Stability of Galerkin and Inertial Algorithms with variable time step size

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Abstract

This paper provides Galerkin and Inertial Algorithms for solving a class of nonlinear evolution equations. Spatial discretization can be performed by either spectral or finite element methods; time discretization is done by Euler explicit or Euler semi-implicit difference schemes with variable time step size. Moreover, the boundedness and stability of these algorithms are studied. By comparison, we find that the boundedness and stability of Inertial Algorithm are superior to the ones of Galerkin Algorithm in the case of explicit scheme and the boundedness and stability of two algorithms are same in the case of semi-implicit scheme. © 2002 Elsevier Science B.V. All rights reserved.

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0. Introduction

In the study of the long time behavior of dynamical systems, the theory of inertial manifolds has been developed considerably during the recent years. For a dissipative dynamical system, all solutions of the system converge as \( t \to \infty \) to its global attractor, which is in general finite dimensional (Hausdoff or fractal) and could be geometrically very complicated (fractal).

The inertial manifold (see [6,13]), whenever it exists, is a positively invariant finite-dimensional Lipschitz manifold which attracts exponentially all the trajectories, whereas the convergence of the...
trajectories toward the attractor can be very slow. Although the existence of inertial manifolds for some dynamical systems, for instance the 2-D Navier–Stokes equations (NSE), is still unknown, it has been proven that the approximate inertial manifolds (see [4,5,14–17]) provide better approximations to the solution than the flat manifold. Recently, Inertial Algorithms or nonlinear Galerkin methods corresponding to these approximate inertial manifolds have been introduced and studied in [1,7–12,16]. The numerical treatment and analysis of the evolution differential equations is important and considered by many authors, see, for example, [2,3] for the linear evolution equations and [7,8,12,16] for the nonlinear evolution equations.

We consider in this paper the classical Galerkin and Inertial Algorithms with variable time step size. Here the dissipative term is treated implicitly to avoid severe time step constraints while keeping the nonlinear terms explicit or semi-implicit so that the corresponding discrete systems are easily invertable. It is well known that Galerkin and Inertial Algorithms of semi-implicit type are bounded and stable under the same less restriction on the time step size. Moreover, Galerkin and Inertial Algorithms of explicit type are bounded and stable under much restriction on the time step size; while the restriction of Inertial Algorithm is less than ones of Galerkin Algorithm, which depends only on coarse grid parameter $H$ and time integration step numbers $m$, however, ones of Galerkin algorithm depends on fine grid parameter $h < H$ and time integration step numbers $m$. Hence, in the case of explicit type, Inertial Algorithm is superior to Galerkin Algorithm.

1. Nonlinear evolution equations

Let $Y$ be a Hilbert space with the scalar product $(\cdot, \cdot)$ and the norm $| \cdot |$. The class of evolution equations that we shall consider has the form

$$\frac{du}{dt} + vAu + B(u,u) + Cu = f$$

with the initial condition

$$u(0) = U_0.$$  (1.2)

The unknown function $u$ is a map from $R^+$ into $Y$. The basic dissipative operator $A$ is a linear self-adjoint unbounded operator in $Y$ with domain $D(A)$. We assume that $A$ is positive closed and that $A^{-1}$ is compact. One can then define the powers $A^s$ of $A$ for $s \in R$; $A^s$ maps $D(A^s)$ into $Y$ and $D(A^s)$ is a Hilbert space when endowed with the norm $|A^s \cdot|$. We set $V = D(A^{1/2})$ and we denote by $\| \cdot \| = |A^{1/2} \cdot|$ the norm on $V$. Since $A^{-1}$ is compact and self-adjoint, there exists an orthonormal basis $\{w_j\}$ of $Y$ consisting of the eigenvectors of $A$

$$Aw_j = \lambda_j w_j, \quad 0 < \lambda_1 \leq \lambda_2 \leq \cdots, \quad \lambda_j \to \infty, \text{ as } j \to \infty.$$  (1.3)

Here $B(\cdot,\cdot)$ is a bilinear operator from $V \times V$ into $V'$, $C$ is a linear operator from $V$ into $Y$ and $f \in L^\infty(R^+; Y)$, $U_0 \in Y$. We denote by $b(\cdot,\cdot,\cdot)$ the trilinear form on $V \times V \times V$ given by

$$b(u,v,w) = \langle B(u,v), w \rangle \quad \forall u, v, w \in V,$$
we assume that

\[ b(u,v,w) = -b(u,w,v) \quad \forall u,v,w \in V, \quad (1.4) \]

\[ |b(u,v,w)| \leq c_0(|u| \|u\| \|w\|)^{1/2} \|v\| \quad \forall u,v,w \in V, \quad (1.5) \]

\[ |Cu| \leq c_1 \|u\| \quad \forall u \in V, \quad (1.6) \]

where \( c_0, c_1 \) are positive constants. Finally, we require \( \nu A + C \) to be positive, i.e., there exists \( \alpha > 0 \) such that

\[ \langle (\nu A + C)u, u \rangle \geq \alpha \|u\|^2 \quad \forall u \in V. \quad (1.7) \]

2. Incremental subspaces

From now on, \( h \) will be a real positive parameter tending to 0 and \( V_h \) will be a finite-dimensional subspace of \( V \). We consider then another subspace \( V_H \subset V_h \) corresponding to the parameter \( H > h \). We denote by \( W_h \) a supplementary (incremental subspace) of \( V_H \) in \( V_h \),

\[ V_h = V_H + W_h. \quad (2.1) \]

Then any \( u_h \in V_h \) can be uniquely written as

\[ u_h = y + z, \quad y \in V_H, \quad z \in W_h. \quad (2.2) \]

For reasons which will become clear hereafter, \( y \) and \( z \) will be called the large and the small eddies components of \( u_h \). The examples of decompositions (2.1) will be given below; before doing that, we state the main hypotheses related to three finite-dimensional subspaces \( V_h, V_H \) and \( W_h \):

**Inverse inequality:**

\[ S_1(h)\|v\| \leq |v| \quad \forall v \in V_h, \quad S_1(H)\|v\| < |v| \quad \forall v \in V_H. \quad (2.3) \]

**Enhanced Cauchy–Schwartz inequality:**

\[ |(A^{1/2}v, A^{1/2}w)| \leq (1 - \delta)\|v\|\|w\| \quad \forall v \in V_H, \quad w \in W_h. \quad (2.4) \]

**Enhanced poincare inequality:**

\[ |w| \leq S_2(H)\|w\| \quad \forall w \in W_h, \quad (2.5) \]

where \( 0 < \delta \leq 1, S_1(h) \to 0, S_2(H) \to 0 \) as \( h, H \to 0 \), respectively, and \( S_1(h) \leq S_1(H) \leq 1 \) as \( h < H \leq H_0, H_0 > 0 \) is a constant.

We now give three important decompositions of the form (2.1).

(i) **Spectral discretization:** For the orthonormal basis \( \{w_j\} \) of \( Y \) and integers \( m, n \), we introduce the finite-dimensional subspaces

\[ V_h = \text{span}\{w_1, \ldots, w_m, \ldots, w_{m+n}\}, \quad V_H = \text{span}\{w_1, \ldots, w_m\}, \]
where \( h = 1/(m+n) \), \( H = 1/m \). In this case, the decomposition of \( V_h \) can be rewritten as \( V_h = V_H + W_h \), where

\[
W_h = \text{span}\{w_{m+1}, \ldots, w_{m+n}\}.
\]

According to the definitions of \( V_H \), \( W_h \) and \( V_h \), we have

\[
\|u\|^2 \leq \lambda_{m+n} |u|^2 \quad \forall u \in V_h, \quad \lambda_{m+1} |u|^2 \leq \|u\|^2 \quad \forall u \in W_h.
\]

\[
(A^{1/2}v, A^{1/2}w) = 0 \quad \forall v \in V_H, \quad w \in W_h.
\]

Hence, we find that

\[
S_1(h) = \lambda_{m+n}^{-1/2}, \quad S_2(H) = \lambda_{m+1}^{-1/2}, \quad \delta = 1.
\]

(ii) **Hierarchical basis finite element discretization:** For simplicity, we restrict here the discussion to the case where \( \Omega \) has polygonal boundaries, but the results can be easily extended to a general curved domain, by introducing an approximate boundary \( \partial \Omega_h \). We are given an admissible triangulation \( \tau_H \) of \( \Omega \) made of triangles \( K \) with diameters bounded by \( H \); a finer triangulation \( \tau_h \) is deduced from \( \tau_H \) by subdividing each triangle \( K \) into \( d^2 \) similar triangles \( K' \), where \( h = H/d \). We denote by \( V_h \) (resp. \( V_H \) the spaces of continuous piecewise linear functions from \( \Omega \) into \( \mathbb{R} \), which are linear on each \( K \in \tau_H \) (resp. \( K \in \tau_h \)) and vanish outside the set \( \Omega \) covered by the triangles

\[
\Omega_h = \bigcup_{K' \in \tau_h} K' = \bigcup_{K \in \tau_H} K.
\]

Let \( e_h \) denote the set of vertices of triangles \( K' \in \tau_h \) and let \( \hat{e}_h \) be the subset of \( e_h \) consisting of interior nodes, i.e., \( \hat{e}_h = e_h \setminus (e_h \cap \partial \Omega) \). We define in a similar manner the corresponding sets \( e_H, \hat{e}_H \) for \( \tau_H \). The canonical (nodal) basis of \( V_h \) consists of the functions \( w_{hM} \in V_h, \ M \in \hat{e}_h \), such that \( w_{hM}(M) = 1 \) and \( w_{hM}(P) = 0 \ \forall P \in e_h, \ P \neq M \). The canonical basis of \( V_H \) is defined in a similar manner. The space \( W_h \) is spanned by the \( \psi_{hM} \in V_h, \ M \in \hat{e}_h \setminus e_H \), such that \( \psi_{hM}(M) = 1 \) and \( \psi_{hM}(P) = 0 \ \forall P \in e_h, \ P \neq M \). These functions \( \psi_{hM} \) constitute a basis of \( W_h \). The union of the basis of \( V_H \) and \( W_h \) provides a basis of \( V_h \) different from the nodal basis; this basis is inherited from \( V_H \) and it is called the hierarchical basis. We may use also sometime the expression-induced basis. For this decomposition of \( V_h \), Marion and Temam [10] has proven that hypotheses (2.3)–(2.5) are satisfied with \( 0 < \delta < 1 \) and

\[
S_1(h) = O(h), \quad S_2(H) = O(H).
\]

(iii) **Orthogonal projection finite element discretization:** For the above finite element spaces \( V_h \) and \( V_H \), we introduce the \( L^2 \)-orthogonal projection \( P_H : Y \to V_H \) defined by

\[
(P_H v, v_H) = (v, v_H) \quad \forall v \in Y, \ v_H \in V_H,
\]

then \( W_h = (I - P_H) V_h \) is a convenient supplementary of \( V_H \) in \( V_h \). In this case, Marion and Xu [11] and Ait Ou Ammi and Marion [1] have proven that hypotheses (2.3)–(2.5) are satisfied with \( 0 < \delta < 1 \) and \( S_1(h) = O(h), \ S_2(H) = O(H) \).
3. Galerkin and inertial algorithms

The usual semi-discrete Galerkin Algorithm consists of finding $u_h(t) \in V_h$ such that

$$
\left( \frac{du_h}{dt}, v_h \right) + a(u_h, v_h) + c(u_h, v_h) + b(u_h, u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h, 
$$

(3.1)

$$
u_h(0) = u_{0h},
$$

(3.2)

where $a(u, v) = (Au, v)$, $c(u, v) = (Cu, v) \forall u, v \in V_h$. Here, $u_{0h}$ is an approximation of $U_0$ in $V_h$. By taking into account the decompositions of $u_h$ and $v_h$

$$
u_h = y + z, \quad y \in V_H, \quad z \in W_h, \quad v_h = v + w, \quad v \in V_H, \quad w \in W_h
$$

and neglecting some small terms in $z$ and $w$, we obtain the semi-discrete Inertial Algorithm: Find

$$
u^h = y + z, \quad y \in V_H, \quad z \in W_h
$$

(3.3)

such that

$$
\left( \frac{dy}{dt}, v \right) + a(y + z, v) + c(y + z, v) + b(y, y, v) + b(y, z, v)

+ b(z, y, v) = (f, v) \quad \forall v \in V_H,
$$

(3.4)

$$
\left( \frac{dz}{dt}, w \right) + a(y + z, w) + c(y + z, w) + b(y, y, w) + b(y, z, w) = (f, w) \quad \forall w \in W_H,
$$

(3.5)

$$
y(0) = u_{0H}, \quad z(0) = u_{0h} - u_{0H}.
$$

(3.6)

We now describe the time discretizations of (3.1)–(3.2) and (3.4)–(3.6) by the Euler explicit and Euler semi-implicit difference schemes. Thus, two Galerkin and two Inertial Algorithms will be proposed. These algorithms are implicit in the linear terms and explicit or semi-implicit in the nonlinear terms.

We set $0 = t_0 < t_1 < t_2 < \cdots$, to be the division of $[0, \infty)$ the time step size $\Delta t_{k+1} = t_{k+1} - t_k$, $k = 0, 1, \ldots,$ and

$$
f^{k+1} = \frac{1}{\Delta t_{k+1}} \int_{t_k}^{t_{k+1}} f(t) \, dt,
$$

then the solution sequences $\{u_k\}$ and $\{u^k\}$, corresponding to Galerkin and Inertial Algorithms, are defined respectively as follows.

**Galerkin Algorithm 1** (Explicit scheme).

$$
\frac{1}{\Delta t_{k+1}} (u_{k+1} - u_k, v) + a(u_{k+1}, v) + c(u_{k+1}, v) + b(u_k, u_k, v) = (f^{k+1}, v) \quad \forall v \in V_h.
$$

(3.7)

**Galerkin Algorithm 2** (Semi-implicit scheme).

$$
\frac{1}{\Delta t_{k+1}} (u_{k+1} - u_k, v) + a(u_{k+1}, v) + c(u_{k+1}, v) + b(u_k, u_{k+1}, v) = (f^{k+1}, v) \quad \forall v \in V_h,
$$

(3.8)
where
\[ u_0 = u_{0h}. \] (3.9)

**Inertial Algorithm 1** (Explicit scheme).
\[
\frac{1}{\Delta t_{k+1}} (y^{k+1} - y^k, v) + a(y^{k+1} + z^{k+1}, v) + c(y^{k+1} + z^{k+1}, v) + b(y^k, y^k, v) \\
+ b(y^k, z^{k+1}, v) + b(z^{k+1}, y^k, v) = (f^{k+1}, v) \quad \forall v \in V_H,
\] (3.10)

\[
\frac{1}{\Delta t_{k+1}} (z^{k+1} - z^k, w) + a(y^k + z^{k+1}, w) + c(y^k + z^{k+1}, w) \\
+ b(y^k, y^k, w) = (f^{k+1}, w) \quad \forall w \in W_h.
\] (3.11)

**Inertial Algorithm 2** (Semi-implicit scheme).
\[
\frac{1}{\Delta t_{k+1}} (y^{k+1} - y^k, v) + a(y^{k+1} + z^{k+1}, v) + c(y^{k+1} + z^{k+1}, v) + b(y^k, y^{k+1}, v) \\
+ b(y^k, z^{k+1}, v) + b(z^{k+1}, y^k, v) = (f^{k+1}, v) \quad \forall v \in V_H,
\] (3.12)

\[
\frac{1}{\Delta t_{k+1}} (z^{k+1} - z^k, w) + a(y^{k+1} + z^{k+1}, w) + c(y^{k+1} + z^{k+1}, w) \\
+ b(y^k, y^k, w) = (f^{k+1}, w) \quad \forall w \in W_h.
\] (3.13)

where
\[ u^k = y^k + z^k, \quad y^0 = y(0), \quad z^0 = z(0). \] (3.14)

For convenience, we usual define \( \Delta t_0 = \Delta t > 0. \)

**Remark 1.** In the case of Galerkin Algorithms, for each \( k \) and \( u_k \), the existence and uniqueness of a solution \( u_{k+1} \) to (3.7) or (3.8) follows from (1.7) and Lax–Milgram’s theorem. Here again, the existence for each \( k \) of the pair \( (y^{k+1}, z^{k+1}) \), solution of (3.10)–(3.11) or (3.12)–(3.13) follows from (1.7) and Lax–Milgram’s theorem.

In order to study the boundedness of these schemes, the following Gronwall lemma is needed.

**Lemma 3.1.** Assume \( \gamma, \beta > 0 \) and \( \eta^k \geq 0, k \leq J + 1 \) such that
\[
(1 + \gamma \Delta t_{k+1}) \eta^{k+1} \leq \eta^k + \beta \Delta t_{k+1}, \quad k \leq J,
\] (3.15)

then
\[ \eta^{J+1} \leq \eta^0 + \beta \gamma^{-1}. \] (3.16)

For proof, the reader can refer to [7,12,16].
4. Boundedness analysis

Assume that \( U_0 \in Y \) and \( f \in L^\infty(R^+; V') \), then we can obtain the boundedness of Galerkin Algorithms and Inertial Algorithms. First, we give the boundedness estimates of the discrete solution sequence \( \{u_k\} \) corresponding to Galerkin Algorithms.

**Theorem 4.1.** If \( \Delta t_k, k \geq 0 \), satisfy

\[
\Delta t_{k+1} \leq \Delta t_k \leq \min \left\{ \frac{2}{\alpha \lambda_1}, \frac{\alpha}{2} c_0^{-2} M^{-2} S_1^2(h) \right\} \quad \text{for } i = 1
\]

\[
\Delta t_{k+1} \leq \Delta t_k \leq \frac{2}{\alpha \lambda_1} \quad \text{for } i = 2,
\]

then the solution sequence \( \{u_k\} \), generated by Galerkin Algorithm \( i \) is bounded in the following sense:

\[
|u_m|^2 + \frac{\alpha}{2} \Delta t_m \|u_m\|^2 \leq M^2 \quad \forall m \geq 0,
\]

\[
\alpha \sum_{k=1}^{m} \|u_k\|^2 \Delta t_k \leq M_1^2 + \frac{\alpha}{2} \int_0^{t_m} \|f(t)\|_*^2 dt \quad \forall m \geq 1,
\]

where \( f_* = \sup_{t \geq 0} \|f(t)\|_* \), \( \|f\|_* = \sup_{v \in V} (f, v)/\|v\| \) and

\[
M_1^2 = \max\{|u_0|^2, |u_0|^2 + |u_0 - u_{0H}|^2\} + \frac{\alpha}{2} \Delta t \|u_0\|^2,
\]

\[
M^2 = M_1^2 + \lambda_1^{-1} \tilde{\delta}^{-1} \left( \frac{2}{\alpha} \right)^2 f_*^2.
\]

For the sake of brevity, we omit this proof, as it is classical and more simple than the proof of Theorem 4.2.

Next, we shall give the boundedness estimates of the solution sequence \( \{u^k\} \) corresponding to Inertial Algorithms.

**Theorem 4.2.** Assume that \( \Delta t_k, k \geq 0 \), satisfy

\[
\Delta t_{k+1} \leq \Delta t_k \leq \min \left\{ \frac{2}{\alpha \lambda_1}, \frac{\alpha \delta S_1^2(H)}{18(v^2 + \tilde{c}^2 + c_0^2 c_2 M^2)} \right\} \quad \text{for } i = 1
\]

\[
\Delta t_{k+1} \leq \Delta t_k \leq \frac{2}{\alpha \lambda_1} \quad \text{for } i = 2,
\]

then the sequence \( \{u^k\} \), generated by Inertial Algorithm \( i, i = 1, 2 \), is bounded in the following sense:

\[
|y^m|^2 + |z^m|^2 + \alpha \delta \|y^m\|^2 \Delta t_m \leq M^2 \quad \forall m \geq 0,
\]

\[
\alpha \sum_{k=1}^{m} \|u^k\|^2 \Delta t_k \leq M_1^2 + \frac{\alpha}{2} \int_0^{t_m} \|f(t)\|_*^2 dt \quad \forall m \geq 1,
\]

\[
M_2^2 = M_1^2 + \lambda_1^{-1} \delta^{-1} \left( \frac{2}{\alpha} \right)^2 f_*^2 + \frac{\alpha}{2} \int_0^{t_m} \|f(t)\|_*^2 dt \quad \forall m \geq 1.
\]
where \( u^k = y^k + z^k \) and
\[
\bar{c} = \max \left\{ 1, c_1 \frac{S_2(H)}{S_1(H)}, \left( \frac{S_2(H)}{S_1(H)} \right)^{1/2} \right\}.
\]

**Proof.** Taking \( v = y^{k+1} \) in (3.10), \( w = z^{k+1} \) in (3.11) and adding the corresponding relations and making use of (1.4), one finds
\[
\frac{1}{2\Delta t_{k+1}} (|y^{k+1}|^2 + |z^{k+1}|^2 - |y^k|^2 - |z^k|^2 + |y^{k+1} - y^k|^2 + |z^{k+1} - z^k|^2)
\]
\[
+ a(y^k + z^k, y^{k+1} + z^{k+1}) + c(y^k + z^k, y^{k+1} + z^{k+1})
\]
\[
+ a(y^k - y^{k+1}, z^k + z^{k+1}) + c(y^k - z^{k+1}, y^{k+1} + z^{k+1}) + b(z^{k+1}, y^k, y^{k+1} - y^k)
\]
\[
+ b(y^k, y^{k+1} - y^k) + b(y^k, z^{k+1}, y^{k+1} - y^k) = (f^{k+1}, y^{k+1} + z^{k+1}).
\]
(4.9)

Using again (1.4)–(1.7) and (2.3)–(2.5), we have
\[
a(u^{k+1}, u^{k+1}) + c(u^{k+1}, u^{k+1}) \geq \alpha \|u^{k+1}\|^2,
\]
(4.10)
\[
|a(y^k - y^{k+1}, z^k + z^{k+1}) + c(y^k - y^{k+1}, z^k + z^{k+1})|
\]
\[
\leq (\nu S^{-1}_1(H)||z^{k+1}|| + \bar{c}||z^{k+1}||)||y^{k+1} - y^k||
\]
\[
\leq \frac{\alpha \delta}{8} ||z^{k+1}||^2 + \frac{4}{\delta \alpha} (v^2 + \bar{c}^2) S^{-2}_1(H)||y^{k+1} - y^k||^2,
\]
(4.11)
\[
|b(y^k, y^k, y^{k+1} - y^k)| \leq \frac{\alpha \delta}{4} ||y^k||^2 + \frac{1}{\delta \alpha} c_0^2 S^{-2}_1(H)||y^{k+1} - y^k||^2,
\]
(4.12)
\[
|b(y^k, z^{k+1}, y^{k+1} - y^k)| + |b(z^{k+1}, y^k, y^{k+1} - y^k)|
\]
\[
\leq \frac{\alpha \delta}{8} ||z^{k+1}||^2 + \frac{8}{\delta \alpha} c_0^2 \bar{c}^2 S^{-2}_1(H)||y^k||^2||y^{k+1} - y^k||^2,
\]
(4.13)
\[
|(f, u^{k+1})| \leq \frac{\alpha}{4} ||u^{k+1}||^2 + \frac{1}{\alpha} ||f||^2_u,
\]
(4.14)
\[
(||y^k||^2 + ||z^k||^2) \leq ||y^k + z^k||^2 \quad \forall k \geq 0.
\]
(4.15)

Thus, (4.5) with (4.9)–(4.15) imply
\[
|y^{k+1}|^2 + |z^{k+1}|^2 - (|y^k|^2 + |z^k|^2) + \alpha ||y^{k+1} + z^{k+1}||^2 \Delta t_{k+1}
\]
\[
+ \left(1 - \frac{8}{\delta \alpha} (v^2 + \bar{c}^2) + \frac{18}{\delta \alpha} c_0^2 \bar{c}^2 ||y^k||^2\right) \Delta t S^{-2}_1(H)||y^{k+1} - y^k||^2
\]
\[
+ \frac{\alpha \delta}{2} ||y^{k+1}||^2 \Delta t_{k+1} - \frac{\alpha \delta}{2} ||y^k||^2 \Delta t_{k+1} \leq \frac{2}{\alpha} \int_{t_k}^{t_{k+1}} ||f(t)||^2_u dt.
\]
(4.16)
Similarly, (3.12) and (3.13) yield
\[
|y^{k+1}|^2 + |z^{k+1}|^2 - (|y^k|^2 + |z^k|^2) + \frac{3}{2} z_k y^{k+1} + z^{k+1} \Delta t_{k+1} \leq \frac{2}{\alpha} \int_{t_k}^{t_{k+1}} \|f(t)\|^2 \, dt,
\]  
thanks to (1.4).

By setting
\[
\eta^k = |y^k| + |z^k|^2 + \frac{\alpha \delta}{2} \|y^k\|^2 \Delta t_k,
\]
we derive from (4.16) that
\[
\eta^{k+1} - \eta^k + \alpha \|y^{k+1} + z^{k+1}\|^2 \Delta t_{k+1} + (1 - B(\Delta t, H, |y^k|))|y^{k+1} - y^k|^2 
\leq \frac{2}{\alpha} \Delta t_{k+1} f_\infty^2,
\]  
(4.18)

where
\[
B(\Delta t, H, \beta) = \left(\frac{18}{\alpha \delta}(v^2 + c^2) + \frac{18}{\alpha \delta} c_0^2 c^2 \beta^2\right) S_1^{-2}(H) \Delta t.
\]

Under condition (4.5), we proceed to prove (4.7) by induction. Thanks to
\[
|y^0|^2 + |z^0|^2 + \frac{\alpha \delta}{2} \|y^0\|^2 \Delta t \leq M^2,
\]
inequality (4.7) is true for \(m = 0\). Assume that (4.7) is true for \(m = 0, 1, \ldots, J\), i.e.,
\[
|y^m|^2 + |z^m|^2 + \frac{\alpha \delta}{2} \|y^m\|^2 \Delta t_m \leq M^2,
\]  
(4.19)

we want to establish it for \(m = J + 1\). Thus, we derive from (4.5) and (4.19) that
\[
1 - B(\Delta t, H, |y^k|) \geq 1 - B(\Delta t, H, M) \geq 0 \quad \forall k \leq J.
\]  
(4.20)

Moreover, (1.3), (2.4) and (4.5) imply
\[
\|y^{k+1} + z^{k+1}\|^2 \geq \frac{\lambda_1}{2} (|y^{k+1}|^2 + |z^{k+1}|^2) + \frac{\delta}{2} \|y^{k+1}\|^2 
\geq \frac{\lambda_1}{2} (|y^{k+1}|^2 + |z^{k+1}|^2 + \frac{\alpha \delta}{2} \|y^{k+1}\|^2 \Delta t_{k+1}) = \frac{\lambda_1}{2} \eta^{k+1}.
\]  
(4.21)

Hence, (4.16) with (4.20)–(4.21) yield
\[
\left(1 + \frac{\alpha \delta \lambda_1}{2} \Delta t_{k+1}\right) \eta^{k+1} \leq \eta^k + \frac{2}{\alpha} f_\infty^2 \Delta t_{k+1}, \quad k \leq J.
\]  
(4.22)

Applying Lemma 3.1, one finds
\[
\eta^{J+1} \leq \eta^0 + \lambda_1^{-1} \delta^{-1} \left(\frac{2}{\alpha}\right)^2 f_\infty^2 \leq M^2.
\]  
(4.23)
This completes the inductive step and proves that (4.7) is true for Inertial Algorithm 1. Furthermore, thanks to (4.7) and (4.20), (4.15) and (4.16) yield
\[
|y^{k+1}|^2 + |z^{k+1}| - |y^k|^2 - |z^k|^2 + \|y^{k+1} + z^{k+1}\|^2 \Delta t_{k+1}
+ \frac{\rho \delta}{2} \Delta t_{k+1}(\|y^{k+1}\|^2 - \|z^k\|^2) \leq \frac{2}{\varepsilon} \int_{t_k}^{t_{k+1}} \|f(t)\|_*^2 \, dt, \tag{4.24}
\]
summing (4.24) for \(k = 0, 1, \ldots, m - 1\), we obtain (4.8).

Similarly, by using (4.6) and (4.17), one can prove that (4.7)–(4.8) are true for inertial Algorithm 2. This completes the proof of Theorem 4.2.

5. Stability analysis

It is well known that the approximate accuracy of numerical solution of differential equation depends on the discrete error of numerical algorithm and its roundoff error. For simplicity, we study here the roundoff error corresponding to the roundoff approximations \(u_{0h} + E_0\) and \(f^k + \hat{\zeta}^k\) of the initial data \(u_{0h}\) and the external force term \(f^k, k \geq 1\). We set \(u_k + E_k\) and \((y^k + e^k, z^k + \hat{\zeta}^k)\) to be the roundoff approximations of the discrete solution \(u_k, (y^k, z^k), k \geq 0\). Here the continuous dependent relation of the roundoff error to small disturbances \(E_0\) and \(\hat{\zeta}^k, k \geq 1\), is called the stability of numerical algorithm, which is different from the concept of stability in [8, 12, 16].

According to Galerkin Algorithms 1, 2 and Inertial Algorithms 1, 2, we obtain the roundoff error equations corresponding to these algorithms, where \(E_0 = e^0 + e^0\).

Galerkin Algorithm 1.

\[
\frac{1}{\Delta t_{k+1}}(E_{k+1} - E_k, v) + a(E_{k+1}, v) + c(E_{k+1}, v) + b(u_k, E_k, v)
+ b(E_k, u_k, v) + b(E_k, E_k, v) = \left(\hat{z}^{k+1}_v, v\right) \quad \forall v \in V_h. \tag{5.1}
\]

Galerkin Algorithm 2.

\[
\frac{1}{\Delta t_{k+1}}(E_{k+1} - E_k, v) + a(E_{k+1}, v) + c(E_{k+1}, v) + b(u_k, E_{k+1}, v)
+ b(E_k, u_{k+1}, v) + b(E_k, E_{k+1}, v) = \left(\hat{z}^{k+1}_v, v\right) \quad \forall v \in V_h. \tag{5.2}
\]

Inertial Algorithm 1.

\[
\frac{1}{\Delta t_{k+1}}(e^{k+1} - e^k, v) + \frac{1}{\Delta t_{k+1}}(e^{k+1} - e^k, w) + a(e^{k+1} + \hat{e}^{k+1}, v + w)
+ c(e^{k+1} + \hat{e}^{k+1}, v + w) + a(e^k - \hat{e}^{k+1}, w) + c(e^k - \hat{e}^{k+1}, w)
+ b(y^k, e^k, v + w) + b(e^k, y^k, v + w) + b(e^k, e^k, v + w)
+ b(y^k, e^{k+1}, v) + b(e^k, z^{k+1}, v) + b(e^k, e^{k+1}, v)
\]
\begin{equation}
+b(e^{k+1}, y^k, v) + b(z^{k+1}, e^k, v) + b(e^{k+1}, e^k, v) \\
= (\xi^{k+1}, v + w) \quad \forall v \in V_H, \ w \in W_h.
\end{equation}

**Inertial Algorithm 2.**

\begin{equation}
\frac{1}{\Delta t_{k+1}} (e^{k+1} - e^k, v) + \frac{1}{\Delta t_{k+1}} (e^{k+1} - e^k, w) + a(e^{k+1} + e^{k+1}, v + w) \\
+ b(y^k, e^{k+1}, v + w) + b(z^{k+1}, e^k, v + w) + b(z^k, e^{k+1}, v + w) + b(z^{k+1}, e^k, v + w) \\
+ b(y^k, e^{k+1}, v + w) + b(y^k, e^{k+1}, v) + b(e^k, z^{k+1}, v) + b(e^k, e^{k+1}, v) \\
+ b(z^k, e^{k+1}, v) + b(e^k, z^{k+1}, v) + b(e^k, e^{k+1}, v) = (\xi^{k+1}, v + w) \quad \forall v \in V_H, \ w \in W_h.
\end{equation}

Taking \( v = E_{k+1} \) in (5.1) and (5.2), \( v = e^{k+1} \), \( w = e^{k+1} \) in (5.3) and (5.4) we then derive from (1.4) and (1.7) that

**Galerkin Algorithm 1.**

\begin{equation}
\frac{1}{2\Delta t_{k+1}} (|E_{k+1}|^2 - |E_k|^2 + |E_{k+1} - E_k|^2) + \alpha \|E_{k+1}\|^2 + b(u_k, E_k, E_{k+1}) \\
+ b(E_k, u_{k+1}, E_{k+1}) + b(E_k, E_k, E_{k+1} - E_k) \leq (\xi^{k+1}, E_{k+1}).
\end{equation}

**Galerkin Algorithm 2.**

\begin{equation}
\frac{1}{2\Delta t_{k+1}} (|E_{k+1}|^2 - |E_k|^2 + |E_{k+1} - E_k|^2) + \alpha \|E_{k+1}\|^2 \\
+ b(E_k, u_{k+1}, E_{k+1}) \leq (\xi^{k+1}, E_{k+1}).
\end{equation}

**Inertial Algorithm 1.**

\begin{equation}
\frac{1}{2\Delta t_{k+1}} (|e^{k+1}|^2 - |e^k|^2 + |e_{k+1} - e_k|^2) + (|e^{k+1} - e^k|^2 + |e^{k+1} - e^k|^2) \\
+ \alpha \|e^{k+1} + e^{k+1}\|^2 + a(e^k - e^{k+1}, e^{k+1}) + c(e^k - e^{k+1}, e^{k+1}) \\
+ b(y^k, e^{k+1} + e^{k+1}) + b(e^k, y^k, e^{k+1} + e^{k+1}) + b(e^{k+1}, y^k, e^{k+1}) \\
+ b(y^k, e^{k+1} + e^{k+1}) + b(e^k, z^{k+1}, e^{k+1}) + b(z^{k+1}, e^k, e^{k+1}) \\
+ b(e^k, e^k, e^{k+1} - e^k) + b(e^k, e^{k+1}, e^{k+1} - e^k) + b(e^{k+1}, e^k, e^{k+1} - e^k) \\
\leq (\xi^{k+1}, e^{k+1} + e^{k+1}).
\end{equation}
Inertial Algorithm 2.

\[
\frac{1}{2\Delta t_{k+1}}(\|e^{k+1}\|^2 + |\xi^{k+1}|^2 - |e^k|^2 - |\xi^k|^2 + |e^{k+1} - e^k|^2) + \alpha \|e^{k+1} + \xi^{k+1}\|^2 \\
+ b(\xi^k, y^{k+1} + \xi^{k+1}, e^{k+1}) + b(\xi^k, y^{k+1}, e^{k+1}) + b(\xi^k, y^{k+1}, e^{k+1}) \\
= (\xi^{k+1}, e^{k+1} + \xi^{k+1}).
\]  

(5.8)

Thanks to (1.4)–(1.5) and from (2.3), we have

\[
|b(u_k, E_k, E_{k+1})| + |b(E_k, u_k, E_{k+1})| \leq 2c_0|u_k||E_k||E_{k+1}|^{1/2}||E_{k+1}||
\leq \frac{\alpha}{8}(\|E_{k+1}\|^2 + \|E_k\|^2) + 2\left(\frac{4}{\alpha}\right)^3 c_0^4|u_k|^2\|E_k\|^2|E_{k+1} - E_k|^2,
\]  

(5.9)

\[
|b(E_k, E_k, E_{k+1} - E_k)| \leq \frac{\alpha}{8}\|E_k\|^2 + \frac{2}{\alpha}c_0^2S^{-2}(h)|E_k|^2|E_{k+1} - E_k|^2,
\]  

(5.10)

\[
|(\xi^{k+1}, E_{k+1})| \leq \frac{\alpha}{8}\|E_{k+1}\|^2 + \frac{2}{\alpha}\|\xi^{k+1}\|^2_s,
\]  

(5.11)

\[
|b(E_k, u_{k+1}, E_{k+1})| \leq \frac{\alpha}{8}\|E_{k+1}\|^2 + \frac{\alpha}{8}\|E_k\|^2 + \left(\frac{2}{\alpha}\right)^3 c_0^4|u_{k+1}|^2\|u_{k+1}\|^2|E_k|^2.
\]  

(5.12)

Combining (5.5) with (5.9)–(5.11), (5.6) with (5.11)–(5.12), we find the following error estimates:

Galerkin Algorithm 1.

\[
|E_{k+1}|^2 - |E_k|^2 + \left(1 - \frac{8}{\alpha}c_0^2S^{-2}(h)\Delta t_{k+1}|E_k|^2\right)|E_{k+1} - E_k|^2 + \frac{\alpha}{2}(\|E_{k+1}\|^2 - \|E_k\|^2)\Delta t_{k+1}
+ \alpha\|E_{k+1}\|^2|E_k|^2
\leq \frac{8}{\alpha}\|\xi^{k+1}\|^2_s\Delta t_{k+1} + 4\left(\frac{4}{\alpha}\right)^3 c_0^4|u_k|^2 + \frac{c_0^2}{\alpha}\|u_k\|^2|E_k|^2\Delta t_{k+1}.
\]  

(5.13)

Galerkin Algorithm 2.

\[
|E_{k+1}|^2 - |E_k|^2 + \alpha\|E_{k+1}\|^2|E_k|^2
\leq \frac{8}{\alpha}\|\xi^{k+1}\|^2_s\Delta t_{k+1} + 4\left(\frac{2}{\alpha}\right)^3 c_0^4|u_{k+1}|^2\|u_{k+1}\|^2|E_k|^2\Delta t_{k+1}.
\]  

(5.14)

Moreover, we will infer the similar inequalities for Inertial Algorithms. Thanks to (1.4)–(1.6) and from (2.3)–(2.5), we have

\[
|a(\xi^k - e^{k+1}, \xi^{k+1}) + c(\xi^k - e^{k+1}, \xi^{k+1})|
\leq \frac{\alpha\delta}{16}\|\xi^{k+1}\|^2 + \frac{8}{\alpha\delta}(\nu^2 + \xi^2)S^{-2}(H)|e^{k+1} - e^k|^2,
\]  

(5.15)
\[ |b(y^k, e^k, e^{k+1} + e^{k+1}) + b(e^k, y^k, e^{k+1} + e^{k+1})| \]
\[ \leq \frac{\alpha}{16} ||e^{k+1} + e^{k+1}\|^2 + \frac{\alpha \delta}{16} ||e^k\|^2 + \frac{4}{\alpha \delta} \left( \frac{16}{\alpha} \right)^2 c_0^2 |y^k|^2 |y^k|^2 |e^k|^2 , \]  
(5.16)

\[ |b(e^{k+1}, y^k, e^{k+1} + e^{k+1})| \leq \frac{\alpha \delta}{16} (||e^{k+1}\|^2 + ||e^{k+1}\|^2) \]
\[ + \frac{4}{\alpha \delta} \left( \frac{16}{\alpha} \right)^2 c_0^2 |y^k|^2 |y^k|^2 |e^{k+1}|^2 , \]  
(5.17)

\[ |b(e^k, z^{k+1}, e^{k+1}) + b(z^{k+1}, e^k, e^{k+1})| \leq \frac{\alpha \delta}{16} (||e^{k+1}\|^2 + ||e^k\|^2) \]
\[ + \left( \frac{4}{\alpha \delta} \right) \left( \frac{16}{\alpha} \right)^2 c_0^4 |z^{k+1}|^2 |z^{k+1}|^2 |e^{k+1}|^2 , \]  
(5.18)

\[ |b(e^k, e^{k+1}, e^{k+1} - e^k) + b(e^{k+1}, e^k, e^{k+1} - e^k)| \]
\[ \leq \frac{\alpha \delta}{16} ||e^{k+1}\|^2 + \frac{16}{\alpha \delta} c_0^2 e^2 s^{-2}(H) |e^k|^2 |e^{k+1} - e^k|^2 , \]  
(5.19)

\[ |b(e^k, e^k, e^{k+1} - e^k)| \leq \frac{\alpha \delta}{16} ||e^k\|^2 + \frac{4}{\alpha \delta} c_0^2 s^{-2}(H) |e^k|^2 |e^{k+1} - e^k|^2 , \]  
(5.20)

\[ |(z^{k+1} + e^{k+1}, e^{k+1})| \leq \frac{\alpha}{8} ||e^{k+1} + e^{k+1}\|^2 + \frac{\alpha}{8} ||z^{k+1}\|^2 \]  
(5.21)

\[ |b(e^k, y^{k+1}, z^{k+1}, e^{k+1})| \leq \frac{\alpha \delta}{16} ||e^{k+1}\|^2 + \frac{\alpha \delta}{16} ||e^k\|^2 \]
\[ + \left( \frac{4}{\alpha \delta} \right)^3 c_0^4 |y^{k+1}| + z^{k+1}|^2 |y^{k+1}| + z^{k+1}|^2 |e^k|^2 , \]  
(5.22)

\[ |b(e^k, y^{k+1}, e^{k+1}) + b(e^k, y^{k+1}, e^{k+1})| \leq \frac{\alpha \delta}{16} (||e^{k+1}\|^2 + ||e^{k+1}\|^2) \]
\[ + \left( \frac{4}{\alpha \delta} \right)^3 c_0^4 |y^{k+1}|^2 |y^{k+1}|^2 (|e^k|^2 + |e^k|^2) \]
\[ + \frac{\alpha \delta}{16} (||e^k\|^2 + ||e^k\|^2) . \]  
(5.23)

Combining (5.7) with (5.15)–(5.21), (5.8) with (5.21)–(5.23), we obtain

Inertial Algorithm 1.

\[ |e^{k+1}|^2 + |e^{k+1}|^2 - |e^k|^2 - |e^k|^2 + \alpha ||e^{k+1} + e^{k+1}\|^2 \Delta t_{k+1} + |e^{k+1} - e^k|^2 \]
\[ + (1 - B(\Delta t_{k+1}, H, |e^k|)) |e^{k+1} - e^k|^2 + \frac{3}{8} \alpha \delta (||e^{k+1}\|^2 - ||e^k\|^2) \Delta t_{k+1} \]
\[
\Delta t_{k+1} + \|y^{k+1}\|_{2}^{2} + \|z^{k+1}\|_{2}^{2} \leq \frac{4}{\alpha} \left\| \Delta t_{k+1} \right\| + \frac{8}{\alpha \delta} \left( \frac{16}{\alpha \delta} \right)^{2} c_{0}^{4} \left( \|e^{k+1}\|^{2} + \|z^{k+1}\|^{2} \right) \Delta t_{k+1} \\
+ \frac{8}{\alpha \delta} \left( \frac{16}{\alpha \delta} \right)^{2} c_{0}^{4} \left( \|z^{k+1}\|^{2} \right) \Delta t_{k+1}.
\]

(5.24)

**Inertial Algorithm 2.**

\[
|e^{k+1}|^{2} + |x^{k+1}|^{2} - |e^{k}|^{2} - |x^{k}|^{2} + \alpha |e^{k+1} + x^{k+1}|^{2} \Delta t_{k+1} \\
+ \frac{3}{8} \alpha (|e^{k+1} + x^{k+1}|^{2} - |e^{k} + x^{k}|^{2} \Delta t)_{k+1} \\
\leq \frac{4}{\alpha} \left\| z^{k+1} \right\|_{2}^{2} \Delta t_{k+1} \\
+ 2 \left( \frac{4}{\alpha} \right) c_{0}^{4} (|y^{k+1} + z^{k+1}|^{2} y^{k+1} + z^{k+1}|^{2} + |y^{k+1}|^{2} y^{k+1}|^{2}) \\
\times (|e^{k}|^{2} + |x^{k}|^{2}) \Delta t_{k+1}.
\]

(5.25)

Here

\[
B(\Delta t_{k+1}, H, \beta) = \left( \frac{16}{\alpha \delta} (v^{2} + \bar{v}^{2}) + \frac{40}{\alpha \delta} c_{0}^{2} \bar{c}^{2} \beta^{2} \right) S^{-2}(H) \Delta t_{k+1}.
\]

To study the stability of above schemes, we will need a discrete version of the Gronwall lemma in a slightly more general form than usually used in the literature [14]. For the sake of completeness, we supply its simple proof.

**Lemma 5.1.** Let \( a_{k}, b_{k}, c_{k}, d_{k}, \gamma_{k}, k \geq 0 \), and \( \beta \) be nonnegative real numbers such that

\[
a_{k+1} + b_{k+1} \Delta t_{k+1} + \gamma_{k+1} \Delta t_{k+1} - \gamma_{k} \Delta t_{k} \leq (1 + d_{k} \Delta t_{k}) a_{k} + c_{k+1} \Delta t_{k+1} \quad \forall k \leq J,
\]

(5.26)

then

\[
a_{J} + \sum_{k=0}^{J-1} b_{k} \Delta t_{k} \leq \exp \left( \sum_{k=0}^{J-1} d_{k} \Delta t_{k} \right) \left\{ a_{0} + (b_{0} + \gamma_{0}) \Delta t + \sum_{k=0}^{J-1} c_{k} \Delta t_{k} \right\}.
\]

(5.27)

**Proof.** Using recursively relation (5.26), we get

\[
a_{J+1} + \sum_{k=0}^{J} b_{k} \Delta t_{k} \leq \prod_{k=0}^{J} (1 + d_{k} \Delta t_{k}) \left\{ a_{0} + (b_{0} + \gamma_{0}) \Delta t + \sum_{k=1}^{J} c_{k} \Delta t_{k} \right\},
\]

on the other hand, since \( (1 + x) \leq e^{x} \forall x \in \mathbb{R} \), we find

\[
\prod_{k=0}^{J} (1 + d_{k} \Delta t_{k}) \leq \exp \left( \sum_{k=0}^{J} d_{k} \Delta t_{k} \right) = \exp \left( \sum_{k=0}^{J} d_{k} \Delta t_{k} \right).
\]

Now, we give the stability results of Galerkin and Inertial Algorithms. \( \square \)
Theorem 5.1. If $\Delta t \leq 2/\lambda_1$ and $\Delta t_k$, $k \geq 0$, satisfy

$$\Delta t_k = \min \left\{ \Delta t, \frac{\tau}{2} c_0^{-2} M^{-2} S_1^2(h), \frac{\tau}{8} c_0^{-2} \theta_{k-1}^{-2} S_1^2(h) \right\} \quad \text{for } i = 1,$$

$$\Delta t_k = \Delta t \quad \text{for } i = 2,$$

then Galerkin Algorithm $i$, $i = 1, 2$, is stable, i.e.,

$$|E_m|^2 + \sum_{k=0}^{m} \|E_k\|^2 \Delta t_k \leq \theta_m^2 \quad \forall m \geq 0,$$

where $\zeta^0 = 0$, $\theta_{-1} = 0$ and

$$\theta_m^2 = \exp \left( \sum_{k=0}^{m} d_k \Delta t_k \right) \left\{ \frac{4}{\tau} \sum_{k=0}^{m} \|\zeta_k\|^2 \Delta t_k + |E_0|^2 + \frac{3}{2} \zeta \|E_0\|^2 \Delta t \right\},$$

$$d_k = \begin{cases} 4(\frac{\tau}{2})^3 c_0^4 |u_k|^2 |u_k|^2 & \text{for } i = 1, \\ 2(\frac{\tau}{2})^3 c_0^4 |u_{k+1}|^2 |u_{k+1}|^2 & \text{for } i = 2. \end{cases}$$

This proof is classical and simple, it can be omitted.

Theorem 5.2. If $\Delta t \leq 2/\lambda_1$, and $\Delta t_k$, $k \geq 0$, satisfy

$$\Delta t_k = \min \left\{ \Delta t, \frac{\tau \delta S_1^2(H)}{18(\nu^2 + \zeta^2 + c_0^2 \zeta^2 M^2)}, \left( \frac{\tau \delta}{16} \right)^3 c_0^{-4} M^{-4} S_1^2(H), \right\} \quad \text{for } i = 1,$$

$$\Delta t_k = \Delta t \quad \text{for } i = 2,$$

then Inertial Algorithm $i$, $i = 1, 2$, is stable, i.e.,

$$|e_m|^2 + |e^m|^2 + \sum_{k=0}^{m} \|e^k + \varepsilon^k\|^2 \Delta t_k \leq \theta_m^2,$$

where $\theta_{-1} = 0$ and

$$\theta_m^2 = \exp \left( \sum_{k=0}^{m} d_k \Delta t_k \right) \left\{ |e_0|^2 + |e^0|^2 + \frac{3}{2} \zeta \|E_0\|^2 \Delta t + \sum_{k=0}^{m} \|\varepsilon_k\|^2 \Delta t_k \right\},$$

$$d_k = \begin{cases} (\frac{16}{\tau \delta})^3 c_0^4 (|y^k|^2 ||y^k||^2 + |z^{k+1}|^2 ||z^{k+1}||^2) & \text{for } i = 1, \\ 2(\frac{4}{\tau \delta})^3 c_0^4 (|y^{k+1} + z^{k+1}|^2 ||y^{k+1} + z^{k+1}||^2 + |y^{k+1}|^2 ||y^{k+1}||^2) & \text{for } i = 2. \end{cases}$$
Proof. We proceed to prove (5.33) by induction for Inertial Algorithm 1. Thanks to
\[ |e^0|^2 + |e^0|^2 + x|e^0 + e^0|^2 \Delta t = |e^0|^2 + |e^0|^2 + x\|E_0\|^2 \Delta t \leq \theta_0^2, \]
(5.33) is true for \( m = 0 \). Assume
\[ (5.33) \text{ is true for } m = 0, 1, \ldots, J, \] (5.34)
we want to prove that (5.33) is true for \( m = J + 1 \).
According to (5.31), (5.34), (2.3) and Theorem 4.2, one finds that for all \( k = 0, 1, \ldots, J \)
\[ 1 - B(\Delta t_{k+1}, H, |e^k|) \geq 1 - B(\Delta t_{k+1}, H, \theta_k) \geq 0, \] (5.35)
\[ |e^{k+1}|^2 \leq 2|e^k|^2 + 2|e^{k+1} - e^k|^2, \] (5.36)
\[ \left( \frac{16}{\alpha \delta} \right)^3 c_0^4 \|y^k\|^2 \|y^k\|^2 |e^{k+1} - e^k|^2 \Delta t_{k+1} \leq \left( \frac{16}{\alpha \delta} \right)^3 c_0^4 |y^k| S_1^{-2}(H) \Delta t_{k+1} |e^{k+1} - e^k|^2 \]
\[ \leq \left( \frac{16}{\alpha \delta} \right)^3 c_0^4 M^4 S_1^{-2}(H) \Delta t_{k+1} |e^{k+1} - e^k|^2 \]
\[ \leq |e^{k+1} - e^k|^2, \] (5.37)
thus, (5.24) gives
\[ |e^{k+1}|^2 + |e^{k+1}|^2 - |e^k|^2 - |e^k|^2 + x\|e^{k+1} + e^{k+1}\|^2 \Delta t_{k+1} \]
\[ + \frac{3}{8} \alpha \delta (\|e^{k+1}\|^2 - \|e^k\|^2) \Delta t_{k+1} \leq \frac{4}{\alpha} \|e^{k+1}\|^2 \Delta t_{k+1} \]
\[ + \left( \frac{16}{\alpha \delta} \right)^3 c_0^4 (\|y^k\|^2 \|y^k\|^2 + \|z^{k+1}\|^2 \|z^{k+1}\|^2) (|e^k|^2 + |e^k|^2) \Delta t_{k+1}. \] (5.38)
Noticing \( \Delta t_{k+1} \leq \Delta t_k \), we obtain
\[ |e^{k+1}|^2 + |e^{k+1}|^2 + x\|e^{k+1} + e^{k+1}\|^2 \Delta t_{k+1} + \frac{3}{8} \alpha \delta \|e^{k+1}\|^2 \Delta t_{k+1} \]
\[ - \frac{3}{8} \alpha \delta \|e^k\|^2 \Delta t_k \leq (1 + d_k \Delta t_k) (|e^k|^2 + |e^k|^2) + \frac{4}{\alpha} \|e^{k+1}\|^2 \Delta t_{k+1}. \] (5.39)
Applying Lemma 5.1 to (5.39) with
\[ a_k = |e^k|^2 + |e^k|^2, \quad b_k = x\|e^k + e^k\|^2, \quad c_k = \frac{4}{\alpha} \|e^k\|^2, \] (5.40)
\[ \gamma_k = \frac{3}{8} \alpha \delta \|e^k\|^2, \quad d_k = \left( \frac{16}{\alpha \delta} \right)^3 c_0^4 (\|y^k\|^2 \|y^k\|^2 + \|z^{k+1}\| \|z^{k+1}\|), \] (5.41)
we obtain that (5.33) is true for \( m = J + 1 \).
Moreover, we apply Lemma 5.1 to (5.25) with \( a_k, b_k, c_k \) like those in (5.40) and
\[ \gamma_k = \frac{3}{8} \alpha \|e^k + e^k\|^2, \quad d_k = 2 \left( \frac{4}{\alpha} \right)^3 c_0^4 (|y^{k+1}|^2 + |z^{k+1}|^2 |y^{k+1}| + |z^{k+1}|^2 + |y^{k+1}|^2 |y^{k+1}|^2), \] (5.42)
then (5.33) is proved for Inertial Algorithm 2, and this completes the proof of Theorem 5.2. \( \square \)
6. Conclusions

In this paper, we have presented the explicit and semi-implicit Inertial Algorithms for the nonlinear evolution equations. In order to check the effectiveness of Inertial Algorithms, we also present the explicit and semi-implicit Galerkin Algorithms. One of our main results is that the boundedness and stability of the semi-implicit Inertial Algorithm and Galerkin Algorithm are the same, which admits a large equal time step size independent of time integral step number \( m \) and spatial discrete parameters \( h, H \).

Another important result is that the boundedness and stability of the explicit Inertial Algorithm is superior to the explicit Galerkin Algorithm. This means the restriction of Inertial Algorithm on the time step size is less than that of Galerkin Algorithm on the time step size. In fact, for the boundedness condition of Inertial Algorithm 1, (4.5) gives

\[
\Delta t_k \leq \alpha_1 S_1^2(H)
\]  
(6.1)

and for the boundedness condition of Galerkin Algorithm 1, (4.1) gives

\[
\Delta t_k \leq \alpha_2 S_1^2(h),
\]  
(6.2)

when \( h < H \) is sufficiently small. Moreover, on the stability analysis, (5.31) gives the stability condition of Inertial Algorithm 1:

\[
\Delta t_k \leq \frac{S_2^2(H)}{\beta_1 + \gamma_1 \theta_k^2_{k-1}}
\]  
(6.3)

and (5.28) gives the stability condition of Galerkin Algorithm 1:

\[
\Delta t_k \leq \frac{S_1^2(h)}{\beta_2 + \gamma_2 \theta_k^2_{k-1}},
\]  
(6.4)

when \( k \) is sufficiently large. Hence, (6.1) and (6.3) show that Inertial Algorithm 1 admits large time step size \( \Delta t_k \) and (6.2), (6.4) show that Galerkin Algorithm 1 admits small time step size \( \Delta t_k \), thanks to \( S_1(h) < S_1(H) \).

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References


