliable the network is [13,14,18]. So, we expect $\lambda_k(G)$ to be as large as possible. Clearly, the optimization of $\lambda_k(G)$ requires an upper bound first. For any positive integer k, let

$$\xi_k(G) = \min\{|[X, \overline{X}]| : |X| = k, G[X] \text{ is connected}\}.$$

It has been shown that $\lambda_k(G) \leq \xi_k(G)$ holds for many graphs [5,15,20]. A connected graph G is called a λ_k-optimal graph if $\lambda_k(G) = \xi_k(G)$. Furthermore, G is called a super k-restricted edge-connected graph, in short, a super-λ_k graph, if every λ_k-cut of G isolates one connected subgraph of order k, that is, every λ_k-cut of G is a set of edges adjacent to a certain connected subgraph of order k. Clearly, $\lambda_1(G) = \lambda(G)$, $\lambda_2(G) = \lambda'(G)$, $\xi_1(G) = \delta(G)$ and $\xi_2(G) = \xi(G)$. Moreover, λ-optimality, super-λ property, λ'-optimality and super-λ' property are λ_1-optimality, super-λ_1 property, λ_2-optimality and super-λ_2 property, respectively. Let G be a λ_k-connected graph with $\lambda_k(G) \leq \xi_k(G)$. By definition, if G is a super-λ_k graph, then G must be a λ_k-optimal graph. However, the converse is not true. For example, a cycle of length $\nu(\nu \geq 2k + 2)$ is a λ_k-optimal graph that is not super-λ_k.

In Section 3, first, we prove that G is λ_k-optimal if $|N_G(u) \cap N_G(v)| \geq 2k - 1$ for all pairs u, v of nonadjacent vertices, and G is super-λ_k if G is not in a special class of graphs and $|N_G(u) \cap N_G(v)| \geq 2k$ for all pairs u, v of nonadjacent vertices. Next, we show that some known results are consequences of our results and give examples to show that our results are best possible in some sense.

In Section 4, we turn our attention to the analogous concept of k-isoperimetric edge connectivity. The k-isoperimetric edge connectivity of G is defined as

$$\gamma_k(G) = \min\{|[X, \overline{X}]| : X \subseteq V(G), |X| \geq k, |\overline{X}| \geq k\}.$$

Clearly, $\gamma_k(G)$ exists for any positive integer $k \leq |V(G)|/2$. An edge cut $S = [X, \overline{X}]$ is called a γ_k-cut if $|S| = \gamma_k(G)$ and $X \subseteq V(G)$, $|X| \geq k$, $|\overline{X}| \geq k$. Let

$$\beta_k(G) = \min\{|[X, \overline{X}]| : X \subseteq V(G), |X| = k\}.$$

Then, it is obvious that $\gamma_k(G) \leq \beta_k(G)$. A graph G is called a γ_k-optimal graph if $\gamma_k(G) = \beta_k(G)$. Moreover, G is called a super-γ_k graph if every γ_k-cut $S = [X, \overline{X}]$ of G has the property that either $|X| = k$ or $|\overline{X}| = k$. It is easy to see that a super-γ_k graph must be a γ_k-optimal graph, but the converse is not true. Several researchers [8,21] have studied γ_k-optimal graphs. In Section 4, we will give some conditions, which are similar to those given in Section 3, for graphs to be γ_k-optimal or super-γ_k.

2. Preliminaries

We start with a simple but useful observation.

Observation 2.1 ([9]). Let G be a graph of order $\nu \geq 2$. Then each pair u, v of nonadjacent vertices satisfies $|N(u) \cap N(v)| \geq 1$ if and only if $D(G) \leq 2$.

Proposition 2.1. *Let k be a positive integer and let G be a graph with at least one pair of nonadjacent vertices. If*

$$|N(u) \cap N(v)| \geq 2k - 1$$

for each pair u, v of nonadjacent vertices, then G is λ_k-connected and $\lambda_k(G) \leq \xi_k(G)$.

Proof. Clearly, $|V(G)| \geq 2k+1$. By Observation 2.1, $D(G) \leq 2$ and hence G is connected. The case $k = 1$ is trivial, so we only consider the case $k \geq 2$. Since G is connected and $|V(G)| \geq 2k+1$, $\xi_k(G)$ exists. Let U be a subset of $V(G)$ such that $|U| = k$, $G[U]$ is connected and $\xi_k(G) = |[U, \overline{U}]|$. If u, v are two nonadjacent vertices in $\overline{U}$, then $|N(u) \cap N(v)| \geq 2k - 1$. Since $|N(u) \cap N(v) \cap U| \leq |U| = k$ and $k \geq 2$, we have $|N(u) \cap N(v) \cap \overline{U}| \geq k - 1 > 0$. Therefore, $G[\overline{U}]$ is connected and thus $[U, \overline{U}]$ is a k-restricted edge cut, which implies that G is λ_k-connected and $\lambda_k(G) \leq |[U, \overline{U}]| = \xi_k(G)$. The proof is complete. □

Let G be a λ_k-connected graph and let S be a λ_k-cut of G. By the minimality of S, the graph $G - S$ consists of exactly two components, say G_1 and G_2. Let $X = V(G_1)$. Then $\overline{X} = V(G_2)$ and $S = [X, \overline{X}]$. Denote $X^0 = \{x \in X : |N(x) \cap \overline{X}| \leq k - 1\}$, $\overline{X}^0 = \{y \in \overline{X} : |N(y) \cap X| \leq k - 1\}$. Without loss of generality, assume

$$\min\{|N(x) \cap \overline{X}| : x \in X\} \geq \min\{|N(y) \cap X| : y \in \overline{X}\}. \tag{1}$$

We will use such notation and this assumption in this section and next section.

The main goal of this section is to give some useful properties of $G[X]$ and $G[\overline{X}]$. By reason of symmetry we only discuss $G[X]$.

Lemma 2.1. *Let G be a λ_k-connected graph with $\lambda_k(G) \leq \xi_k(G)$ and let $S = [X, \overline{X}]$ be a λ_k-cut of G.*

(i) *If there exists a connected subgraph H of order k in $G[X]$ with the property that*

$$\sum_{v \in X \setminus V(H)} |N(v) \cap V(H)| \leq \sum_{v \in X \setminus V(H)} |N(v) \cap \overline{X}|,$$

then G is λ_k-optimal.

(ii) *There exists no connected subgraph H of order k in $G[X]$ with the property that*

$$\sum_{v \in X \setminus V(H)} |N(v) \cap V(H)| < \sum_{v \in X \setminus V(H)} |N(v) \cap \overline{X}|.$$

Proof. The hypotheses of Lemma 2.1(i) imply

$$\begin{aligned}
\xi_k(G) &\leq |[V(H), \overline{V(H)}]| \\
&= |[V(H), X \setminus V(H)]| + |[V(H), \overline{X}]| \\
&= \sum_{v \in X \setminus V(H)} |N(v) \cap V(H)| + |[V(H), \overline{X}]| \\
&\leq \sum_{v \in X \setminus V(H)} |N(v) \cap \overline{X}| + |[V(H), \overline{X}]| \\
&= |[X \setminus V(H), \overline{X}]| + |[V(H), \overline{X}]| \\
&= |[X, \overline{X}]| = |S| = \lambda_k(G).
\end{aligned}$$

Since $\lambda_k(G) \leq \xi_k(G)$, we deduce that $\lambda_k(G) = \xi_k(G)$ and hence G is λ_k-optimal. The proof of (i) is complete. (ii) can be easily seen from the proof of (i). □

Corollary 2.1. *Let G be a λ_k-connected graph with $\lambda_k(G) \leq \xi_k(G)$ and let $S = [X, \overline{X}]$ be a λ_k-cut. If there exists a vertex x^* in X such that $|N(x^*) \cap \overline{X}| \geq k + 1$, then there exists no connected subgraph H of order k in $G[X] - x^*$*

with the property that

$$\sum_{v\in X\setminus(V(H)\cup\{x^*\})}|N(v)\cap V(H)|\leq\sum_{v\in X\setminus(V(H)\cup\{x^*\})}|N(v)\cap\overline{X}|.$$

Proof. Suppose, on the contrary, that there exists such a subgraph H. Since $|N(x^*)\cap\overline{X}|\geq k+1>k=|V(H)|\geq|N(x^*)\cap V(H)|$, it follows that $\sum_{v\in X\setminus V(H)}|N(v)\cap V(H)|<\sum_{v\in X\setminus V(H)}|N(v)\cap\overline{X}|$, contrary to Lemma 2.1 (ii). $\square$

Lemma 2.2. *Suppose that G is a λ_k-connected graph and $S=[X,\overline{X}]$ is a λ_k-cut of G. Let X^* be a subset of X such that $|X^*|\geq k$, $X^0\subseteq X^*$ and $G[X^*]$ is connected. If there exists a connected subgraph H' of $G[X^*]$ such that $|V(H')|\leq k$ and $X^0\subseteq V(H')$, then there exists a connected subgraph H of order k in $G[X^*]$ such that*

$$\sum_{v\in X^*\setminus V(H)}|N(v)\cap V(H)|\leq\sum_{v\in X^*\setminus V(H)}|N(v)\cap\overline{X}|.$$

Proof. If there exists such a subgraph H', then, by the connectedness of $G[X^*]$, there exists a connected subgraph H of $G[X^*]$ such that $|V(H)|=k$, $X^0\subseteq V(H')\subseteq V(H)$ and hence $X^*\setminus V(H)\subseteq X^*\setminus X^0$. By the definition of X^0, $|N(v)\cap\overline{X}|\geq k$ for any $v\in X^*\setminus V(H)$. It follows that $\sum_{v\in X^*\setminus V(H)}|N(v)\cap V(H)|\leq\sum_{v\in X^*\setminus V(H)}|V(H)|=k|X^*\setminus V(H)|\leq\sum_{v\in X^*\setminus V(H)}|N(v)\cap\overline{X}|$. $\square$

Combining Lemmas 2.1(i) and 2.2, we get the following corollary.

Corollary 2.2. *Let G be a λ_k-connected graph with $\lambda_k(G)\leq\xi_k(G)$ and let $S=[X,\overline{X}]$ be a λ_k-cut. If there exists a connected subgraph H of $G[X]$ such that $|V(H)|\leq k$ and $X^0\subseteq V(H)$, then G is λ_k-optimal. In particular, if $X^0=\emptyset$, then G is λ_k-optimal.*

Lemma 2.3. *Let X^* be a subset of X such that $|X^*|\geq k$, $X^0\subseteq X^*$ and $G[X^*]$ is connected and let $G^*=G[X^*\cup\overline{X}]$. If $X^0\neq\emptyset$, $\overline{X}^0\neq\emptyset$ and $|N_{G^*}(u)\cap N_{G^*}(v)|\geq 2k-1$ for all pairs u,v of nonadjacent vertices in G^*, then there exists a connected subgraph H of order k in $G[X^*]$ such that*

$$\sum_{v\in X^*\setminus V(H)}|N(v)\cap V(H)|\leq\sum_{v\in X^*\setminus V(H)}|N(v)\cap\overline{X}|.$$

Proof. Suppose, on the contrary, that $G[X^*]$ contains no connected subgraph H of order k in $G[X^*]$ such that $\sum_{v\in X^*\setminus V(H)}|N(v)\cap V(H)|\leq\sum_{v\in X^*\setminus V(H)}|N(v)\cap\overline{X}|$. Then we have the following nine claims.

Claim 1. $xy\in E(G^*)$ for any $x\in X^0$, $y\in\overline{X}^0$.

By contradiction. Suppose that there exist $x\in X^0$ and $y\in\overline{X}^0$ such that $xy\notin E(G^*)$. Then $2k-1\leq|N_{G^*}(x)\cap N_{G^*}(y)|=|N_{G^*}(x)\cap N_{G^*}(y)\cap X^*|+|N_{G^*}(x)\cap N_{G^*}(y)\cap\overline{X}|\leq|N_{G^*}(y)\cap X^*|+|N_{G^*}(x)\cap\overline{X}|\leq|N(y)\cap X|+|N(x)\cap\overline{X}|\leq 2(k-1)=2k-2$, a contradiction.

Claim 2. $1\leq|X^0|\leq k-1$ and $1\leq|\overline{X}^0|\leq k-1$ (and therefore $k\geq 2$).

Since $X^0\neq\emptyset$ and $\overline{X}^0\neq\emptyset$, we have $1\leq|X^0|$ and $1\leq|\overline{X}^0|$. Let y be a vertex in $\overline{X}^0$. Then, by Claim 1, $X^0\subseteq N_{G^*}(y)$ and hence $X^0\subseteq N_{G^*}(y)\cap X^*$. It follows that $|X^0|\leq|N_{G^*}(y)\cap X^*|\leq|N(y)\cap X|\leq k-1$. Similarly, we also have $|\overline{X}^0|\leq k-1$.

Claim 3. $|X^0|>\lceil\frac{k}{2}\rceil$ and there exists a connected subgraph H' of order at most k in $G[X^*]$ such that $|V(H')\cap X^0|\geq\lceil\frac{k}{2}\rceil$.

Let $t=\lceil\frac{k}{2}\rceil$ and let $U=\{x_1,\ldots,x_t\}$ be a subset of X^* such that U contains as many vertices in X^0 as possible. Clearly, $X^0\cap U\neq\emptyset$, and if $|X^0|\leq\lceil\frac{k}{2}\rceil$, then $X^0\subseteq U$; if $|X^0|>\lceil\frac{k}{2}\rceil$, then $U\subseteq X^0$. Without loss of generality, assume that $x_1\in X^0$.

By Claim 2, we have $k\geq 2$. If $k=2$, then let H' be the graph with vertex set $\{x_1\}$. It is easy to see that H' is a connected subgraph of $G[X^*]$, $|V(H')|\leq k$ and $U=\{x_1\}=V(H')$. If $k\geq 3$, then $t\geq 2$. For any $i=2,\ldots,t$, if $x_1x_i\in E(G^*)$, then let $u_i=x_i$. If $x_1x_i\notin E(G^*)$, then, since $|N_{G^*}(x_1)\cap N_{G^*}(x_i)|\geq 2k-1$ and

$|N_{G^*}(x_1) \cap N_{G^*}(x_i) \cap \overline{X}| \le |N(x_1) \cap \overline{X}| \le k-1$, we have $|N_{G^*}(x_1) \cap N_{G^*}(x_i) \cap X^*| \ge k > 0$ and hence we may let u_i be a vertex in $N_{G^*}(x_1) \cap N_{G^*}(x_i) \cap X^*$. Let $U^* = U \cup \{u_2, \ldots, u_t\}$ and let $H' = G[U^*]$. Then H' is a connected subgraph of $G[X^*]$, $|V(H')| = |U^*| \le t + (t-1) \le k$ and $U \subseteq V(H')$.

Suppose that $|X^0| \le \lceil \frac{k}{2} \rceil$. Then $X^0 \subseteq U \subseteq V(H')$. By Lemma 2.2, there exists a connected subgraph H of order k in $G[X^*]$ such that $\sum_{v \in X^* \setminus V(H)} |N(v) \cap V(H)| \le \sum_{v \in X^* \setminus V(H)} |N(v) \cap \overline{X}|$, contrary to the initial assumption. Therefore, $|X^0| > \lceil \frac{k}{2} \rceil$ and hence $U \subseteq X^0$. This implies that $|V(H') \cap X^0| \ge |U| = t = \lceil \frac{k}{2} \rceil$. The proof of Claim 3 is complete.

Denote $|X^0| = k - s$. Then, by Claims 2 and 3, $1 \le s < \lfloor \frac{k}{2} \rfloor$.

Claim 4. $|N(x) \cap \overline{X}| \ge k - s$ for any $x \in X^0$.

Let $y' \in \overline{X}$ be a vertex such that $|N(y') \cap X| = \min\{|N(y) \cap X| : y \in \overline{X}\}$. Clearly, $y' \in \overline{X}^0$. By Claim 1, $X^0 \subseteq N_{G^*}(y') \cap X^*$ and hence $k - s = |X^0| \le |N_{G^*}(y') \cap X^*| \le |N(y') \cap X|$. By the assumption (1), we have $|N(x) \cap \overline{X}| \ge \min\{|N(y) \cap X| : y \in \overline{X}\} = |N(y') \cap X|$ and thus $|N(x) \cap \overline{X}| \ge k - s$ for any $x \in X^0$.

Claim 5. $|X^*| \ge 2k - s$.

If $G[X^0]$ is connected, then let $H' = G[X^0]$. By Lemma 2.2, a contradiction to the initial assumption is obtained. Therefore, $G[X^0]$ is not connected. Let x_1, x_2 be two vertices in two different components of $G[X^0]$. Then $|N_{G^*}(x_1) \cap N_{G^*}(x_2)| \ge 2k-1$, $|N_{G^*}(x_1) \cap N_{G^*}(x_2) \cap X^0| = 0$ and $|N_{G^*}(x_1) \cap N_{G^*}(x_2) \cap \overline{X}| \le |N(x_1) \cap \overline{X}| \le k-1$. It follows that $|N_{G^*}(x_1) \cap N_{G^*}(x_2) \cap (X^* \setminus X^0)| \ge k$ and hence $|X^* \setminus X^0| \ge k$. Consequently, $|X^*| = |X^0| + |X^* \setminus X^0| \ge 2k - s$.

Claim 6. $|N(v) \cap X^0| \le k - 2s$ for any $v \in X^* \setminus X^0$.

By contradiction. Suppose that there exists a vertex $v \in X^* \setminus X^0$ such that $|N(v) \cap X^0| \ge k - 2s + 1$. Then $|N_{G^*}(v) \cap X^0| \ge k - 2s + 1$. Let $U_1 = N_{G^*}(v) \cap X^0, U_2 = X^0 \setminus U_1 = \{x_1, x_2, \ldots, x_t\}$. Then $t = |U_2| = k - s - |U_1| \le s - 1$. For any $i = 1, 2, \ldots, t$, since $vx_i \notin E(G^*)$, we have $|N_{G^*}(v) \cap N_{G^*}(x_i)| \ge 2k-1$. Combining this with $|N_{G^*}(v) \cap N_{G^*}(x_i) \cap \overline{X}| \le |N(x_i) \cap \overline{X}| \le k-1$, we have $|N_{G^*}(v) \cap N_{G^*}(x_i) \cap X^*| \ge k$ and hence we may pick a vertex u_i in $N_{G^*}(v) \cap N_{G^*}(x_i) \cap X^*$. Let $U = U_1 \cup U_2 \cup \{u_1, \ldots, u_t\} \cup \{v\}$. Then $X^0 \subseteq U$. Clearly, $G^*[U]$ is connected and $|U| \le |U_1 \cup U_2| + t + 1 \le (k-s) + (s-1) + 1 = k$. By Lemma 2.2, a contradiction to the initial assumption is obtained. The proof of Claim 6 is then complete.

Let $X^1 = \{x \in X^* \setminus X^0 : xy \in E(G^*)$ for any $y \in \overline{X}^0\}$ and let $X^2 = X^* \setminus (X^0 \cup X^1)$.

Claim 7. $|X^1| \le s - 1$.

By Claim 1 and the definitions of X^0 and X^1, for any $y \in \overline{X}^0$, $k - s + |X^1| = |X^0| + |X^1| = |X^0 \cup X^1| = |N(y) \cap (X^0 \cup X^1)| \le |N(y) \cap X| \le k - 1$. Therefore, $|X^1| \le s - 1$.

Claim 8. $|N(x) \cap \overline{X}| \ge k + s$ for any $x \in X^2$.

For any $x \in X^2$, by definition, there exists a vertex $y \in \overline{X}^0$ such that $xy \notin E(G^*)$. By Claim 1 and the choice of y, $|N(y) \cap (X^* \setminus X^0)| = |N(y) \cap X^*| - |N(y) \cap X^0| = |N(y) \cap X^*| - |X^0| \le (k-1) - (k-s) = s - 1$. By Claim 6, $|N(x) \cap X^0| \le k - 2s$. Therefore, we have

$$\begin{aligned} 2k - 1 &\le |N_{G^*}(x) \cap N_{G^*}(y)| \\ &= |N_{G^*}(x) \cap N_{G^*}(y) \cap \overline{X}| + |N_{G^*}(x) \cap N_{G^*}(y) \cap X^0| + |N_{G^*}(x) \cap N_{G^*}(y) \cap (X^* \setminus X^0)| \\ &\le |N_{G^*}(x) \cap \overline{X}| + |N_{G^*}(x) \cap X^0| + |N_{G^*}(y) \cap (X^* \setminus X^0)| \\ &\le |N(x) \cap \overline{X}| + |N(x) \cap X^0| + |N(y) \cap (X^* \setminus X^0)| \\ &\le |N(x) \cap \overline{X}| + (k - 2s) + (s - 1). \end{aligned}$$

This implies that $|N(x) \cap \overline{X}| \ge k + s$.

Claim 9. $|N(x) \cap \overline{X}| \ge k$ for any $x \in X^1$.

Since $X^1 \cap X^0 = \emptyset$, Claim 9 follows from the definition of X^0.

We continue now with the proof of this lemma. By Claim 3, there exists a connected subgraph H' of order at most k in $G[X^*]$ such that $|V(H') \cap X^0| \ge \lceil \frac{k}{2} \rceil$. By the connectedness of $G[X^*]$, there exists a connected subgraph H of $G[X^*]$ such that $|V(H)| = k$ and $V(H') \subseteq V(H)$. It follows that $|V(H) \cap X^0| \ge \lceil \frac{k}{2} \rceil$.

Let $t_0 = |(X^* \setminus V(H)) \cap X^0|, t_1 = |(X^* \setminus V(H)) \cap X^1|$ and $t_2 = |(X^* \setminus V(H)) \cap X^2|$. Then $t_0 = |X^0| - |V(H) \cap X^0| = (k - s) - |V(H) \cap X^0| \le \lfloor \frac{k}{2} \rfloor - s$. Combining this with Claims 5 and 7, we have $t_2 = |X^*| - |V(H)| - t_1 - t_0 \ge |X^*| - |V(H)| - |X^1| - t_0 \ge (2k - s) - k - (s - 1) - (\lfloor \frac{k}{2} \rfloor - s) = \lceil \frac{k}{2} \rceil - s + 1 > t_0$.

Combining this with Claims 4, 8 and 9, we have

$$\begin{aligned}\sum_{v\in X^*\setminus V(H)} |N(v)\cap \overline{X}| &= \sum_{i=0}^{2}\sum_{v\in (X^*\setminus V(H))\cap X^i} |N(v)\cap \overline{X}| \\ &\geq (k-s)t_0 + kt_1 + (k+s)t_2 \\ &> k(t_2+t_1+t_0) \\ &= k|X^*\setminus V(H)| \\ &\geq \sum_{v\in X^*\setminus V(H)} |N(v)\cap V(H)|.\end{aligned}$$

This is contrary to the initial assumption. The proof is then complete. □

3. Sufficient conditions for λ_k-optimality or super-λ_k property

We start with a simple result.

Proposition 3.1. *Let k be a positive integer. If G is a complete graph with order at least $2k$, then G is super-λ_k.*

Proof. It is easy to verify that G is λ_k-connected and $\lambda_k(G) \leq \xi_k(G)$. Suppose, on the contrary, that G is not super-λ_k. Then there exists a λ_k-cut $S = [X, \overline{X}]$ such that $|X| \geq k+1$ and $|\overline{X}| \geq k+1$. Let U be a subset of X such that $|U| = k$ and let $H = G[U]$. Then H is connected and $\sum_{v\in X\setminus V(H)} |N(v)\cap V(H)| = k|X\setminus V(H)| < (k+1)|X\setminus V(H)| \leq \sum_{v\in X\setminus V(H)} |N(v)\cap \overline{X}|$, a contradiction to Lemma 2.1(ii). □

Theorem 3.1. *Let k be a positive integer and let G be a graph with order at least $2k$. If*

$$|N(u)\cap N(v)| \geq 2k-1$$

for all pairs u, v of nonadjacent vertices, then G is λ_k-optimal.

Proof. If G contains no nonadjacent vertices, then G is a complete graph with order at least $2k$. By Proposition 3.1, G is super-λ_k and so G is λ_k-optimal. Therefore, we only consider the case that there exist nonadjacent vertices in G below. By Proposition 2.1, G is λ_k-connected and $\lambda_k(G) \leq \xi_k(G)$. Let $S = [X, \overline{X}]$ be an arbitrary λ_k-cut of G. By definition, $|X| \geq k$, $|\overline{X}| \geq k$.

If $X^0 = \emptyset$ or $\overline{X}^0 = \emptyset$, then, by Corollary 2.2, G is λ_k-optimal. Suppose that both $X^0 \neq \emptyset$ and $\overline{X}^0 \neq \emptyset$. By Lemma 2.3 (regarding X as the X^* in Lemma 2.3), there exists a connected subgraph H of order k in $G[X]$ such that $\sum_{v\in X\setminus V(H)} |N(v)\cap V(H)| \leq \sum_{v\in X\setminus V(H)} |N(v)\cap \overline{X}|$. By Lemma 2.1(i), G is λ_k-optimal. □

Recalling that $\lambda_1(G) = \lambda(G)$ and $\xi_1(G) = \delta(G)$, Corollary 3.1 follows from Observation 2.1 and Theorem 3.1.

Corollary 3.1 (*Plesnik [16] 1975*). *If G is a graph of diameter $D(G) \leq 2$, then $\lambda(G) = \delta(G)$.*

Recalling that $\lambda'(G) = \lambda_2(G)$ and λ'-optimality is λ_2-optimality, we have the following corollary.

Corollary 3.2 (*Hellwig and Volkmann [9] 2004*). *Let G be a λ'-connected graph. If*

$$|N(u)\cap N(v)| \geq 3$$

for all pairs u, v of nonadjacent vertices, then G is λ'-optimal.

Observation 3.1. Let G be a graph of order ν and let $u, v \in V(G)$ be a pair of nonadjacent vertices. For any integer $l \geq -1$, if $d_G(u) + d_G(v) \geq \nu + l$, then $|N(u)\cap N(v)| \geq l+2$.

Corollary 3.3 (*Zhang and Yuan [21] 2007*). *Let k be a positive integer, and G a connected graph on $\nu \geq 2k$ vertices. Suppose that $d_G(u) + d_G(v) \geq \nu + 2k - 3$, for every pair of nonadjacent vertices u and v in G. Then G is λ_k-optimal.*

Next we introduce a class of graphs $W(p, k)$ that will show that Theorem 3.1 is an improvement of Corollary 3.3.

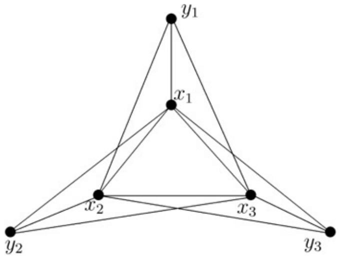

Fig. 1. The graph $W(3, 2)$.

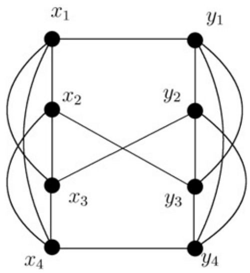

Fig. 2. A graph in $L_2(4, 1)$.

Example 3.1. Let k and p be fixed positive integers with $p \geq 3$; let $C = \{x_1, x_2, \ldots, x_{2k-1}\}$, $I = \{y_1, y_2, \ldots, y_p\}$; and define $W(p, k)$ as follows. The vertices of $W(k, p)$ are $x_1, x_2, \ldots, x_{2k-1}, y_1, y_2, \ldots, y_p$, where two vertices u and v are adjacent if at least one of u and v is in C. The graph $W(3, 2)$ is shown in Fig. 1. If u, v is a pair of nonadjacent vertices in $W(k, p)$, then $u, v \in I$. It is easy to see that $|N(u) \cap N(v)| = |C| = 2k - 1$ and $d_G(u) + d_G(v) = 2(2k - 1) = 4k - 2$. By Theorem 3.1, G is λ_k-optimal. But, since $p \geq 3$, we have $d_G(u) + d_G(v) = 4k - 2 < (2k - 1 + 3) + (2k - 3) \leq (|C| + |I|) + (2k - 3) = |V| + 2k - 3$ and hence Corollary 3.3 does not show that G is λ_k-optimal.

The next example shows that Theorem 3.1 is best possible in the sense that the condition $|N(u) \cap N(v)| \geq 2k-2$ for all pairs u, v of nonadjacent vertices does not imply λ_k-optimality. Before giving the example, we introduce another class of graphs.

Given two integers $p > k > 0$, let H_1 and H_2 be two disjoint complete graphs with $V(H_1) = \{x_1, \ldots, x_p\}$ and $V(H_2) = \{y_1, \ldots, y_p\}$, respectively. Let F_k be the set of all k-regular bipartite graphs with bipartition $(V(H_1), V(H_2))$, and let the graph class $L_2(p, k) = \{H_1 \cup H_2 \cup H_3 : H_3 \in F_k\}$. A graph in $L_2(4, 1)$ is shown in Fig. 2.

Example 3.2. Let $G = H_1 \cup H_2 \cup H_3$ be a graph in $L_2(p, k - 1)$, where $p > k$. Then $|N(u) \cap N(v)| = 2k - 2$ for all pairs u, v of nonadjacent vertices in G. Let U be a subset of $V(G)$ such that $|U| = k$. Denote $t = |U \cap V(H_1)|$, $s = |U \cap V(H_2)|$. Then $t + s = k$ and $|[U, \overline{U}]| = |[U \cap V(H_1), \overline{U} \cap V(H_1)]| + |[U \cap V(H_1), \overline{U} \cap V(H_2)]| + |[U \cap V(H_2), \overline{U} \cap V(H_1)]| + |[U \cap V(H_2), \overline{U} \cap V(H_2)]| \geq t(p-t) + t((k-1)-s) + s((k-1)-t) + s(p-s) = k(p-1)$. If $U \subseteq V(H_1)$, then $G[U]$ is connected and $|[U, \overline{U}]| = |[U, V(H_1) \setminus U]| + |[U, V(H_2)]| = k(p-k) + k(k-1) = k(p-1)$. Therefore, $\xi_k(G) = k(p-1)$. Clearly, $E(H_3) = [V(H_1), V(H_2)]$ is a k-restricted edge cut and hence $\lambda_k(G) \leq |E(H_3)| = p(k-1)$. It follows that $\lambda_k(G) < \xi_k(G)$ from $p > k$. Therefore, G is not λ_k-optimal.

Theorem 3.1 shows that the condition $|N(u) \cap N(v)| \geq 2k - 1$ for all pairs u, v of nonadjacent vertices guarantees the graph G is λ_k-optimal, but even a stronger condition that $|N(u) \cap N(v)| \geq 2k$ for all pairs u, v of nonadjacent vertices cannot guarantee G is super-λ_k. We give such an example below.

Example 3.3. Let $G \in L_2(p, k)$, where $p > k$. By a similar method as in the above example, we have $\xi_k(G) = kp$ and $|N(u) \cap N(v)| = 2k$ for all pairs u, v of nonadjacent vertices. By Theorem 3.1, G is λ_k-optimal and hence $\lambda_k(G) = \xi_k(G) = kp$. We also have $|E(H_3)| = |[V(H_1), V(H_2)]| = kp$, which implies that $[V(H_1), V(H_2)]$ is a λ_k-cut. Combining this with $|V(H_1)| = |V(H_2)| = p > k$, we conclude that G is not super-λ_k.

Theorem 3.2. *Let k be a positive integer and let G be a graph with order at least $2k$. If*

$$|N(u) \cap N(v)| \geq 2k$$

for all pairs u, v of nonadjacent vertices, then G either is super-λ_k or is in $L_2(\frac{v}{2}, k)$.

Proof. If G contains no nonadjacent vertices, then, by Proposition 3.1, we are done. Therefore, we only consider the case that there exist nonadjacent vertices in G below. By Theorem 3.1, G is λ_k-optimal. That is, $\lambda_k(G) = \xi_k(G)$. Suppose that G is not super-λ_k. Then there exists a λ_k-cut $S = [X, \overline{X}]$ such that $|X| \geq k+1$ and $|\overline{X}| \geq k+1$.

Claim 1. $X^0 = \emptyset$ or $\overline{X}^0 = \emptyset$.

By contradiction. Suppose that both $X^0 \neq \emptyset$ and $\overline{X}^0 \neq \emptyset$. Similar to the proof of Claim 1 in Lemma 2.3, we have $xy \in E(G)$ for any $x \in X^0$, $y \in \overline{X}^0$.

Since $|X| \geq k+1$ and $|N(y) \cap X| \leq k-1$ for any $y \in \overline{X}^0$, there exist $x^* \in X$ and $y^* \in \overline{X}^0$ such that $x^*y^* \notin E(G)$. Combining this with the fact that $xy \in E(G)$ for any $x \in X^0$, $y \in \overline{X}^0$, we have $x^* \in X \setminus X^0$. Since $|N(x^*) \cap N(y^*)| \geq 2k$ and $|N(x^*) \cap N(y^*) \cap X| \leq |N(y^*) \cap X| \leq k-1$, it follows that $|N(x^*) \cap \overline{X}| \geq k+1$. Let $X^* = X - x^*$, $G^* = G[X^* \cup \overline{X}]$. Then $G^* = G - x^*$. Clearly, $X^0 \subseteq X^*$ and $|N_{G^*}(u) \cap N_{G^*}(v)| \geq 2k-1$ for any pair u, v of nonadjacent vertices in G^*.

We will show that $G[X^*]$ is connected. Suppose that $G[X^*]$ is not connected and let $u \in X^0 \subseteq X^*$. Then we may choose a vertex $v \in X^*$ such that u, v are in different components of $G[X^*]$. Since $|N_{G^*}(u) \cap N_{G^*}(v)| \geq 2k-1$ and $|N_{G^*}(u) \cap N_{G^*}(v) \cap \overline{X}| \leq |N(u) \cap \overline{X}| \leq k-1$, we have $|N_{G^*}(u) \cap N_{G^*}(v) \cap X^*| \geq k > 0$ and hence u, v are connected in $G[X^*]$, which contradicts the choice of v. Therefore, $G[X^*]$ is connected.

By Lemma 2.3, there exists a connected subgraph H of order k in $G[X^*]$ such that $\sum_{v \in X^* \setminus V(H)} |N(v) \cap V(H)| \leq \sum_{v \in X^* \setminus V(H)} |N(v) \cap \overline{X}|$. Recalling $|N(x^*) \cap \overline{X}| \geq k+1$ and $X^* = X - x^*$, we obtain a contradiction to Corollary 2.1. The proof of Claim 1 is complete.

By Claim 1, without loss of generality, we may assume that $X^0 = \emptyset$. That is, $|N(x) \cap \overline{X}| \geq k$ for any $x \in X$.

Claim 2. $|N(x) \cap \overline{X}| = k$ for any $x \in X$.

By contradiction. Suppose that there is a vertex u in X such that $|N(u) \cap \overline{X}| > k$. If $G[X] - u$ is connected or $k = 1$, then there exists a connected subgraph H of $G[X] - u$ such that $|V(H)| = k$. Since $|N(v) \cap V(H)| \leq |V(H)| = k \leq |N(v) \cap \overline{X}|$ for any $v \in X \setminus (V(H) \cup \{u\})$, it follows that $\sum_{v \in X \setminus (V(H) \cup \{u\})} |N(v) \cap V(H)| \leq k|X \setminus (V(H) \cup \{u\})| \leq \sum_{v \in X \setminus (V(H) \cup \{u\})} |N(v) \cap \overline{X}|$, contrary to Corollary 2.1. Therefore, $G[X] - u$ is not connected and $k \geq 2$.

For any $x \in X - u$, we may choose a vertex $x' \in X - u$ such that x, x' are in different components of $G[X] - u$. So $|N(x) \cap N(x')| \geq 2k$ and $N(x) \cap N(x') \cap X \subseteq \{u\}$, which implies that $|N(x) \cap N(x') \cap \overline{X}| \geq 2k-1$ and hence $|N(x) \cap \overline{X}| \geq 2k-1$. Let H be a connected subgraph of $G[X]$ such that $|V(H)| = k$ and $u \in V(H)$. It follows that $\sum_{x \in X \setminus V(H)} |N(x) \cap V(H)| \leq k|X \setminus V(H)| < (2k-1)|X \setminus V(H)| \leq \sum_{x \in X \setminus V(H)} |N(x) \cap \overline{X}|$. This is contrary to Lemma 2.1(ii). Therefore, $|N(x) \cap \overline{X}| = k$ for any $x \in X$.

Claim 3. $\overline{X}^0 = \emptyset$.

By contradiction. Suppose that there exists $y \in \overline{X}^0$. Then, since $|X| > k$ and $|N(y) \cap X| \leq k-1$, there exists $x \in X$ such that $xy \notin E(G)$. By Claim 2, $|N(x) \cap \overline{X}| = k$. It follows that $2k \leq |N(x) \cap N(y)| = |N(x) \cap N(y) \cap X| + |N(x) \cap N(y) \cap \overline{X}| \leq |N(y) \cap X| + |N(x) \cap \overline{X}| \leq (k-1) + k = 2k-1$, a contradiction.

Similar to Claim 2, we have:

Claim 4. $|N(y) \cap X| = k$ for any $y \in \overline{X}$.

Claim 5. $G[X]$ is complete.

Let u, v be two arbitrary vertices in X. First, we will show $G[X] - u$ is connected. For any $x_1, x_2 \in X - u$, if $x_1x_2 \notin E(G)$, then $|N(x_1) \cap N(x_2)| \geq 2k$. Suppose that $k = 1$. By Claim 4, $|N(y) \cap X| = k = 1$ for any $y \in \overline{X}$, which implies that $N(x_1) \cap N(x_2) \cap \overline{X} = \emptyset$. It follows that $|N(x_1) \cap N(x_2) \cap (X - u)| \geq 2k - 1 > 0$. Therefore, $G[X] - u$ is connected. Suppose that $k \geq 2$. Then, by Claim 2, $|N(x_1) \cap N(x_2) \cap \overline{X}| \leq |N(x_1) \cap \overline{X}| = k$. It follows that $|N(x_1) \cap N(x_2) \cap (X - u)| \geq k - 1 > 0$. Therefore, $G[X] - u$ is connected. Clearly, there exists a connected subgraph H of $G[X] - u$ of order k such that $v \in V(H)$. It follows that $\sum_{w \in X \setminus V(H)} |N(w) \cap V(H)| \leq \sum_{w \in X \setminus V(H)} |V(H)| = k|X \setminus V(H)| = \sum_{w \in X \setminus V(H)} |N(w) \cap \overline{X}|$. Combining this with Lemma 2.1(ii), we have $\sum_{w \in X \setminus V(H)} |N(w) \cap V(H)| = \sum_{w \in X \setminus V(H)} |N(w) \cap \overline{X}|$ and hence $\sum_{w \in X \setminus V(H)} |N(w) \cap V(H)| = \sum_{w \in X \setminus V(H)} |V(H)|$. It follows that

$|N(w) \cap V(H)| = |V(H)|$ and so $wz \in E(G)$ for any $w \in X \setminus V(H)$, $z \in V(H)$. In particular, $uv \in E(G)$. By the arbitrariness of u, v, we conclude that $G[X]$ is complete.

Similar to Claim 5, we have:

Claim 6. $G[\overline{X}]$ is complete.

By Claims 2, 4, 5 and 6, it follows that ν is even and G is a graph in $L_2(\frac{\nu}{2}, k)$ and $\frac{\nu}{2} = |X| \geq k+1$. The proof is complete. □

Recalling that for any $G \in L_2(\frac{\nu}{2}, k)$, $|N(u) \cap N(v)| = 2k$ holds for every pair u, v of nonadjacent vertices. Therefore, we have the following corollary.

Corollary 3.4. *Let k be a positive integer and let G be a graph with order at least $2k$. If*

$$|N(u) \cap N(v)| \geq 2k+1$$

for all pairs u, v of nonadjacent vertices, then G is super-λ_k.

By Observation 3.1 and Theorem 3.2, we have the following corollary.

Corollary 3.5. *Let k be a positive integer and let G be a graph with order $\nu \geq 2k$. If*

$$d_G(u) + d_G(v) \geq \nu + 2k - 2$$

for all pairs u, v of nonadjacent vertices, then G either is super-λ_k or is in $L_2(\frac{\nu}{2}, k)$.

By Observation 3.1 and Corollary 3.4, we have the following corollary.

Corollary 3.6. *Let k be a positive integer and let G be a graph with order $\nu \geq 2k$. If*

$$d_G(u) + d_G(v) \geq \nu + 2k - 1$$

for all pairs u, v of nonadjacent vertices, then G is super-λ_k.

Recalling that super-λ property is super-λ_1 property, we have the following corollary.

Corollary 3.7 (*Lesniak [11] 1974*). *Let G be a connected graph of order ν. If $d_G(u) + d_G(v) \geq \nu + 1$ for all pairs u, v of nonadjacent vertices, then G is super-λ.*

The conditions that G is not in $L_2(\frac{\nu}{2}, k)$ and $|N(u) \cap N(v)| \geq 2k-1$ for all pairs u, v of nonadjacent vertices cannot guarantee G is super-λ_k, which is shown by the next example. In this sense, Theorem 3.2 is best possible.

Example 3.4. Given two integers $k \geq 2, p \geq 3$, let H_1 and H_2 be two disjoint complete graphs with $V(H_1) = \{x_1, \ldots, x_{pk}\}$ and $V(H_2) = \{y_1, \ldots, y_{p(k-1)}\}$, respectively. Let H_3 be a bipartite graph with bipartition $(V(H_1), V(H_2))$ such that $|N(x_i) \cap V(H_2)| = k-1$, $|N(y_j) \cap V(H_1)| = k$ for any $i = 1, \ldots, pk$; $j = 1, \ldots, p(k-1)$. Let $G = H_1 \cup H_2 \cup H_3$. Then $|N(u) \cap N(v)| = 2k-1$ for all pairs u, v of nonadjacent vertices. Let U be an arbitrary subset of $V(G)$ such that $|U| = k$ and let $t = |U \cap V(H_1)|$. By a similar method as in Example 3.2, we have $|[U, \overline{U}]| \geq t(p-1) + pk(k-1) \geq pk(k-1)$. Clearly, if $U \subseteq V(H_2)$, then $G[U]$ is connected and $|[U, \overline{U}]| = pk(k-1)$. So, $\xi_k(G) = pk(k-1)$. By Theorem 3.1, G is λ_k-optimal. That is, $\lambda_k(G) = \xi_k(G) = pk(k-1)$. Clearly, $|E(H_3)| = |[V(H_1), V(H_2)]| = pk(k-1)$, and H_1, H_2 are connected. It follows that $[V(H_1), V(H_2)]$ is a λ_k-cut. Since $|V(H_1)| > k$, $|V(H_2)| > k$, we conclude that G is not super-λ_k.

4. Sufficient conditions for γ_k-optimality or super-γ_k property

Suppose that G is a graph of order $\nu \geq 2k$ and $S = [X, \overline{X}]$ is a γ_k-cut of G. Denote $X^0 = \{x \in X : |N(x) \cap \overline{X}| \leq k-1\}$, $\overline{X}^0 = \{y \in \overline{X} : |N(y) \cap X| \leq k-1\}$. Without loss of generality, assume that $\min\{|N(x) \cap \overline{X}| : x \in X\} \geq \min\{|N(y) \cap X| : y \in \overline{X}\}$. We will use this assumption and such notation in this section.

By a similar method as in Section 2, the following results can be shown.

Lemma 4.1. *Let G be a graph of order $\nu \geq 2k$ and let $S = [X, \overline{X}]$ be a γ_k-cut of G.*
(i) *If there exists a subset U of X such that $|U| = k$ and*

$$\sum_{v\in X\setminus U} |N(v)\cap U| \leq \sum_{v\in X\setminus U} |N(v)\cap \overline{X}|,$$

then G is γ_k-optimal.
(ii) *There exists no subset U of X such that $|U| = k$ and*

$$\sum_{v\in X\setminus U} |N(v)\cap U| < \sum_{v\in X\setminus U} |N(v)\cap \overline{X}|.$$

Corollary 4.1. *Let G be a graph of order $\nu \geq 2k$ and let $S = [X, \overline{X}]$ be a γ_k-cut of G. If there exists a vertex x^* in X such that $|N(x^*)\cap \overline{X}| \geq k+1$, then there exists no subset U of $X - x^*$ such that $|U| = k$ and*

$$\sum_{v\in X\setminus (U\cup\{x^*\})} |N(v)\cap U| \leq \sum_{v\in X\setminus (U\cup\{x^*\})} |N(v)\cap \overline{X}|.$$

Lemma 4.2. *Let X^* be a subset of X such that $|X^*| \geq k$ and $X^0 \subseteq X^*$, and let $G^* = G[X^*\cup\overline{X}]$. If $X^0 \neq \emptyset$, $\overline{X}^0 \neq \emptyset$ and $|N_{G^*}(u)\cap N_{G^*}(v)| \geq 2k-1$ for all pairs u, v of nonadjacent vertices in G^*, then there exists a subset U of X^* such that $|U| = k$ and*

$$\sum_{v\in X^*\setminus U} |N(v)\cap U| \leq \sum_{v\in X^*\setminus U} |N(v)\cap \overline{X}|.$$

Proof. Similar to the proof of Claim 2 in Lemma 2.3, we have $1 \leq |X^0| \leq k-1$. Since $|X^*| \geq k$ and $X^0 \subseteq X^*$, there exists a subset U of X^* such that $X^0 \subseteq U$ and $|U| = k$. By the definition of X^0, $|N(v)\cap\overline{X}| \geq k$ for any $v \in X^*\setminus U$. It follows that $\sum_{v\in X^*\setminus U} |N(v)\cap U| \leq \sum_{v\in X^*\setminus U} |U| = k|X^*\setminus U| \leq \sum_{v\in X^*\setminus U} |N(v)\cap\overline{X}|$. The proof is complete. □

Similar to Proposition 3.1, we have the following result.

Proposition 4.1. *Let k be a positive integer. If G is a complete graph with order at least $2k$, then G is super-γ_k.*

Theorem 4.1. *Let k be a positive integer and let G be a graph with order at least $2k$. If*

$$|N(u)\cap N(v)| \geq 2k-1$$

for all pairs u, v of nonadjacent vertices, then G is γ_k-optimal.

Proof. By Proposition 4.1, we only consider the case that G is not a complete graph. Let $S = [X, \overline{X}]$ be an arbitrary γ_k-cut. By definition, $|X| \geq k$, $|\overline{X}| \geq k$. Suppose that $X^0 = \emptyset$ or $\overline{X}^0 = \emptyset$. Without loss of generality, assume that $X^0 = \emptyset$. Then $|N(x)\cap\overline{X}| \geq k$ for any $x \in X$. Let U be a subset of X such that $|U| = k$. It follows that $\sum_{v\in X\setminus U} |N(v)\cap U| \leq \sum_{v\in X\setminus U} |U| = k|X\setminus U| \leq \sum_{v\in X\setminus U} |N(v)\cap\overline{X}|$. By Lemma 4.1(i), G is γ_k-optimal. Suppose that both $X^0 \neq \emptyset$ and $\overline{X}^0 \neq \emptyset$. Then, by Lemmas 4.1(i) and 4.2, G is γ_k-optimal. The proof is complete. □

Corollary 4.2 (*Zhang and Yuan [21] 2007*). *Let G be a connected graph on $\nu \geq 2k$ vertices. Suppose that*

$$d_G(u) + d_G(v) \geq \nu + 2k - 3$$

for every pair of nonadjacent vertices u and v in G. Then G is γ_k-optimal.

Example 3.1 also shows that Theorem 4.1 is an improvement of Corollary 4.2.

Similar to Theorem 3.2, we have the following theorem.

Theorem 4.2. *Let k be a positive integer and let G be a graph with order at least $2k$. If*

$$|N(u) \cap N(v)| \geq 2k$$

for all pairs u, v of nonadjacent vertices, then G either is super-γ_k or is in $L_2(\frac{\nu}{2}, k)$.

Proof. By Proposition 4.1, we only consider the case that G is not a complete graph. By Theorem 4.1, G is γ_k-optimal. That is, $\gamma_k(G) = \beta_k(G)$. Suppose that G is not super-γ_k. Then there exists a γ_k-cut $S = [X, \overline{X}]$ such that $|X| \geq k+1$ and $|\overline{X}| \geq k+1$.

Claim 1. $X^0 = \emptyset$ or $\overline{X}^0 = \emptyset$.

By contradiction. Suppose that both $X^0 \neq \emptyset$ and $\overline{X}^0 \neq \emptyset$. Then, similar to the proof in Theorem 3.2, there exists $x^* \in X$ such that $|N(x^*) \cap \overline{X}| \geq k+1$. Let $X^* = X - x^*$, $G^* = G[X^* \cup \overline{X}]$. Then $G^* = G - x^*$. Clearly, $X^0 \subseteq X^*$ and $|N_{G^*}(u) \cap N_{G^*}(v)| \geq 2k-1$ for all pairs u, v of nonadjacent vertices in G^*. By Lemma 4.2, there exists a subset U of X^* such that $|U| = k$ and $\sum_{v \in X^* \setminus U} |N(v) \cap U| \leq \sum_{v \in X^* \setminus U} |N(v) \cap \overline{X}|$. This is contrary to Corollary 4.1. The proof of Claim 1 is complete.

By Claim 1, without loss of generality, we may assume that $X^0 = \emptyset$. Then $|N(x) \cap \overline{X}| \geq k$ for any $x \in X$.

Claim 2. $|N(x) \cap \overline{X}| = k$ for any $x \in X$.

By contradiction. Suppose that there is a vertex u in X such that $|N(u) \cap \overline{X}| > k$. Then, since $|X| \geq k+1$, there exists a subset U of $X^* = X - u$ such that $|U| = k$. It follows that $\sum_{v \in X^* \setminus U} |N(v) \cap U| \leq \sum_{v \in X^* \setminus U} |U| = k|X^* \setminus U| \leq \sum_{v \in X^* \setminus U} |N(v) \cap \overline{X}|$. This is contrary to Corollary 4.1.

Claim 3. $G[X]$ is complete.

Let u, v be two arbitrary vertices in X and let U be an arbitrary subset of $X - u$ such that $|U| = k$ and $v \in U$. It follows that $\sum_{w \in X \setminus U} |N(w) \cap U| \leq \sum_{w \in X \setminus U} |U| = k|X \setminus U| = \sum_{w \in X \setminus U} |N(w) \cap \overline{X}|$. Combining this with Lemma 4.1(ii), we have $\sum_{w \in X \setminus U} |N(w) \cap U| = \sum_{w \in X \setminus U} |N(w) \cap \overline{X}|$ and hence $\sum_{w \in X \setminus U} |N(w) \cap U| = \sum_{w \in X \setminus U} |U|$. It follows that $|N(w) \cap U| = |U|$ and hence $wz \in E(G)$ for any $w \in X \setminus U, z \in U$. In particular, $uv \in E(G)$. By the arbitrariness of u, v, we conclude that $G[X]$ is complete.

Claim 4. $\overline{X}^0 = \emptyset$.

By contradiction. Suppose that there exists $y \in \overline{X}^0$. Then, since $|X| > k$ and $|N(y) \cap X| \leq k-1$, there exists $x \in X$ such that $xy \notin E(G)$. By Claim 2, $|N(x) \cap \overline{X}| = k$. It follows that $2k \leq |N(x) \cap N(y)| = |N(x) \cap N(y) \cap X| + |N(x) \cap N(y) \cap \overline{X}| \leq |N(y) \cap X| + |N(x) \cap \overline{X}| \leq (k-1) + k = 2k-1$, a contradiction.

Similar to Claims 2 and 3, we have $|N(y) \cap X| = k$ for any $y \in \overline{X}$ and $G[\overline{X}]$ is complete. It follows that ν is even and $G \in L_2(\frac{\nu}{2}, k)$. The proof is complete. $\square$

Similarly, it can been shown that Theorems 4.1 and 4.2 are best possible in some sense by Examples 3.2 and 3.4.

References

[1] C. Balbuena, A. Carmona, J. Fàbrega, M.A. Fiol, Superconnectivity of bipartite digraphs and graphs, Discrete Math. 197–198 (1999) 61–75.
[2] C. Balbuena, X. Marcote, P. García-Vázquez, On restricted connectivities of permutation graphs, Networks 45 (2005) 113–118.
[3] C. Balbuena, P. García-Vázquez, X. Marcote, Sufficient conditions for λ'-optimality in graphs with girth g, J. Graph Theory 52 (2006) 73–86.
[4] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, The Macmillan Press Ltd, New York, 1976.
[5] P. Bonsma, N. Ueffing, L. Volkmann, Edge-cuts leaving components of order at least three, Discrete Math. 256 (2002) 431–439.
[6] A.H. Esfahanian, S.L. Hakimi, On computing a conditional edge-connectivity of a graph, Inform. Process. Lett. 27 (1988) 195–199.
[7] J. Fàbrega, M.A. Fiol, On the extraconnectivity of graphs, Discrete Math. 155 (1996) 49–57.
[8] Y.O. Hamidoune, A.S. Lladó, O. Serra, R. Tindell, On isoperimetric connectivity in vertex-transitive graphs, SIAM J. Discrete Math. 13 (2000) 139–144.
[9] A. Hellwig, L. Volkmann, Sufficient conditions for λ'-optimality in graphs of diameter 2, Discrete Math. 283 (2004) 113–120.
[10] A. Hellwig, L. Volkmann, Sufficient conditions for graphs to be λ'-optimal, super-edge-connected, and maximally edge-connected, J. Graph Theory 48 (2005) 228–246.
[11] L. Lesniak, Results on the edge-connectivity of graphs, Discrete Math. 8 (1974) 351–354.
[12] Q.L. Li, Q. Li, Reliability analysis of circulant graphs, Networks 31 (1998) 61–65.
[13] J.X. Meng, Optimally super-edge-connected transitive graphs, Discrete Math. 260 (2003) 239–248.
[14] J.X. Meng, Y.H. Ji, On a kind of restricted edge connectivity of graphs, Discrete Appl. Math. 117 (2002) 183–193.
[15] J.P. Ou, Edge cuts leaving components of order at least m, Discrete Math. 305 (2005) 365–371.

[16] J. Plesnik, Critical graphs of given diameter, Acta Fac. Rerum Natur. Univ. Commenian Math. 30 (1975) 71–93.
[17] T. Soneoka, H. Nakada, M. Imase, C. Peyrat, Sufficient conditions for maximally connected dense graphs, Discrete Math. 63 (1987) 53–66.
[18] M. Wang, Q. Li, Conditional edge connectivity properties, reliability comparison and transitivity of graphs, Discrete Math. 258 (2002) 205–214.
[19] J.M. Xu, K.L. Xu, On restricted edge-connectivity of graphs, Discrete Math. 243 (2002) 291–298.
[20] Z. Zhang, J.J. Yuan, A proof of an inequality concerning k-restricted edge connectivity, Discrete Math. 304 (2005) 128–134.
[21] Z. Zhang, J.J. Yuan, Degree conditions for restricted-edge connectivity and isoperimetric-edge-connectivity to be optimal, Discrete Math. 307 (2007) 293–298.