Sufficient conditions for super \( k \)-restricted edge connectivity in graphs of diameter 2\(^{\dagger} \)

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Abstract

For a connected graph \( G = (V, E) \), an edge set \( S \subseteq E \) is a \( k \)-restricted edge cut if \( G - S \) is disconnected and every component of \( G - S \) has at least \( k \) vertices. The \( k \)-restricted edge connectivity of \( G \), denoted by \( \lambda_k(G) \), is defined as the cardinality of a minimum \( k \)-restricted edge cut. Let \( \xi_k(G) = \min \{|X, \bar{X}| : |X| = k, G[X] \text{ is connected}\} \). \( G \) is \( \lambda_k \)-optimal if \( \lambda_k(G) = \xi_k(G) \). Moreover, \( G \) is super-\( \lambda_k \) if every minimum \( k \)-restricted edge cut of \( G \) isolates one connected subgraph of order \( k \). In this paper, we prove that if \( |N_G(u) \cap N_G(v)| \geq 2k - 1 \) for all pairs \( u, v \) of nonadjacent vertices, then \( G \) is \( \lambda_k \)-optimal; and if \( |N_G(u) \cap N_G(v)| \geq 2k \) for all pairs \( u, v \) of nonadjacent vertices, then \( G \) is either super-\( \lambda_k \) or in a special class of graphs. In addition, for \( k \)-isoperimetric edge connectivity, which is closely related with the concept of \( k \)-restricted edge connectivity, we show similar results.

Keywords: Restricted edge connectivity; Isoperimetric edge connectivity; \( \lambda_k \)-optimality; Diameter

1. Terminology and introduction

For graph-theoretical terminology and notation not defined here we follow [4]. We consider finite, undirected and simple graphs \( G \) with the vertex set \( V(G) \) and the edge set \( E(G) \). For any vertex \( v \) in \( G \), we define the neighbour set of \( v \) in \( G \) to be the set of all vertices adjacent to \( v \); this set is denoted by \( N(v) = N_G(v) \). If \( G' \) is a subgraph of \( G \) and \( v \) is a vertex of \( G' \), we define \( N_G'(v) = N_G(v) \cap V(G') \). The degree \( d_G(v) \) of a vertex \( v \) in \( V(G) \) equals the number of vertices in \( N_G(v) \). Let \( \delta(G) \) denote the minimum degree in \( G \). For \( U \subseteq V(G) \) let \( G[U] \) be the subgraph induced by \( U \). For subsets \( U \) and \( U' \) of \( V(G) \), we denote by \( [U, U'] \) the set of edges with one end in \( U \) and the other in \( U' \). If vertices \( u \) and \( v \) are connected in \( G \), the distance between \( u \) and \( v \) in \( G \), denoted by \( d_G(u, v) \), is the length of a shortest path from \( u \) to \( v \) in \( G \); if there is no path connecting \( u \) and \( v \) we define \( d_G(u, v) \) to be infinite. The diameter of \( G \), denoted by \( D(G) \), is the maximum distance between two vertices of \( G \). Let \( G_1 \) and \( G_2 \) be two graphs. The union \( G_1 \cup G_2 \) of \( G_1 \) and \( G_2 \) is the graph with vertex set \( V(G_1) \cup V(G_2) \) and edge set \( E(G_1) \cup E(G_2) \).

The edge connectivity \( \lambda(G) \) of a graph \( G \) is the minimum cardinality of an edge cut of \( G \). It is well known that \( \lambda(G) \leq \delta(G) \). A graph \( G \) is \( \lambda \)-optimal if \( \lambda(G) = \delta(G) \). Furthermore, \( G \) is super-\( \lambda \) if every minimum edge cut consists

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of edges adjacent to a vertex of minimum degree. As a more refined index than the edge connectivity, restricted edge connectivity was proposed by Esfahanian and Hakimi [6]. A set of edges $S$ in a connected graph $G$ is called a restricted edge cut if $G - S$ is disconnected and contains no isolated vertex. If such an edge cut exists, then the restricted edge connectivity of $G$, denoted by $\lambda'(G)$, is defined to be the minimum number of edges over all restricted edge cuts of $G$. A graph is called $\lambda'$-connected if it contains restricted edge cuts. Esfahanian and Hakimi [6] showed that each connected graph $G$ of order $\nu(G) \geq 4$ except a star $K_{1,v-1}$ is $\lambda'$-connected and satisfies $\lambda(G) \leq \lambda'(G) \leq \xi(G)$, where $\xi(G) = \min\{d_G(u) + d_G(v) - 2: uv \in E(G)\}$ is the minimum edge degree of $G$. A graph $G$ is $\lambda'$-optimal if $\lambda'(G) = \xi(G)$. Moreover, $G$ is super-$\lambda'$ if every minimum restricted edge cut of $G$ isolates one edge, that is, every minimum restricted edge cut of $G$ is a set of edges adjacent to a certain edge with minimum edge degree in $G$. There has been much research on $\lambda$-optimal graphs, super-$\lambda$ graphs, $\lambda'$-optimal graphs and super-$\lambda'$ graphs (cf. e.g. [1–3,6,7,9,12,16–19]).

Generally, for a connected graph $G$, an edge set $S \subseteq E(G)$ is called a $k$-restricted edge cut of $G$ if $G - S$ is disconnected and every component of $G - S$ has at least $k$ vertices. The $k$-restricted edge connectivity of $G$, denoted by $\lambda_k(G)$, is defined as the cardinality of a minimum $k$-restricted edge cut. A minimum $k$-restricted edge cut is called a $\lambda_k$-cut. By definition, if $S$ is a $\lambda_k$-cut, then $|S| = \lambda_k(G)$. It should be pointed out that not all connected graphs have $k$-restricted edge cuts. A connected graph $G$ is called $\lambda_k$-connected if $\lambda_k(G)$ exists. It is easy to see that if $G$ is $\lambda_k$-connected for $k \geq 2$, then $G$ is also $\lambda_{k-1}$-connected and $\lambda_{k-1}(G) \leq \lambda_k(G)$. Sufficient conditions for graphs to be $\lambda_k$-connected were given by several authors [5,6,15,20]. In view of recent studies on $k$-restricted edge connectivity, it seems that the larger $\lambda_k(G)$ is, the more reliable the network is [13,14,18]. So, we expect $\lambda_k(G)$ to be as large as possible. Clearly, the optimization of $\lambda_k(G)$ requires an upper bound first. For any positive integer $k$, let

$$\xi_k(G) = \min\{|\{X, \overline{X}\}: |X| = k, G[X] \text{ is connected}\}.$$  

It has been shown that $\lambda_k(G) \leq \xi_k(G)$ holds for many graphs [5,15,20]. A connected graph $G$ is called a $\lambda_k$-optimal graph if $\lambda_k(G) = \xi_k(G)$. Furthermore, $G$ is called a super $k$-restricted edge-connected graph, in short, a super-$\lambda_k$ graph, if every $\lambda_k$-cut of $G$ isolates one connected subgraph of order $k$, that is, every $\lambda_k$-cut of $G$ is a set of edges adjacent to a certain connected subgraph of order $k$. Clearly, $\lambda_1(G) = \lambda(G)$, $\lambda_2(G) = \lambda'(G)$, $\xi_1(G) = \delta(G)$ and $\xi_2(G) = \xi(G)$. Moreover, $\lambda$-optimality, super-$\lambda$ property, $\lambda'$-optimality and super-$\lambda'$ property are $\lambda_1$-optimality, super-$\lambda_1$ property, $\lambda_2$-optimality and super-$\lambda_2$ property, respectively. Let $G$ be a $\lambda_k$-connected graph with $\lambda_k(G) \leq \xi_k(G)$. By definition, if $G$ is a super-$\lambda_k$ graph, then $G$ must be a $\lambda_k$-optimal graph. However, the converse is not true. For example, a cycle of length $\nu(v) \geq 2k + 2$ is a $\lambda_{k'}$-optimal graph that is not super-$\lambda_k$.

In Section 3, first, we prove that $G$ is $\lambda_k$-optimal if $|N_G(u) \cap N_G(v)| \geq 2k - 1$ for all pairs $u, v$ of nonadjacent vertices, and $G$ is super-$\lambda_k$ if $G$ is not in a special class of graphs and $|N_G(u) \cap N_G(v)| \geq 2k$ for all pairs $u, v$ of nonadjacent vertices. Next, we show that some known results are consequences of our results and give examples to show that our results are best possible in some sense.

In Section 4, we turn our attention to the analogous concept of $k$-isoperimetric edge connectivity. The $k$-isoperimetric edge connectivity of $G$ is defined as

$$\gamma_k(G) = \min\{|\{X, \overline{X}\}: X \subseteq V(G), |X| \geq k, |\overline{X}| \geq k\}.$$  

Clearly, $\gamma_k(G)$ exists for any positive integer $k \leq |V(G)|/2$. An edge cut $S = \{X, \overline{X}\}$ is called a $\gamma_k$-cut if $|S| = \gamma_k(G)$ and $X \subseteq V(G)$, $|X| \geq k$, $|\overline{X}| \geq k$. Let

$$\beta_k(G) = \min\{|\{X, \overline{X}\}: X \subseteq V(G), |X| = k\}.$$  

Then, it is obvious that $\gamma_k(G) \leq \beta_k(G)$. A graph $G$ is called a $\gamma_k$-optimal graph if $\gamma_k(G) = \beta_k(G)$. Moreover, $G$ is called a super-$\gamma_k$ graph if every $\gamma_k$-cut $S = \{X, \overline{X}\}$ of $G$ has the property that either $|X| = k$ or $|\overline{X}| = k$. It is easy to see that a super-$\gamma_k$ graph must be a $\gamma_k$-optimal graph, but the converse is not true. Several researchers [8,21] have studied $\gamma_k$-optimal graphs. In Section 4, we will give some conditions, which are similar to those given in Section 3, for graphs to be $\gamma_k$-optimal or super-$\gamma_k$.

2. Preliminaries

We start with a simple but useful observation.
Observation 2.1 ([9]). Let $G$ be a graph of order $v \geq 2$. Then each pair $u, v$ of nonadjacent vertices satisfies $|N(u) \cap N(v)| \geq 1$ if and only if $D(G) \leq 2$.

Proposition 2.1. Let $k$ be a positive integer and let $G$ be a graph with at least one pair of nonadjacent vertices. If

$$|N(u) \cap N(v)| \geq 2k - 1$$

for each pair $u, v$ of nonadjacent vertices, then $G$ is $\lambda_k$-connected and $\lambda_k(G) \leq \xi_k(G)$.

Proof. Clearly, $|V(G)| \geq 2k + 1$. By Observation 2.1, $D(G) \leq 2$ and hence $G$ is connected. The case $k = 1$ is trivial, so we only consider the case $k \geq 2$. Since $G$ is connected and $|V(G)| \geq 2k + 1$, $\xi_k(G)$ exists. Let $U$ be a subset of $V(G)$ such that $|U| = k$, $G[U]$ is connected and $\xi_k(G) = |[U, U]|$. If $u, v$ are two nonadjacent vertices in $U$, then $|N(u) \cap N(v)| \geq 2k - 1$. Since $|N(u) \cap N(v) \cap U| \leq |U| = k$ and $k \geq 2$, we have $|N(u) \cap N(v) \cap U| \geq k - 1 > 0$. Therefore, $G[U]$ is connected and thus $[U, U]$ is a $k$-restricted edge cut, which implies that $G$ is $\lambda_k$-connected and $\lambda_k(G) \leq |[U, U]| = \xi_k(G)$. The proof is complete. □

Let $G$ be a $\lambda_k$-connected graph and let $S$ be a $\lambda_k$-cut of $G$. By the minimality of $S$, the graph $G - S$ consists of exactly two components, say $G_1$ and $G_2$. Let $X = V(G_1)$. Then $X = V(G_2)$ and $S = [X, \overline{X}]$. Denote $X^0 = \{x \in X : |N(x) \cap \overline{X}| \leq k - 1\}$, $\overline{X}^0 = \{y \in \overline{X} : |N(y) \cap X| \leq k - 1\}$. Without loss of generality, assume

$$\min\{|N(x) \cap \overline{X}| : x \in X\} \geq \min\{|N(y) \cap X| : y \in \overline{X}\}. \quad (1)$$

We will use such notation and this assumption in this section and next section.

The main goal of this section is to give some useful properties of $G[X]$ and $G[\overline{X}]$. By reason of symmetry we only discuss $G[X]$.

Lemma 2.1. Let $G$ be a $\lambda_k$-connected graph with $\lambda_k(G) \leq \xi_k(G)$ and let $S = [X, \overline{X}]$ be a $\lambda_k$-cut of $G$.

(i) If there exists a connected subgraph $H$ of order $k$ in $G[X]$ with the property that

$$\sum_{v \in X \setminus V(H)} |N(v) \cap V(H)| \leq \sum_{v \in X \setminus V(H)} |N(v) \cap \overline{X}|,$$

then $G$ is $\lambda_k$-optimal.

(ii) There exists no connected subgraph $H$ of order $k$ in $G[X]$ with the property that

$$\sum_{v \in X \setminus V(H)} |N(v) \cap V(H)| < \sum_{v \in X \setminus V(H)} |N(v) \cap \overline{X}|.$$

Proof. The hypotheses of Lemma 2.1(i) imply

$$\xi_k(G) \leq |[V(H), \overline{V(H)}]|$$

$$= |[V(H), X \setminus V(H)]| + |[V(H), \overline{X}]|$$

$$= \sum_{v \in X \setminus V(H)} |N(v) \cap V(H)| + |[V(H), \overline{X}]|$$

$$\leq \sum_{v \in X \setminus V(H)} |N(v) \cap \overline{X}| + |[V(H), \overline{X}]|$$

$$= |[X \setminus V(H), \overline{X}]| + |[V(H), \overline{X}]|$$

$$= |[X, \overline{X}]| = |S| = \lambda_k(G).$$

Since $\lambda_k(G) \leq \xi_k(G)$, we deduce that $\lambda_k(G) = \xi_k(G)$ and hence $G$ is $\lambda_k$-optimal. The proof of (i) is complete. (ii) can be easily seen from the proof of (i). □

Corollary 2.1. Let $G$ be a $\lambda_k$-connected graph with $\lambda_k(G) \leq \xi_k(G)$ and let $S = [X, \overline{X}]$ be a $\lambda_k$-cut. If there exists a vertex $x^*$ in $X$ such that $|N(x^*) \cap \overline{X}| \geq k + 1$, then there exists no connected subgraph $H$ of order $k$ in $G[X] - x^*$.
Suppose that $G$ is a $\lambda_k$-connected graph and $S = [X, \overline{X}]$ is a $\lambda_k$-cut of $G$. Let $X^*$ be a subset of $X$ such that $|X^*| \geq k$, $X^0 \subseteq X^*$ and $G[X^*]$ is connected. If there exists a connected subgraph $H'$ of $G[X^*]$ such that $|V(H')| \leq k$ and $X^0 \subseteq V(H')$, then there exists a connected subgraph $H$ of order $k$ in $G[X^*]$ such that $|H| \geq k$, $X^0 \subseteq H$, $H \subseteq V(H)$ and $X \subseteq H$. It follows that $\sum_{v \in X^\ast \cap V(H)} |N(v) \cap V(H)| \leq \sum_{v \in X^\ast \cap V(H)} |N(v) \cap \overline{X}|$. □

**Lemma 2.2.** Suppose, on the contrary, that there exists such a subgraph $H$. Since $|N(x^*) \cap \overline{X}| \geq k + 1 > k = |V(H)| \geq |N(x^*) \cap V(H)|$, it follows that $\sum_{v \in V(H)} |N(v) \cap V(H)| < \sum_{v \in X \cap V(H)} |N(v) \cap \overline{X}|$, contrary to Lemma 2.1 (ii). □

**Lemma 2.3.** Let $X^*$ be a subset of $X$ such that $|X^*| \geq k$, $X^0 \subseteq X^*$ and $G[X^*]$ is connected and let $G^* = G[X^* \cup \overline{X}]$. If $X^0 \neq \emptyset$, $\overline{X}^0 \neq \emptyset$ and $|N_G^*(u) \cap N_G^*(v)| \geq 2k - 1$ for all pairs $u, v$ of nonadjacent vertices in $G^*$, then there exists a connected subgraph $H$ of order $k$ in $G[X^*]$ such that $\sum_{v \in X \cap V(H)} |N(v) \cap V(H)| \leq \sum_{v \in X \cap V(H)} |N(v) \cap \overline{X}|$. □

Combining Lemmas 2.1(i) and 2.2, we get the following corollary.

**Corollary 2.2.** Let $G$ be a $\lambda_k$-connected graph with $\lambda_k(G) \leq \xi_k(G)$ and let $S = [X, \overline{X}]$ be a $\lambda_k$-cut. If there exists a connected subgraph $H$ of $G[X]$ such that $|V(H)| \leq k$ and $X^0 \subseteq V(H)$, then $G$ is $\lambda_k$-optimal. In particular, if $X^0 = \emptyset$, then $G$ is $\lambda_k$-optimal.

**Lemma 2.4.** Suppose, on the contrary, that there exists such a subgraph $H$. Since $|N(x^*) \cap \overline{X}| \geq k + 1 > k = |V(H)| \geq |N(x^*) \cap V(H)|$, it follows that $\sum_{v \in V(H)} |N(v) \cap V(H)| < \sum_{v \in X \cap V(H)} |N(v) \cap \overline{X}|$, contrary to Lemma 2.1 (ii). □

**Claim 1.** $xy \in E(G^*)$ for any $x \in X^0, y \in \overline{X}^0$.

By contradiction. Suppose that there exist $x \in X^0$ and $y \in \overline{X}^0$ such that $xy \notin E(G^*)$. Then $2k - 1 \leq |N_G^*(x) \cap N_G^*(y)| = |N_G^*(x) \cap N_G^*(y) \cap X^*| + |N_G^*(x) \cap N_G^*(y) \cap \overline{X}| \leq |N_G^*(y) \cap X^*| + |N_G^*(x) \cap \overline{X}| \leq |N(y) \cap X| + |N(x) \cap \overline{X}| \leq 2(k - 1) = 2k - 2$, a contradiction.

**Claim 2.** $1 \leq |X^0| \leq k - 1$ and $1 \leq |\overline{X}^0| \leq k - 1$ (and therefore $k \geq 2$).

Since $X^0 \neq \emptyset$ and $\overline{X}^0 \neq \emptyset$, we have $1 \leq |X^0|$ and $1 \leq |\overline{X}^0|$. Let $y$ be a vertex in $\overline{X}^0$. Then, by Claim 1, $X^0 \subseteq N_G(y)$ and hence $X^0 \subseteq N_G(y) \cap X^*$. It follows that $|X^0| \leq |N_G(y) \cap X^*| \leq |N(y) \cap X| \leq k - 1$. Similarly, we also have $|\overline{X}^0| \leq k - 1$.

**Claim 3.** $|X^0| > \lfloor \frac{k}{2} \rfloor$ and there exists a connected subgraph $H'$ of order at most $k$ in $G[X^*]$ such that $|V(H') \cap X^0| \geq \lfloor \frac{k}{2} \rfloor$.

Let $t = \lfloor \frac{k}{2} \rfloor$ and let $U = \{x_1, \ldots, x_t\}$ be a subset of $X^*$ such that $U$ contains as many vertices in $X^0$ as possible. Clearly, $X^0 \cap U \neq \emptyset$, and if $|X^0| \leq \lfloor \frac{k}{2} \rfloor$, then $X^0 \subseteq U$; if $|X^0| > \lfloor \frac{k}{2} \rfloor$, then $U \subseteq X^0$. Without loss of generality, assume that $x_1 \in X^0$.

By Claim 2, we have $k \geq 2$. If $k = 2$, then let $H'$ be the graph with vertex set $\{x_1\}$. It is easy to see that $H'$ is a connected subgraph of $G[X^*]$, $|V(H')| \leq k$ and $U = \{x_1\} = V(H')$. If $k \geq 3$, then $t \geq 2$. For any $i = 2, \ldots, t$, if $x_1x_i \in E(G^*)$, then let $u_i = x_i$. If $x_1x_i \notin E(G^*)$, then, since $|N_G(x_1) \cap N_G(x_i)| \geq 2k - 1$ and
Lemma 2.2

Let $|N_G^*(x_1) \cap N_G^*(x_2)| \leq |N(x_1) \cap \overline{X}| \leq k - 1$, we have $|N_G^*(x_1) \cap N_G^*(x_2) \cap X^*| \geq k > 0$ and hence we may let $u_i$ be a vertex in $N_G^*(x_1) \cap N_G^*(x_2) \cap X^*$. Let $U^* = U \cup \{u_2, \ldots, u_t\}$ and let $H' = G[U^*]$. Then $H'$ is a connected subgraph of $G[X^*], |V(H')| = |U^*| \leq t + (t - 1) \leq k$ and $U \subseteq V(H')$.

Suppose that $|X| < \lceil \frac{k}{2} \rceil$. Then $X^* \subseteq U \subseteq V(H')$. By Lemma 2.2, there exists a connected subgraph $H$ of order $k$ in $G[X^*]$ such that $\sum_{v \in X^* \setminus V(H)} |N(v) \cap V(H)| \leq \sum_{v \in X^* \setminus V(H)} |N(v) \cap \overline{X}|$, contrary to the initial assumption. Therefore, $|X^*| > \lceil \frac{k}{2} \rceil$ and hence $U \subseteq X^*$. This implies that $|V(H') \cap X^*| \geq |U| = t = \lceil \frac{k}{2} \rceil$. The proof of Claim 3 is complete.

Denote $|X| = k - s$. Then, byClaims 2 and 3, $1 \leq s < \lfloor \frac{k}{2} \rfloor$.

Claim 4. $|N(x) \cap \overline{X}| \geq k - s$ for any $x \in X^*$.

Let $y' \in \overline{X}$ be a vertex such that $|N(y') \cap X| = \min\{|N(y) \cap X| : y \in \overline{X}\}$. Clearly, $y' \in \overline{X}$. By Claim 1, $X^* \subseteq N_G^*(y') \cap X^*$ and hence $k - s = |X| \leq |N_G^*(y') \cap X^*| \leq |N(y') \cap X|$. By the assumption (I), we have $|N(x) \cap \overline{X}| \geq \min\{|N(y) \cap X| : y \in \overline{X}\} = |N(y') \cap X|$ and thus $|N(x) \cap \overline{X}| \geq k - s$ for any $x \in X^*$.

Claim 5. $|X^*| \geq 2k - s$.

If $G[X^*]$ is connected, then let $H' = G[X^*]$. By Lemma 2.2, a contradiction to the initial assumption is obtained. Therefore, $G[X^*]$ is not connected. Let $x_1, x_2$ be two vertices in two different components of $G[X^*]$. Then $|N_G^*(x_1) \cap N_G^*(x_2)| \geq 2k - 1$, $|N_G^*(x_1) \cap N_G^*(x_2) \cap X^*| = 0$ and $|N_G^*(x_1) \cap N_G^*(x_2) \cap \overline{X}| \leq |N(x_1) \cap \overline{X}| \leq k - 1$. It follows that $|N_G^*(x_1) \cap N_G^*(x_2) \cap (X^* \setminus X^0)| \geq k$ and hence $|X^* \setminus X^0| \geq k$. Consequently, $|X^*| = |X^0| + |X^* \setminus X^0| \geq 2k - s$.

Claim 6. $|N(v) \cap X^0| \leq k - 2s$ for any $v \in X^* \setminus X^0$.

By contradiction. Suppose that there exists a vertex $v \in X^* \setminus X^0$ such that $|N(v) \cap X^0| \geq k - 2s + 1$. Then $|N_G^*(v) \cap X^0| \geq k - 2s + 1$. Let $U_1 = N_G^*(v) \cap X^0, U_2 = X^0 \setminus U_1 = \{x_1, x_2, \ldots, x_t\}$. Then $t = |U_2| = k - s - |U_1| \leq s - 1$. For any $i = 1, 2, \ldots, s$, $v \in \overline{X}$. Hence $|N_G^*(v) \cap N_G^*(x_i)| \geq 2k - 1$. Combining this with $|N_G^*(v) \cap N_G^*(x_i) \cap \overline{X}| \leq |N(x_i) \cap \overline{X}| \leq k - 1$, we have $|N_G^*(v) \cap N_G^*(x_i) \cap X^0| \geq k$ and hence we may pick a vertex $u_i$ in $N_G^*(v) \cap N_G^*(x_i) \cap X^0$. Let $U = U_1 \cup U_2 \cup \{u_1, \ldots, u_t\}$. Then $X^0 \subseteq U$. Clearly, $G^4[U]$ is connected and $|U| \leq |U_1 \cup U_2| + t + 1 \leq (k - s) + (s - 1) + 1 = k$. By Lemma 2.2, a contradiction to the initial assumption is obtained. The proof of Claim 6 is then complete.

Let $X^1 = \{x \in X^* \setminus X^0 : xy \in E(G^*) \text{ for any } y \in \overline{X}\}$ and let $X^2 = X^* \setminus (X^0 \cup X^1)$.

Claim 7. $|X^1| \leq s - 1$.

By Claim 1 and the definitions of $X^0$ and $X^1$, for any $y \in \overline{X}$, $k - s + |X^1| = |X^0| + |X^1| = |X^0 \cup X^1| = |N(y) \cap (X^0 \cup X^1)| \leq |N(y) \cap X| \leq k - 1$. Therefore, $|X^1| \leq s - 1$.

Claim 8. $|N(x) \cap \overline{X}| \geq k + s$ for any $x \in X^2$.

For any $x \in X^2$, by definition, there exists a vertex $y \in \overline{X}$ such that $xy \notin E(G^*)$. By Claim 1 and the choice of $y$, $|N(y) \cap (X^* \setminus X^0)| = |N(y) \cap X^*| - |N(y) \cap X^0| \geq |N(y) \cap X^*| - |X^0| \geq (k - 1) - (k - s) = s - 1$. By Claim 6, $|N(x) \cap X^0| \leq k - 2s$. Therefore, we have

$$2k - 1 \leq |N_G^*(x) \cap N_G^*(y)| = |N_G^*(x) \cap N_G^*(y) \cap \overline{X}| + |N_G^*(x) \cap N_G^*(y) \cap X^0| + |N_G^*(x) \cap N_G^*(y) \cap (X^* \setminus X^0)| \leq |N_G^*(x) \cap \overline{X}| + |N_G^*(x) \cap X^0| + |N_G^*(y) \cap (X^* \setminus X^0)| \leq |N(x) \cap \overline{X}| + |N(x) \cap X^0| + |N(y) \cap (X^* \setminus X^0)| \leq |N(x) \cap \overline{X}| + (k - 2s) + (s - 1).$$

This implies that $|N(x) \cap \overline{X}| \geq k + s$.

Claim 9. $|N(x) \cap \overline{X}| \geq k$ for any $x \in X^1$.

Since $X^1 \cap X^0 = \emptyset$, Claim 9 follows from the definition of $X^0$.

We continue with the proof of this lemma. By Claim 3, there exists a connected subgraph $H'$ of order at most $k$ in $G[X^*]$ such that $|V(H') \cap X^0| \geq \lceil \frac{k}{2} \rceil$. By the connectedness of $G[X^*]$, there exists a connected subgraph $H$ of $G[X^*]$ such that $|V(H)| = k$ and $V(H') \subseteq V(H)$. It follows that $|V(H) \cap X^0| \geq \lceil \frac{k}{2} \rceil$.

Let $t_0 = |(X^* \setminus V(H)) \cap X^0|, t_1 = |(X^* \setminus V(H)) \cap X^1|$ and $t_2 = |(X^* \setminus V(H)) \cap X^2|$. Then $t_0 = |X^0| - |V(H) \cap X^0| = (k - s) - |V(H) \cap X^0| \leq \lceil \frac{k}{2} \rceil - s$. Combining this with Claims 5 and 7, we have $t_2 = |X^*| - |V(H)| - t_1 - t_0 \geq |X^*| - |V(H)| - |X^1| - t_0 \geq (2k - s) - k - (s - 1) - (\lceil \frac{k}{2} \rceil - s) = \lceil \frac{k}{2} \rceil - s + 1 > t_0$. 

Combining this with Claims 4, 8 and 9, we have

\[
\sum_{v \in X^* \setminus V(H)} |N(v) \cap \overline{X}| = \sum_{i=0}^{2} \sum_{v \in (X^* \setminus V(H)) \cap X^i} |N(v) \cap \overline{X}|
\geq (k-s)t_0 + kt_1 + (k+s)t_2
> k(t_2 + t_1 + t_0)
= k|X^* \setminus V(H)|
\geq \sum_{v \in X^* \setminus V(H)} |N(v) \cap V(H)|.
\]

This is contrary to the initial assumption. The proof is then complete. \(\square\)

3. Sufficient conditions for \(\lambda_k\)-optimality or super-\(\lambda_k\) property

We start with a simple result.

**Proposition 3.1.** Let \(k\) be a positive integer. If \(G\) is a complete graph with order at least \(2k\), then \(G\) is super-\(\lambda_k\).

**Proof.** It is easy to verify that \(G\) is \(\lambda_k\)-connected and \(\lambda_k(G) \leq \xi_k(G)\). Suppose, on the contrary, that \(G\) is not super-\(\lambda_k\). Then there exists a \(\lambda_k\)-cut \(S = [X, \overline{X}]\) such that \(|X| \geq k + 1\) and \(|\overline{X}| \geq k + 1\). Let \(U\) be a subset of \(X\) such that \(|U| = k\) and let \(H = G[U]\). Then \(H\) is connected and \(\sum_{v \in X^* \setminus V(H)} |N(v) \cap V(H)| = k|X \setminus V(H)| < (k + 1)|X \setminus V(H)| \leq \sum_{v \in X^* \setminus V(H)} |N(v) \cap \overline{X}|\), a contradiction to Lemma 2.1(ii). \(\square\)

**Theorem 3.1.** Let \(k\) be a positive integer and let \(G\) be a graph with order at least \(2k\). If

\[|N(u) \cap N(v)| \geq 2k - 1\]

for all pairs \(u, v\) of nonadjacent vertices, then \(G\) is \(\lambda_k\)-optimal.

**Proof.** If \(G\) contains no nonadjacent vertices, then \(G\) is a complete graph with order at least \(2k\). By Proposition 3.1, \(G\) is super-\(\lambda_k\) and so \(G\) is \(\lambda_k\)-optimal. Therefore, we only consider the case that there exist nonadjacent vertices in \(G\) below. By Proposition 2.1, \(G\) is \(\lambda_k\)-connected and \(\lambda_k(G) \leq \xi_k(G)\). Let \(S = [X, \overline{X}]\) be an arbitrary \(\lambda_k\)-cut of \(G\). By definition, \(|X| \geq k, |\overline{X}| \geq k|X|\).

If \(X^0 = \emptyset\) or \(\overline{X}^0 = \emptyset\), then, by Corollary 2.2, \(G\) is \(\lambda_k\)-optimal. Suppose that both \(X^0 \neq \emptyset\) and \(\overline{X}^0 \neq \emptyset\). By Lemma 2.3 (regarding \(X\) as the \(X^*\) in Lemma 2.3), there exists a connected subgraph \(H\) of order \(k\) in \(G[X]\) such that \(\sum_{v \in X^* \setminus V(H)} |N(v) \cap V(H)| \leq \sum_{v \in X \setminus V(H)} |N(v) \cap \overline{X}|\). By Lemma 2.1(i), \(G\) is \(\lambda_k\)-optimal. \(\square\)

Recalling that \(\lambda_1(G) = \lambda(G)\) and \(\xi_1(G) = \delta(G)\), Corollary 3.1 follows from Observation 2.1 and Theorem 3.1.

**Corollary 3.1** (Plesnik [16] 1975). If \(G\) is a graph of diameter \(D(G) \leq 2\), then \(\lambda(G) = \delta(G)\).

Recalling that \(\lambda'(G) = \lambda_2(G)\) and \(\lambda'\)-optimality is \(\lambda_2\)-optimality, we have the following corollary.

**Corollary 3.2** (Hellwig and Volkmann [9] 2004). Let \(G\) be a \(\lambda'\)-connected graph. If

\[|N(u) \cap N(v)| \geq 3\]

for all pairs \(u, v\) of nonadjacent vertices, then \(G\) is \(\lambda'\)-optimal.

**Observation 3.1.** Let \(G\) be a graph of order \(v\) and let \(u, v \in V(G)\) be a pair of nonadjacent vertices. For any integer \(l \geq -1\), if \(d_G(u) + d_G(v) \geq v + l\), then \(|N(u) \cap N(v)| \geq l + 2\).

**Corollary 3.3** (Zhang and Yuan [21] 2007). Let \(k\) be a positive integer, and \(G\) a connected graph on \(v \geq 2k\) vertices. Suppose that \(d_G(u) + d_G(v) \geq v + 2k - 3\), for every pair of nonadjacent vertices \(u\) and \(v\) in \(G\). Then \(G\) is \(\lambda_k\)-optimal.

Next we introduce a class of graphs \(W(p, k)\) that will show that Theorem 3.1 is an improvement of Corollary 3.3.
Theorem 3.1. Let $k$ and $p$ be fixed positive integers with $p \geq 3$; let $C = \{x_1, x_2, \ldots, x_{2k-1}\}$, $I = \{y_1, y_2, \ldots, y_p\}$; and define $W(p, k)$ as follows. The vertices of $W(p, k)$ are $x_1, x_2, \ldots, x_{2k-1}, y_1, y_2, \ldots, y_p$, where two vertices $u$ and $v$ are adjacent if at least one of $u$ and $v$ is in $C$. The graph $W(3, 2)$ is shown in Fig. 1. If $u, v$ is a pair of nonadjacent vertices in $W(p, k)$, then $u, v \in I$. It is easy to see that $|N(u) \cap N(v)| = |C| = 2k - 1$ and $d_G(u) + d_G(v) = 2(2k - 1) = 4k - 2$. By Theorem 3.1, $G$ is $\lambda_k$-optimal. But, since $p \geq 3$, we have $d_G(u) + d_G(v) = 4k - 2 < (2k - 1 + 3) + (2k - 3) \leq (|C| + |I|) + (2k - 3) = |V| + 2k - 3$ and hence Corollary 3.3 does not show that $G$ is $\lambda_k$-optimal.

The next example shows that Theorem 3.1 is best possible in the sense that the condition $|N(u) \cap N(v)| \geq 2k - 2$ for all pairs $u, v$ of nonadjacent vertices does not imply $\lambda_k$-optimality. Before giving the example, we introduce another class of graphs.

Given two integers $p > k > 0$, let $H_1$ and $H_2$ be two disjoint complete graphs with $V(H_1) = \{x_1, \ldots, x_p\}$ and $V(H_2) = \{y_1, \ldots, y_p\}$, respectively. Let $F_k$ be the set of all $k$-regular bipartite graphs with bipartition $(V(H_1), V(H_2))$, and let the graph class $L_2(p, k) = \{H_1 \cup H_2 \cup H_3 : H_3 \in F_k\}$. A graph in $L_2(4, 1)$ is shown in Fig. 2.

Example 3.2. Let $G = H_1 \cup H_2 \cup H_3$ be a graph in $L_2(p, k - 1)$, where $p > k$. Then $|N(u) \cap N(v)| = 2k - 2$ for all pairs $u, v$ of nonadjacent vertices in $G$. Let $U$ be a subset of $V(G)$ such that $|U| = k$. Denote $t = |U \cap V(H_1)|$, $s = |U \cap V(H_2)|$. Then $t + s = k$ and $|U| = |U \cap V(H_1)| + |U \cap V(H_2)| + |U \cap V(H_3)| = t + s + k - 1$. Therefore, $\lambda_k(G) = k(p - 1)$ and hence $G$ is $\lambda_k$-optimal. But, since $p > k$, we have $|U| = k(p - 1) + k(k - 1) = k(p - 1)$. Therefore, $\lambda_k(G) < k(p - 1)$. It follows that $\lambda_k(G) \leq |E(H_3)| = p(k - 1)$. Therefore, $G$ is not $\lambda_k$-optimal.

Theorem 3.1 shows that the condition $|N(u) \cap N(v)| \geq 2k - 1$ for all pairs $u, v$ of nonadjacent vertices guarantees the graph $G$ is $\lambda_k$-optimal, but even a stronger condition that $|N(u) \cap N(v)| \geq 2k$ for all pairs $u, v$ of nonadjacent vertices cannot guarantee $G$ is super-$\lambda_k$. We give such an example below.

Example 3.3. Let $G \in L_2(p, k)$, where $p > k$. By a similar method as in the above example, we have $\xi_k(G) = kp$ and $|N(u) \cap N(v)| = 2k$ for all pairs $u, v$ of nonadjacent vertices. By Theorem 3.1, $G$ is $\lambda_k$-optimal and hence $\lambda_k(G) = \xi_k(G) = kp$. We also have $|E(H_3)| = |[V(H_1), V(H_2)]| = kp$, which implies that $[V(H_1), V(H_2)]$ is a $\lambda_k$-cut. Combining this with $|V(H_1)| = |V(H_2)| = p > k$, we conclude that $G$ is not super-$\lambda_k$. 

![Fig. 1. The graph $W(3, 2)$.](image1)

![Fig. 2. A graph in $L_2(4, 1)$.](image2)
Theorem 3.2. Let \( k \) be a positive integer and let \( G \) be a graph with order at least \( 2k \). If

\[
|N(u) \cap N(v)| \geq 2k
\]

for all pairs \( u, v \) of nonadjacent vertices, then \( G \) either is super-\( \lambda_k \) or is in \( L_2(\frac{\lambda}{k}, k) \).

Proof. If \( G \) contains no nonadjacent vertices, then, by Proposition 3.1, we are done. Therefore, we only consider the case that there exist nonadjacent vertices in \( G \) below. By Theorem 3.1, \( G \) is \( \lambda_k \)-optimal. That is, \( \lambda_k(G) = \xi_k(G) \). Suppose that \( G \) is not super-\( \lambda_k \). Then there exists a \( \lambda_k \)-cut \( S = [X, \overline{X}] \) such that \( |X| \geq k + 1 \) and \( |\overline{X}| \geq k + 1 \).

Claim 1. \( X^0 = \emptyset \) or \( \overline{X}^0 = \emptyset \).

By contradiction. Suppose that both \( X^0 \neq \emptyset \) and \( \overline{X}^0 \neq \emptyset \). Similar to the proof of Claim 1 in Lemma 2.3, we have \( xy \in E(G) \) for any \( x \in X^0, y \in \overline{X}^0 \).

Since \( |X| \geq k + 1 \) and \( |N(y) \cap X| \leq k - 1 \) for any \( y \in \overline{X}^0 \), there exist \( x^* \in X \) and \( y^* \in \overline{X}^0 \) such that \( x^*y^* \notin E(G) \). Combining this with the fact that \( xy \in E(G) \) for any \( x \in X^0, y \in \overline{X}^0 \), we have \( x^* \in X \setminus X^0 \). Since \( |N(x^*) \cap N(y^*)| \geq 2k \) and \( |N(x^*) \cap N(y^*) \cap X| \leq |N(y^*) \cap X| \leq k - 1 \), it follows that \( |N(x^*) \cap \overline{X}| \geq k + 1 \). Let \( X^* = X - x^* \), \( G^* = G[X^* \cup \overline{X}] \). Then \( G^* = G - x^* \). Clearly, \( X^0 \subseteq X^* \) and \( |N_{G^*}(u) \cap N_{G^*}(v)| \geq 2k - 1 \) for any pair \( u, v \) of nonadjacent vertices in \( G^* \).

We will show that \( G[X^*] \) is connected. Suppose that \( G[X^*] \) is not connected and let \( u \in X^0 \subseteq X^* \). Then we may choose a vertex \( v \in X^* \) such that \( u, v \) are in different components of \( G[X^*] \). Since \( |N_{G^*}(u) \cap N_{G^*}(v)| \geq 2k - 1 \) and \( |N_{G^*}(u) \cap N_{G^*}(v) \cap \overline{X}| \leq |N(u) \cap \overline{X}| \leq k - 1 \), we have \( |N_{G^*}(u) \cap N_{G^*}(v) \cap X^*| \geq k > 0 \) and hence \( u, v \) are connected in \( G[X^*] \), which contradicts the choice of \( v \). Therefore, \( G[X^*] \) is connected.

By Lemma 2.3, there exists a connected subgraph \( H \) of order \( k \) in \( G[X^*] \) such that \( \sum_{v \in X^0 \setminus V(H)} |N(v) \cap V(H)| \leq \sum_{v \in X^0 \setminus V(H)} |N(v) \cap \overline{X}| \). Recalling \( |N(x^*) \cap \overline{X}| \geq k + 1 \) and \( X^* = X - x^* \), we obtain a contradiction to Corollary 2.1. The proof of Claim 1 is complete.

By Claim 1, without loss of generality, we may assume \( X^0 = \emptyset \). That is, \( |N(x) \cap \overline{X}| \geq k \) for any \( x \in X \).

Claim 2. \( |N(x) \cap \overline{X}| = k \) for any \( x \in X \).

By contradiction. Suppose that there is a vertex \( u \) in \( X \) such that \( |N(u) \cap \overline{X}| > k \). If \( G[X - u] \) is connected or \( k = 1 \), then there exists a connected subgraph \( H \) of \( G[X] - u \) such that \( |V(H)| = k \). Since \( |N(v) \cap V(H)| \leq |V(H)| = k \leq |N(v) \cap \overline{X}| \) for any \( v \in X \setminus (V(H) \cup \{u\}) \), it follows that \( \sum_{v \in X \setminus (V(H) \cup \{u\})} |N(v) \cap V(H)| \leq k|X \setminus (V(H) \cup \{u\})| \leq \sum_{v \in X \setminus (V(H) \cup \{u\})} |N(v) \cap \overline{X}| \), contrary to Corollary 2.1. Therefore, \( G[X - u] \) is not connected and \( k \geq 2 \).

For any \( x \in X - u \), we may choose a vertex \( x' \in X - u \) such that \( x, x' \) are in different components of \( G[X - u] \). So \( |N(x) \cap N(x')| \geq 2k \) and \( N(x) \cap N(x') \cap X \subseteq \{u\} \), which implies that \( |N(x) \cap N(x') \cap \overline{X}| \geq 2k - 1 \) and hence \( |N(x) \cap \overline{X}| \geq 2k - 1 \). Let \( H \) be a connected subgraph of \( G[X] \) such that \( |V(H)| = k \) and \( u \in V(H) \). It follows that \( \sum_{v \in X \setminus V(H)} |N(v) \cap V(H)| \leq k|X \setminus V(H)| < (2k - 1)|X \setminus V(H)| \leq \sum_{v \in X \setminus V(H)} |N(x) \cap \overline{X}| \). This is contrary to Lemma 2.1(ii). Therefore, \( |N(x) \cap \overline{X}| = k \) for any \( x \in X \).

Claim 3. \( \overline{X}^0 = \emptyset \).

By contradiction. Suppose that there exists \( y \in \overline{X}^0 \). Then, since \( |X| > k \) and \( |N(y) \cap X| \leq k - 1 \), there exists \( x \in X \) such that \( xy \notin E(G) \). By Claim 2, \( |N(x) \cap \overline{X}| = k \). It follows that \( 2k \leq |N(x) \cap N(y)| = |N(x) \cap N(y) \cap X| + |N(x) \cap N(y) \cap \overline{X}| \leq |N(y) \cap X| + |N(x) \cap \overline{X}| \leq (k - 1) + k = 2k - 1 \), a contradiction.

Similar to Claim 2, we have:

Claim 4. \( |N(y) \cap X| = k \) for any \( y \in \overline{X} \).

Claim 5. \( G[X] \) is complete.

Let \( u, v \) be two arbitrary vertices in \( X \). First, we will show \( G[X - u] \) is connected. For any \( x_1, x_2 \in X - u \), if \( x_1x_2 \notin E(G) \), then \( |N(x_1) \cap N(x_2)| \geq 2k \). Suppose that \( k = 1 \). By Claim 4, \( |N(y) \cap X| = k \) for any \( y \in \overline{X} \), which implies that \( N(x_1) \cap N(x_2) \cap \overline{X} = \emptyset \). It follows that \( |N(x_1) \cap N(x_2) \cap (X - u)| \geq 2k - 1 > 0 \). Therefore, \( G[X - u] \) is connected. Suppose that \( k > 2 \). Then, by Claim 2, \( |N(x_1) \cap N(x_2) \cap \overline{X}| = |N(x_1) \cap \overline{X}| = k \). It follows that \( |N(x_1) \cap N(x_2) \cap (X - u)| \geq k - 1 > 0 \). Therefore, \( G[X - u] \) is connected. Clearly, there exists a connected subgraph \( H \) of \( G[X] \) such that \( v \in V(H) \). It follows that \( \sum_{w \in X \setminus V(H)} |N(w) \cap V(H)| \leq \sum_{w \in X \setminus V(H)} |V(H)| = k|X \setminus V(H)| = |\sum_{w \in X \setminus V(H)} |N(w) \cap \overline{X}| \). Combining this with Lemma 2.1(ii), we have \( \sum_{w \in X \setminus V(H)} |N(w) \cap V(H)| = \sum_{w \in X \setminus V(H)} |N(w) \cap \overline{X}| \). Hence \( \sum_{w \in X \setminus V(H)} |N(w) \cap V(H)| = \sum_{w \in X \setminus V(H)} |N(w) \cap \overline{X}| \). It follows that
Let \( k \) be a positive integer and let \( G \) be a graph with order at least \( \) 

**Theorem 3.1** 

\[ X \]

**Example 3.2** 

**Observation 3.1** 

**Theorem 3.2** 

Let \( k \) be a positive integer and let \( G \) be a graph with order at least \( \) 

Given two integers \( u, v \) of nonadjacent vertices, let \( \lambda \) be a super-\( \lambda \)-optimal graph such that \( \lambda \geq 2k \). By Claims 2, 4, 5 and 6, it follows that \( v \) is even and \( G \) is a graph in \( L_2(\frac{v}{2}, k) \) and \( \frac{v}{2} = |X| \geq k + 1 \). The proof is complete. \( \square \)

Recalling that for any \( G \in L_2(\frac{v}{2}, k) \), \( |N(u) \cap N(v)| = 2k \) holds for every pair \( u, v \) of nonadjacent vertices. Therefore, we have the following corollary.

**Corollary 3.4.** Let \( k \) be a positive integer and let \( G \) be a graph with order at least \( 2k \). If 

\[ |N(u) \cap N(v)| \geq 2k + 1 \]

for all pairs \( u, v \) of nonadjacent vertices, then \( G \) is super-\( \lambda \)-k.

By **Observation 3.1** and **Theorem 3.2**, we have the following corollary.

**Corollary 3.5.** Let \( k \) be a positive integer and let \( G \) be a graph with order \( v \geq 2k \). If 

\[ d_G(u) + d_G(v) \geq v + 2k - 2 \]

for all pairs \( u, v \) of nonadjacent vertices, then \( G \) either is super-\( \lambda \)-k or is in \( L_2(\frac{v}{2}, k) \).

By **Observation 3.1** and **Corollary 3.4**, we have the following corollary.

**Corollary 3.6.** Let \( k \) be a positive integer and let \( G \) be a graph with order \( v \geq 2k \). If 

\[ d_G(u) + d_G(v) \geq v + 2k - 1 \]

for all pairs \( u, v \) of nonadjacent vertices, then \( G \) is super-\( \lambda \)-k.

Recalling that super-\( \lambda \) property is super-\( \lambda_1 \) property, we have the following corollary.

**Corollary 3.7** (Lesniak [11] 1974). Let \( G \) be a connected graph of order \( v \). If 

\[ d_G(u) + d_G(v) \geq v + 1 \]

for all pairs \( u, v \) of nonadjacent vertices, then \( G \) is super-\( \lambda \).

The conditions that \( G \) is not in \( L_2(\frac{v}{2}, k) \) and 

\[ |N(u) \cap N(v)| \geq 2k - 1 \]

for all pairs \( u, v \) of nonadjacent vertices cannot guarantee \( G \) is super-\( \lambda \)-k, which is shown by the next example. In this sense, **Theorem 3.2** is best possible.

**Example 3.4.** Given two integers \( k \geq 2, p \geq 3 \), let \( H_1 \) and \( H_2 \) be two disjoint complete graphs with 

\[ V(H_1) = \{x_1, \ldots, x_{pk}\} \] and 

\[ V(H_2) = \{y_1, \ldots, y_{p(k-1)}\} \]

respectively. Let \( H_3 \) be a bipartite graph with bipartition 

\[ (V(H_1), V(H_2)) \] such that 

\[ |N(x_i) \cap V(H_2)| = k - 1, |N(y_j) \cap V(H_1)| = k \]

for any \( i = 1, \ldots, pk; j = 1, \ldots, p(k-1) \). Let 

\[ G = H_1 \cup H_2 \cup H_3 \] Then 

\[ |N(u) \cap N(v)| = 2k - 1 \]

for all pairs \( u, v \) of nonadjacent vertices. Let \( U \) be an arbitrary subset of \( V(G) \) such that 

\[ |U| = k \] and let \( t = |U \cap V(H_1)| \).

By a similar method as in **Example 3.2**, we have 

\[ |U, \bar{U}| \geq t(p-1) + pk(k-1) \geq pk(k-1) \]

Clearly, if \( U \subseteq V(H_2) \), then \( G[U] \) is connected and 

\[ |U, \bar{U}| \geq pk(k-1) \]

By **Theorem 3.1**, \( G \) is \( \lambda_k \)-optimal. That is, 

\[ \lambda_k(G) = \xi_k(G) = pk(k-1) \]

Clearly, \( |E(H_3)| = |\{V(H_1), V(H_2)\}| = pk(k-1) \), and \( H_1, H_2 \) are connected. It follows that 

\[ |V(H_1), V(H_2)| \]

is a \( \lambda_k \)-cut. Since 

\[ |V(H_1)| > k, |V(H_2)| > k \]

we conclude that \( G \) is not super-\( \lambda_k \).

4. **Sufficient conditions for \( \gamma_k \)-optimality or super-\( \gamma_k \) property** 

Suppose that \( G \) is a graph of order \( v \geq 2k \) and 

\[ S = [X, \bar{X}] \] is a \( \gamma_k \)-cut of \( G \). Denote 

\[ X^0 = \{x \in X : |N(x) \cap \bar{X}| \leq k - 1\} \] and 

\[ \bar{X}^0 = \{y \in \bar{X} : |N(y) \cap X| \leq k - 1\} \] Without loss of generality, assume that 

\[ \min\{|N(x) \cap \bar{X} : x \in X| \geq \min\{|N(y) \cap X| : y \in \bar{X}\} \].

We will use this assumption and such notation in this section.
By a similar method as in Section 2, the following results can be shown.

**Lemma 4.1.** Let $G$ be a graph of order $v \geq 2k$ and let $S = [X, \overline{X}]$ be a $\gamma_k$-cut of $G$.

(i) If there exists a subset $U$ of $X$ such that $|U| = k$ and

$$\sum_{v \in X \setminus U} |N(v) \cap U| \leq \sum_{v \in X \setminus U} |N(v) \cap \overline{X}|,$$

then $G$ is $\gamma_k$-optimal.

(ii) There exists no subset $U$ of $X$ such that $|U| = k$ and

$$\sum_{v \in X \setminus U} |N(v) \cap U| < \sum_{v \in X \setminus U} |N(v) \cap \overline{X}|.$$

**Corollary 4.1.** Let $G$ be a graph of order $v \geq 2k$ and let $S = [X, \overline{X}]$ be a $\gamma_k$-cut of $G$. If there exists a vertex $x^*$ in $X$ such that $|N(x^*) \cap \overline{X}| \geq k + 1$, then there exists no subset $U$ of $X - x^*$ such that $|U| = k$ and

$$\sum_{v \in X \setminus (U \cup \{x^*\})} |N(v) \cap U| \leq \sum_{v \in X \setminus (U \cup \{x^*\})} |N(v) \cap \overline{X}|.$$

**Lemma 4.2.** Let $X^*$ be a subset of $X$ such that $|X^*| \geq k$ and $X^0 \subseteq X^*$, and let $G^* = G[X^* \cup \overline{X}]$. If $X^0 \neq \emptyset$, $\overline{X}^0 \neq \emptyset$ and $|N_{G^*}(u) \cap N_{G^*}(v)| \geq 2k - 1$ for all pairs $u, v$ of nonadjacent vertices in $G^*$, then there exists a subset $U$ of $X^*$ such that $|U| = k$ and

$$\sum_{v \in X^* \setminus U} |N(v) \cap U| \leq \sum_{v \in X^* \setminus U} |N(v) \cap \overline{X}|.$$

**Proof.** Similar to the proof of Claim 2 in Lemma 2.3, we have $1 \leq |X^0| \leq k - 1$. Since $|X^*| \geq k$ and $X^0 \subseteq X^*$, there exists a subset $U$ of $X^*$ such that $X^0 \subseteq U$ and $|U| = k$. By the definition of $X^0$, $|N(v) \cap \overline{X}| \geq k$ for any $v \in X^* \setminus U$.

It follows that $\sum_{v \in X^* \setminus U} |N(v) \cap U| \leq \sum_{v \in X^* \setminus U} |N(v) \cap \overline{X}| = k|X^* \setminus U| \leq k|X^* \setminus (U \cup \{x^*\})|$. The proof is complete. \qed

Similar to Proposition 3.1, we have the following result.

**Proposition 4.1.** Let $k$ be a positive integer. If $G$ is a complete graph with order at least $2k$, then $G$ is super-$\gamma_k$.

**Theorem 4.1.** Let $k$ be a positive integer and let $G$ be a graph with order at least $2k$. If

$$|N(u) \cap N(v)| \geq 2k - 1$$

for all pairs $u, v$ of nonadjacent vertices in $G$, then $G$ is $\gamma_k$-optimal.

**Proof.** By Proposition 4.1, we only consider the case that $G$ is not a complete graph. Let $S = [X, \overline{X}]$ be an arbitrary $\gamma_k$-cut. By definition, $|X| \geq k$, $|\overline{X}| \geq k$. Suppose that $X^0 = \emptyset$ or $\overline{X}^0 = \emptyset$. Without loss of generality, assume that $X^0 = \emptyset$. Then $|N(x) \cap \overline{X}| \geq k$ for any $x \in X$. Let $U$ be a subset of $X$ such that $|U| = k$. It follows that $\sum_{v \in X \setminus U} |N(v) \cap U| \leq \sum_{v \in X \setminus U} |N(v) \cap \overline{X}| = k|X \setminus U| \leq k|X^* \setminus (U \cup \{x^*\})|$. By Lemma 4.1(i), $G$ is $\gamma_k$-optimal.

Suppose that both $X^0 \neq \emptyset$ and $\overline{X}^0 \neq \emptyset$. Then, by Lemmas 4.1(i) and 4.2, $G$ is $\gamma_k$-optimal. The proof is complete. \qed

**Corollary 4.2.** (Zhang and Yuan [21] 2007). Let $G$ be a connected graph on $v \geq 2k$ vertices. Suppose that

$$d_G(u) + d_G(v) \geq v + 2k - 3$$

for every pair of nonadjacent vertices $u$ and $v$ in $G$. Then $G$ is $\gamma_k$-optimal.

Example 3.1 also shows that Theorem 4.1 is an improvement of Corollary 4.2.
Similar to Theorem 3.2, we have the following theorem.

**Theorem 4.2.** Let $k$ be a positive integer and let $G$ be a graph with order at least $2k$. If

$$|N(u) \cap N(v)| \geq 2k$$

for all pairs $u, v$ of nonadjacent vertices, then $G$ either is super-$\gamma_k$ or is in $L_2(\frac{v}{2}, k)$.

**Proof.** By Proposition 4.1, we only consider the case that $G$ is not a complete graph. By Theorem 4.1, $G$ is $\gamma_k$-optimal. That is, $\gamma_k(G) = \beta_k(G)$. Suppose that $G$ is not super-$\gamma_k$. Then there exists a $\gamma_k$-cut $S = \{X, \overline{X}\}$ such that $|X| \geq k + 1$ and $|\overline{X}| \geq k + 1$.

**Claim 1.** $X^0 = \emptyset$ or $\overline{X}^0 = \emptyset$.

By contradiction. Suppose that both $X^0 \neq \emptyset$ and $\overline{X}^0 \neq \emptyset$. Then, similar to the proof in Theorem 3.2, there exists $x^* \in X$ such that $|N(x^*) \cap \overline{X}| \geq k + 1$. Let $X^* = X - x^*$, $G^* = G[X^* \cup \overline{X}]$. Then $G^* = G - x^*$. Clearly, $X^0 \subseteq X^*$ and $|N_{G^*}(u) \cap N_{G^*}(v)| \geq 2k - 1$ for all pairs $u, v$ of nonadjacent vertices in $G^*$. By Lemma 4.2, there exists a subset $U$ of $X^*$ such that $|U| = k$ and $\sum_{v \in X^* \setminus U} |N(v) \cap U| \leq \sum_{v \in X^* \setminus U} |N(v) \cap \overline{X}|$. This is contrary to Corollary 4.1. The proof of Claim 1 is complete.

By Claim 1, without loss of generality, we may assume that $X^0 = \emptyset$. Then $|N(x) \cap \overline{X}| \geq k$ for any $x \in X$.

**Claim 2.** $|N(x) \cap \overline{X}| = k$ for any $x \in X$.

By contradiction. Suppose that there is a vertex $u$ in $X$ such that $|N(u) \cap \overline{X}| > k$. Then, since $|X| \geq k + 1$, there exists a subset $U$ of $X^* = X - u$ such that $|U| = k$. It follows that $\sum_{v \in X^* \setminus U} |N(v) \cap U| \leq \sum_{v \in X^* \setminus U} |N(v) \cap \overline{X}|$. This is contrary to Corollary 4.1.

**Claim 3.** $G[X]$ is complete.

Let $u, v$ be two arbitrary vertices in $X$ and let $U$ be an arbitrary subset of $X - u$ such that $|U| = k$ and $v \in U$. It follows that $\sum_{w \in X \setminus U} |N(w) \cap U| = k |X \setminus U| = \sum_{w \in X \setminus U} |N(w) \cap \overline{X}|$. Combining this with Lemma 4.1(ii), we have $\sum_{w \in X \setminus U} |N(w) \cap U| = \sum_{w \in X \setminus U} |N(w) \cap \overline{X}|$ and hence $\sum_{w \in X \setminus U} |N(w) \cap U| = \sum_{w \in X \setminus U} |U|$. It follows that $|N(w) \cap U| = |U|$ and hence $wz \in E(G)$ for any $w \in X \setminus U$, $z \in U$. In particular, $uv \in E(G)$.

**Claim 4.** $\overline{X}^0 = \emptyset$.

By contradiction. Suppose that there exists $y \in \overline{X}^0$. Then, since $|X| > k$ and $|N(y) \cap X| \leq k - 1$, there exists $x \in X$ such that $xy \notin E(G)$. By Claim 2, $|N(x) \cap \overline{X}| = k$. It follows that $2k \leq |N(x) \cap N(y)| = |N(x) \cap N(y) \cap X| \leq |N(y) \cap X| \leq (k - 1) + k = 2k - 1$, a contradiction.

Similar to Claims 2 and 3, we have $|N(y) \cap X| = k$ for any $y \in \overline{X}$ and $G[\overline{X}]$ is complete. It follows that $v$ is even and $G \in L_2(\frac{v}{2}, k)$. The proof is complete. □

Similarly, it can be shown that Theorems 4.1 and 4.2 are best possible in some sense by Examples 3.2 and 3.4.

**References**