Quotient maps, group actions and Lusternik–Schnirelmann category

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Abstract

This note explores connections between Lusternik–Schnirelmann category, quotient maps and
group actions. Category is used to restrict group actions in general and Hamiltonian actions in
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1. Introduction

In [2], Israel Berstein gave an indication of how Lusternik–Schnirelmann category could
be used in the study of compact group actions. Here we wish to examine this subject in
some detail using a mixture of old and new approaches to category. Further, we will also
see how the general results presented here may be applied to interesting situations in, for
example, symplectic geometry.

Recall that the category of a space \( X \) is the least integer \( n \) so that \( X \) may be covered
by open sets \( U_1, \ldots, U_{n+1} \) having the property that each \( U_i \) is contractible to a point
in \( X \). The category of a map \( f : X \to Y \) is the least \( n \) so that \( X \) may be covered by open
sets \( U_1, \ldots, U_{n+1} \) having the property that each \( f|_{U_i} \) is nullhomotopic. Also, the relative
category \( \text{cat}_X(A) \) is defined to be the least integer \( n \) such that there exist \( n+1 \) open sets
in \( X, U_1, \ldots, U_{n+1} \) such that \( A \subset \bigcup U_i \) and each \( U_i \) is contractible to a point in \( X \). Any
of these types of covers \( U_1, \ldots, U_{n+1} \) will be called categorical covers.

As soon as they had defined the notion of category, Lusternik and Schnirelmann [16]
noted the relation to covering dimension, \( \text{cat} X \leq \dim X \). Recall that the order of an open
covering \( \mathcal{V} = \{ V_\alpha \} \) is the least integer \( k \) so that there exist \( k + 1 \) members of \( \mathcal{V} \) with nontrivial intersection, but not \( k + 2 \) such. The covering dimension of a paracompact space \( X \), denoted \( \dim X \), is then the least \( k \) so that any open cover has a refinement of order \( k \). Our main goal is to prove the following result.

**Theorem 1.1.** Let \( p : X \to \overline{X} \) be a quotient map which is closed and suppose that \( X \) is normal and \( \overline{X} \) is paracompact. If \( f : X \to Y \) is a map with \( \operatorname{cat}(f|_{p^{-1}(\overline{x})}) \leq n \) for each \( \overline{x} \in \overline{X} \) and \( Y \) is an ANR, then

\[
\operatorname{cat} f \leq (\dim \overline{X} + 1)(n + 1) - 1.
\]

**Remark 1.2.** If \( f \) is nullhomotopic on each fibre \( p^{-1}(\overline{x}) \), then \( n = 0 \) and the conclusion of Theorem 1.1 reduces to \( \operatorname{cat} f \leq \dim \overline{X} \). Also see the next example and Corollary 1.7 for the case \( f = 1 : X \to X \) where the condition \( \operatorname{cat}(f|_{p^{-1}(\overline{x})}) \leq n \) reduces to \( \operatorname{cat}(\overline{X}) \leq n \).

**Example 1.3.** Let \( S^1 \) act on \( S^2 \) by rotation about the \( z \)-axis. (This is the standard Hamiltonian action of \( S^1 \) on \( S^2 \).) The orbit space is \( S^2/S^1 = I \), an interval, so its dimension is one. Each orbit in \( S^2 \) (i.e., a circle or a point) is contractible in \( S^2 \), so with \( f = 1_{S^2} : S^2 \to S^2 \), we have \( n = 0 \). By Theorem 1.1 (or Corollary 1.7 below), we have

\[
1 = \operatorname{cat}(S^2) \leq (1 + 1)(0 + 1) - 1 = 1,
\]

so the inequality is actually an equality and the theorem is sharp.

This example reflects the general focus of this paper; namely, the relationship between Lusternik-Schnirelmann category and group actions. Previously, such a relationship was known for free actions (i.e., principal bundles) by a general category result for fibrations (see Corollary 4.2). A main point of this paper is then the extension of this relationship to general compact Lie group actions. A first step was taken by Berstein [2] and we shall generalize his result in Corollary 1.5 and Theorem 5.3. To do this, we note the following immediate consequence of Theorem 1.1 which will be proved in Section 3.

**Corollary 1.4.** With the hypotheses of Theorem 1.1, further suppose that \( n = 0 \) and \( Y = K(L, 1) \). If \( f : X \to K(L, 1) \) has \( H^k(f) \neq 0 \) (with any coefficients), then \( \dim \overline{X} \geq k \).

Now we can give Berstein’s result (although we use cohomology instead of homology).

**Corollary 1.5 (Berstein [2]).** If a compact connected semisimple Lie group \( G \) acts on \( X \), \( L \) is torsionfree and \( f : X \to K(L, 1) \) has \( H^k(f) \neq 0 \), then \( \dim X / G \geq k \).

**Proof.** The hypotheses of Theorem 1.1 with \( n = 0 \) are satisfied because \( L \) is torsionfree, a semisimple \( G \) has finite \( \pi_1 \) and \( f \) is classified up to homotopy by its induced \( \pi_1 \)-homomorphism. An application of Corollary 1.4 finishes the proof. \( \square \)

**Remark 1.6.** In fact, the same proof works for the non-semisimple group \( S^1 \) if we add the assumption that the orbit map \( i : S^1 \to X \) gives \( f_\# \circ i_\# = 0 \). Thus, \( f \) would...
be nullhomotopic on each $S^1$-orbit. For semisimple $G$, the finiteness of $\pi_1 G$ implies the finiteness of the fundamental group of an orbit. Hence, $f$ composed with any orbit inclusion is nullhomotopic because the torsionfreeness of $L$ implies that the induced fundamental group homomorphism is zero and, since the target is an Eilenberg–Mac Lane space, homomorphisms of $\pi_1$ determine homotopy class. For the case of $S^1$, see Theorem 5.3 below, where we prove a similar result to that of Berstein, but replacing $\dim X/G$ with the better bound $\text{cat } X/G$ when $X$ is a topological manifold.

We obtain another immediate corollary of the main theorem by taking $f$ to be the identity $1_X : X \to X$ and recalling that $\text{cat } X = \text{cat } X$.

**Corollary 1.7.** Let $p : X \to \overline{X}$ be a quotient map which is closed and suppose $X$ normal, $\overline{X}$ paracompact. If each fibre $p^{-1}(\overline{x})$ is contractible in $X$, then

$$\text{cat } X \leq \dim \overline{X}.$$ 

It follows that, if $\text{cat } X = \dim X$, then $\dim X \leq \dim \overline{X}$.

**Proof.** Let $f$ be the identity map of $X$. For the second part, note that the hypothesis simply replaces $\text{cat } X$ with $\dim X$ in the inequality. ☐

**Remark 1.8.** There is a subtlety in Theorem 1.1 which must be addressed before we proceed to proofs and applications. This subtlety is best displayed in a simple example. Consider the principal action of $S^1$ on $\mathbb{R}P^3$ and let $f = 1_{\mathbb{R}P^3}$ (as in Corollary 1.7). According to the hypotheses of Theorem 1.1 with $n = 0$ or Corollary 1.7, we require the inclusion of each $S^1$-orbit to be nullhomotopic in $\mathbb{R}P^3$. Of course this is not the case for the principal action since, on fundamental groups, we have a surjection $\mathbb{Z} \to \mathbb{Z}/2$. However, if we induce a new (non-principal) $S^1$-action on $\mathbb{R}P^3$ by composing the principal action with the double covering of $S^1$ by itself, we obtain an action such that the orbit map $S^1 \to \mathbb{R}P^3$ is nullhomotopic. At first glance, it may appear that the hypotheses of Corollary 1.7 are satisfied here, so that we would obtain

$$3 = \text{cat } \mathbb{R}P^3 \leq \dim \mathbb{R}P^3 / S^1 = 2,$$

a contradiction. But our first glance is incorrect. Here, it is not a question of whether or not the orbit map is essential or not, but rather a question of whether the orbit itself is contractible in $\mathbb{R}P^3$. But the orbits of the double cover action are exactly the same as those of the principal action—and these are not contractible in $\mathbb{R}P^3$ as we noted. We shall see this situation again when we apply Corollary 1.7 to group actions (Theorem 6.1) in Section 6. In particular, we shall prove a result (Lemma 6.2) which permits us to conclude in certain circumstances that $S^1$-orbits are contractible in the space $M$ when $\pi_1 S^1 \to \pi_1 M$ is trivial. Further, we will then show what is, perhaps, the most interesting result of the paper, Theorem 6.1.

**Theorem.** Suppose $M$ is a topological manifold with $\text{cat } M = \dim M$ and such that the center of $\pi_1 M$ is torsionfree. If a compact connected Lie group $G$ acts on $M$ continuously
and effectively, then $G$ is a torus and $\pi_1(G) \to \pi_1(M)$ is injective (so all isotropy groups are finite).

2. The proof of Theorem 1.1

The following result (in the form we give) is due to Milnor.

**Lemma 2.1.** Let $\mathcal{U} = \{U_a\}$ be an open covering of $X$ of order $n$. Then there is an open covering of $X$ refining $\mathcal{U}$, $\mathcal{G} = \{G_{i\beta}\}$, $i = 1, \ldots, n + 1$, such that $G_{i\beta} \cap G_{i\beta'} = \emptyset$ for $\beta \neq \beta'$. In particular, such a refinement may be found if $X$ is a paracompact space with covering dimension $n$ and $\mathcal{U} = \{U_a\}$ is any open covering of $X$.

**Proof.** Since the order of the covering is $n$, no $x \in X$ can belong to more than $n + 1$ of the $U_a$. Let $\{\phi_a\}$ be a locally finite partition of unity subordinate to the cover $\mathcal{U}$. That is, $\text{supp}(\phi_a) \subseteq U_a$. Now let $B_i$ denote the set of $i$-tuples obtained from the set $\{1, 2, \ldots, n + 1\}$. Given $\beta = (\alpha_1, \ldots, \alpha_i) \in B_i$, set

$$G_{i\beta} = \{x \in X \mid \text{for } j = 1, \ldots, n, \phi_{\alpha_j}(x) > 0 \text{ and for } \alpha \neq \beta, \phi_{\alpha}(x) < \phi_{\alpha_j}(x)\}.$$ 

Since, in a neighborhood of any $x \in X$, only a finite number of $\phi_{\alpha}$ are not identically zero, each $G_{i\beta}$ is open. If $\beta \neq \beta'$, then, by the second condition, $G_{i\beta} \cap G_{i\beta'} = \emptyset$. Also,

$$G_{i\beta} \subseteq \bigcap_{\alpha \in \beta} \text{supp}(\phi_{\alpha}) \subseteq \bigcap_{\alpha \in \beta} U_{a}$$

so that $\mathcal{G} = \{G_{i\beta}\}$ refines $\mathcal{U}$. Now, given $x \in X$, let $(\alpha_1, \ldots, \alpha_m)$ be all the indices such that $\phi_{\alpha_j}(x) > 0$. Then $x \in \bigcap_{i=1}^m U_{a_i}$ and the order of the cover $\mathcal{U}$ is $n$, so we must have $m \leq n$. Without loss of generality, suppose $\phi_{\alpha_1}(x) = \phi_{\alpha_2}(x) = \cdots = \phi_{\alpha_j}(x) > \phi_{\alpha_{j+1}}(x) \geq \phi_{\alpha_{j+2}}(x) \geq \cdots \geq \phi_{\alpha_m}(x)$. This then means that $x \in G_{i(\alpha_1, \ldots, \alpha_m)}$ and $\mathcal{G}$ covers $X$. $\square$

Before we can prove Theorem 1.1, we need two more lemmas. Recall that a subset $V$ as above is *saturated* if it is the inverse image under $p$ of an open set in $\overline{X}$.

**Lemma 2.2.** Let $p: X \to \overline{X}$ be a closed map. If $U \subseteq X$ is an open set with $p^{-1}(\overline{x}) \subseteq U$, then there exists a saturated open set $V$ with $p^{-1}(\overline{x}) \subseteq V \subseteq U$.

**Proof.** Consider $U^c$, the complement of $U$ in $X$. Then $U^c$ is closed as is $p(U^c)$ (since $p$ is a closed map by hypothesis). Let $\overline{V} = (p(U^c))^c$, which is open of course, and take $V = p^{-1}(\overline{V})$. To show $V \subseteq U$, let $x \in V$. Then $p(x) \in (p(U^c))^c$. But this means that $x \in U$ since any $y \in U^c$ has $p(y) \in p(U^c)$, not $(p(U^c))^c$. To show that $p^{-1}(\overline{x}) \subseteq V$, note that $\overline{x} \notin p(U^c)$ (i.e., $p^{-1}(\overline{x}) \subseteq U$), so $\overline{x} \in (p(U^c))^c = \overline{V}$. $\square$

**Remark 2.3.** The hypothesis that $p$ is closed is essential here as the example of the projection from $I \times I - \{(1, x) \mid x > 0\}$ onto the first coordinate shows.
Lemma 2.4. Suppose \( f : X \to Y \) is a map from a normal space \( X \) to an ANR \( Y \). If \( A \) is closed in \( X \) and \( \text{cat}(f|_A) \leq n \), then there exist open sets in \( X \), \( U_1, \ldots, U_{n+1} \), such that \( f|_{U_i} \) is nullhomotopic for each \( i \) and \( A \subset \bigcup_i U_i \).

Proof. The hypothesis \( \text{cat}(f|_A) \leq n \) implies that there exist \( V_1, \ldots, V_{n+1} \) open in \( A \) such that \( A \subset \bigcup_i V_i \) and \( f|_{V_i} \) is nullhomotopic for each \( i \). Now \( A \) is closed in the normal space \( X \), so it is normal as well. Therefore, the open cover \( \{V_i\} \) may be refined to an open cover \( \{W_i\} \) with the property that \( W_i \subset V_i \) for all \( i \). Note that, since \( \overline{W_i} \) is closed in \( A \) and \( A \) is closed in \( X \), then \( \overline{W_i} \) is closed in \( X \) also.

The space \( Y \) is an ANR, so we may apply the homotopy extension property to a nullhomotopy \( H : \overline{W_i} \times I \to Y \) of \( f|_{\overline{W_i}} \) which restricts the one from \( V_i \). Specifically, define a mapping \( \tilde{H} : Q = X \times 0 \cup \overline{W_i} \times I \cup X \times 1 \to Y \) by

\[
\tilde{H}(x, 0) = f(x), \quad \tilde{H}(w, t) = H(w, t), \quad \tilde{H}(x, 1) = y_0,
\]

where \( y_0 \) is a fixed point in \( Y \) and \( w \in \overline{W_i} \). Now, \( Q \) is closed in \( X \times I \) and \( Y \) is an ANR, so there is an extension \( K : U \to Y \) of \( \tilde{H} \) to \( U \), an open neighborhood of \( Q \). Now let \( U_i \) be an open neighborhood of \( \overline{W_i} \) in \( X \) such that \( U_i \times I \subseteq U \). Then \( K \) is a nullhomotopy of \( f \) restricted to the neighborhood \( U_i \) of \( \overline{W_i} \). This may be done for every \( \overline{W_i} \), so \( A \) is covered by the open sets \( U_1, \ldots, U_{n+1} \) on each of which \( f \) is nullhomotopic. \( \square \)

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let \( O_{\tilde{x}} \) denote the fibre \( p^{-1}(\tilde{x}) \). By assumption, \( \text{cat}(f|_{O_{\tilde{x}}}) \leq n \) for all fibres, so, by Lemma 2.4, \( O_{\tilde{x}} \) is covered by open sets \((\tilde{x})\), \( U_1^{\tilde{x}}, \ldots, U_{n+1}^{\tilde{x}} \) on each of which \( f \) is nullhomotopic. By Lemma 2.2, there exists for each \( \tilde{x} \) a saturated open set \( V^{\tilde{x}} \) with

\[
O_{\tilde{x}} \subset V^{\tilde{x}} \subset U^{\tilde{x}} = \bigcup_{i=1}^{n+1} U_i^{\tilde{x}}.
\]

Because \( V^{\tilde{x}} \) is saturated, \( V^{\tilde{x}} = p^{-1}(\overline{V^{\tilde{x}}}) \), where \( \tilde{x} \in \overline{V^{\tilde{x}}} \). Then \( \{\overline{V^{\tilde{x}}} \}_{\tilde{x} \in \overline{X}} \) is a covering of \( \overline{X} \). Now, \( \dim \overline{X} = k \), so any open cover of \( \overline{X} \) may be refined to an open cover of order \( k \). Let \( \{\overline{W^{\tilde{x}}} \} \) be a cover of order \( k \) refining \( \{\overline{V^{\tilde{x}}} \} \). Hence, we have

\[
p^{-1}(\overline{W^{\tilde{x}}}) \subset p^{-1}(\overline{V^{\tilde{x}}}) = V^{\tilde{x}} \subset U^{\tilde{x}}.
\]

By Lemma 2.1, we obtain a refinement of \( \{\overline{W^{\tilde{x}}} \}, \{\overline{G_i} \}, i = 1, \ldots, k + 1 \), which covers \( \overline{X} \) and has the property that each \( \overline{G_i} \) is a disjoint union, \( \bigcup_\beta \overline{G_i}_\beta \), of open sets, each of which lies in some \( \overline{W^{\tilde{x}}} \). Choose one such \( \overline{W^{\tilde{x}}} \) for each \( \overline{G_i}_\beta \).

Let \( \overline{G_i}_\beta = p^{-1}(\overline{G_i}_\beta) \) with \( \overline{G_i} = p^{-1}(\overline{G_i}) = p^{-1}(\bigcup_\beta \overline{G_i}_\beta) = \bigcup_\beta \overline{G_i}_\beta \) for \( i = 1, \ldots, k + 1 \). Suppose \( \overline{G_i}_\beta \subset \overline{W^{\tilde{x}}} \) say. Then

\[
\overline{G_i}_\beta = p^{-1}(\overline{G_i}_\beta) \subset p^{-1}(\overline{W^{\tilde{x}}}) \subset U^{\tilde{x}}.
\]
Let \( G_{i\beta j} = G_{i\beta} \cap U_j \) for \( j = 1, \ldots, n + 1 \). Then, because \( f|_{U_j} \) is nullhomotopic, so is \( f|_{G_{i\beta j}} \) for each \( i \) and \( j \). Define

\[
G_{ij} = \bigcup_\beta G_{i\beta j}.
\]

Of course, by the construction of the \( G_{i\beta} \), we have \( G_{i\beta j} \cap G_{i\beta' j} = \emptyset \) for \( \beta \neq \beta' \). Thus, \( G_{ij} \) is a disjoint union of open sets on each of which \( f \) is nullhomotopic. Therefore, \( f \) is nullhomotopic on \( G_{ij} \) as well. Because \( \{G_i\} \) covers \( X \), so does the collection \( \{G_{ij}\} \) and because \( i = 1, \ldots, k + 1, j = 1, \ldots, n + 1 \), there are \((k + 1)(n + 1)\) open sets in the collection \( \{G_{ij}\} \). The definition of category as one less than the cardinality of such a cover then completes the proof. \( \square \)

3. Detecting elements, strict category weight and orbit spaces

In this section, we recall some properties of the relatively new invariant, strict category weight, denoted swgt. The purpose of swgt is to estimate Lusternik–Schnirelmann category. Here we will also give some results needed in later sections, some easy applications to orbit spaces and the proof of Corollary 1.4. Basic references for swgt are, for example, [18–21]. Recall the following facts.

**Definition 3.1.** The strict category weight of a cohomology class \( u \in H^\ast(X; R) \) (for some coefficient ring \( R \)) is defined by

\[
swgt(u) \geq k \; \text{if and only if} \; \phi^\ast(u) = 0 \; \text{for any} \; \phi : A \to X \; \text{with} \; \text{cat} \phi < k.
\]

Strict category weight has the following properties.

- \( swgt(u) \leq \text{cat} X \) for nonzero \( u \).
- Strict category weight obeys the relation \( swgt(u^n) \geq n \cdot swgt(u) \).
- If \( g : W \to Z \) is a map and \( u \in H^\ast(Z; R) \) with \( g^\ast(u) \neq 0 \), then \( \text{cat} g \geq swgt(u) \).
- If \( u \in H^p(K(\pi, 1); R) \) is nontrivial, then \( swgt(u) = p \).

**Proof of Corollary 1.4.** Let \( u \in H^k(K(L, 1); R) \) have \( H^k(f)(u) \neq 0 \). By the last property above, \( swgt(u) = k \). Then, by the third property, \( \text{cat} f \geq k \) as well. Applying Theorem 1.1, we obtain the result. \( \square \)

We say that a manifold \( M^n \) is a \( \kappa_R \)-manifold (where \( R \) denotes a coefficient group) if there exists a map \( f : M \to K(L, 1) \) such that \( f^\ast : H^n(K(L, 1); R) \to H^n(M; R) \) is surjective. Note that, because of the factorization

\[
\begin{array}{ccc}
\pi_1 M & \xrightarrow{f^\ast} & L \\
\downarrow \pi_1 M/\text{Ker}(f^\ast) & & \\
\end{array}
\]
we may assume that \( f_L : \pi_1 M \to L \) is onto. Also note that the class of \( \kappa_R \)-manifolds includes the classes of hyperaspherical manifolds [8], \( K \)-manifolds [12] and \( \kappa \)-manifolds [14].

**Proposition 3.2.** A \( \kappa_R \)-manifold \( M^n \) has the property that \( \text{cat} M = n = \dim M \).

**Proof.** By the definition of \( \kappa_R \)-manifold, there exists a map \( f : M \to K(L, 1) \) such that \( f^* : H^n(K(L, 1); R) \to H^n(M; R) \) is surjective. Then there is a \( u \in H^n(K(L, 1); R) \) with \( f^*(u) \neq 0 \). By the fourth property above, \( \text{swgt}(u) = n \) and, by the third property,

\[
 n = \dim M \geq \text{cat} M \geq \text{cat} f \geq \text{swgt}(u) = n.
\]

Hence, \( \text{cat} M = n = \dim M \). \( \square \)

**Remark 3.3.** The inequality \( \text{cat} f \leq \text{cat} M \) is true by general facts about category. Also note that the proof of Proposition 3.2 then shows that \( \text{cat} f = n \) if \( M^n \) is a \( \kappa_R \)-manifold with map \( f : M \to K(L, 1) \).

**Definition 3.4.** A class \( u \in H^*(Y; R) \) is called a detecting element for \( Y \) if \( \text{swgt}(u) = \text{cat} Y \).

**Example 3.5.** Suppose \((M^{2n}, \omega)\) is a closed symplectic manifold such that \( \omega \in H^2(M; \mathbb{R}) \), thought of as an element of \( \text{Hom}(H_2(M), \mathbb{R}) \), vanishes on the image of the Hurewicz homomorphism \( h : \pi_2(M) \to H_2(M) \). These manifolds are called symplectically aspherical with non-trivial \( \pi_2 \) (see [11]). Recall by [14] (also see [19]) that this condition on \( \omega \) means that \( f : M \to K(\pi_1(M), 1) \) has \( f^*(\tilde{\omega}) = \omega \) for some \( \tilde{\omega} \in H^2(K(\pi_1(M), 1); \mathbb{R}) \). Because \( \omega \) is a symplectic class, \( \omega^n \neq 0 \). Hence, \( f^*(\omega^n) \neq 0 \) and \( \text{cat} f \geq \text{swgt}(\omega^n) = n \cdot \text{swgt}(\tilde{\omega}) = 2n \) since \( \tilde{\omega} \in H^2(K(\pi_1(M), 1); \mathbb{R}) \). Then \( 2n = \dim M \geq \text{cat} M \geq \text{cat} f \geq 2n \) implies that \( \text{cat} M = \text{swgt}(\omega^n) \), so \( M \) has a detecting element. Of course, this description also shows that \( M \) is a \( \kappa_Q \)-manifold, so Proposition 3.2 also applies.

**Lemma 3.6.** Given \( f : X \to Y \) and \( u \) a detecting element for \( Y \) with \( f^*(u) \neq 0 \), we have \( \text{cat} X \geq \text{cat} Y \).

**Proof.** By the properties listed above, we have \( \text{cat} X \geq \text{swgt}(f^*(u)) \geq \text{swgt}(u) = \text{cat} Y \). \( \square \)

**Proposition 3.7** ([18]). Suppose \( X^n \) and \( Y^n \) are closed manifolds, \( f : X \to Y \) induces an isomorphism on \( H^n(\_; R) \), \( u \) is a detecting element for \( Y \) and \( \text{cat} Y = \dim Y \). Then \( \text{cat} X = \text{cat} Y \).

**Proof.** Since \( f^* \) is injective on cohomology, by Lemma 3.6, \( \text{cat} X \geq \text{cat} Y \). But we also have \( \text{cat} Y = \dim Y = \dim X \geq \text{cat} X \) so we have equality. \( \square \)
In Section 6, we would like to do without the added assumption of having a detecting element. Fortunately, the following result of Rudyak [18] provides a way to do this.

**Theorem 3.8.** Let \( q \geq 1 \) and let \( M^n, n \geq 4, \) be a closed orientable \((q - 1)\)-connected PL-manifold with \( q \text{cat } M = n = \dim M \). Then \( M \) possesses a detecting element.

We may use Rudyak’s result to construct interesting examples of manifolds satisfying \( \text{cat } M = \dim M \). These examples will prove relevant in Section 6. As basic building blocks we may take any manifolds known to satisfy the conditions of Theorem 3.8 such as aspherical (or hyperaspherical) manifolds or symplectically aspherical manifolds with nontrivial \( \pi_2 \) say.

**Corollary 3.9.** Suppose \( M^n \) satisfies the hypotheses of Theorem 3.8 with \( q = 1 \). Then, for any closed manifold \( N^n \),

\[
\text{cat}(M \# N) = n = \dim(M \# N).
\]

**Proof.** We know by Theorem 3.8 that \( \text{cat } M = \dim M \) and \( M \) has a detecting element. Also, the composition

\[
M \# N \to M \lor N \to M
\]

gives an isomorphism on \( H^n(-; \mathbb{R}) \), where the first map collapses the \( S^{n-1} \) where \( M \) and \( N \) are joined and the second map folds the wedge into one summand. By Proposition 3.7, we are done. \( \square \)

In a somewhat different direction, now consider the projection \( p : X \to X/G \) for a compact Lie group \( G \) acting on \( X \). If \( u \) is a detecting element for \( X/G \) such that \( p^*(u) \neq 0 \), then \( \text{cat } X \geq \text{cat } X/G \). If the action has orbits which are contractible in \( X \), then we may take \( f = 1_X \) as in Corollary 1.7 and get \( \text{cat } X \leq \dim X/G \). Hence,

**Proposition 3.10.** Suppose that a compact Lie group \( G \) acts on \( X \) such that orbits are contractible. If \( X/G \) possesses a detecting element \( u \) such that the projection \( p : X \to X/G \) has \( p^*(u) \neq 0 \), then \( \text{cat } X/G \leq \text{cat } X \leq \dim X/G \).

**Example 3.11.** Consider a free action of a finite group \( G \) on \( S^{2n+1} \) with quotient \( S^{2n+1}/G \). It is known that the classifying map \( f : S^{2n+1}/G \to BG = K(G, 1) \) induces a surjection in cohomology with \( \mathbb{Z}/|G| \) coefficients and, by the properties of strict category weight above, that each non-zero element \( u \in H^{2n+1}(BG) \) has \( \text{swgt}(u) = 2n + 1 \) and

\[
2n + 1 = \dim S^{2n+1}/G \geq \text{cat } S^{2n+1}/G \geq \text{cat } f \geq \text{swgt}(u) = 2n + 1.
\]
So, \( S^{2n+1}/G \) has a detecting element, but \( \text{cat } S^{2n+1}/G > \text{cat } S^{2n+1} \). By the proposition, we must have \( p^*(u) = 0 \) for all elements (as is known). Also note that \( S^{2n+1}/G \) is a \( \kappa_{\mathbb{Z}/|G|} \)-manifold as well.
Example 3.12. Consider $M = T^k = T^{k-1} \times S^1$ with $S^1$ action translation on the last factor. Then $M/S^1 = T^{k-1}$, so $\dim M/S^1 = k - 1 < k = \text{cat } M$. This type of example shows that the condition on the orbits is essential to the result.

Example 3.13. We can extend Example 3.11 to connected groups by considering the higher dimensional Hopf action of $S^1$ on $S^{2n+1}$ with orbit space $\mathbb{C}P^n$. The latter is a simply connected symplectic (even Kähler) manifold, so $\text{cat } \mathbb{C}P^n = n > 1 = \text{cat } S^{2n+1}$. For $M = S^{2n+1}$ with the Hopf $S^1$-action then, $\text{cat } M < \text{cat } M/S^1$. Of course the element $\omega^n$ (i.e., the Kähler class) is a detecting element for $\mathbb{C}P^n$, but it does not pull back nontrivially to $S^{2n+1}$.

4. The fibration case

In the situation where we have a fibration instead of a quotient map, an analogous result to Theorem 1.1 replacing $\dim$ with $\text{cat}$ may be proved more easily using the homotopy lifting property.

Theorem 4.1. Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration and $f: E \to Y$ is a map from a normal space $E$ to an ANR $Y$. Then

$$
\text{cat } f \leq (\text{cat } B + 1)(\text{cat } (f \circ i) + 1) - 1.
$$

Proof. Let $\text{cat } (f \circ i) = \text{cat } (f\vert_F) = n$ with categorical cover $V_1, \ldots, V_{n+1}$ and let $\text{cat } (B) = m$ with categorical cover $U_1, \ldots, U_{m+1}$. Consider the diagram obtained from the homotopy lifting property,

$\begin{tikzcd}
F \arrow{d}{p} \arrow{r}{f} & Y \\
F \arrow{d}{G} \arrow{r}{p^{-1}} & B \\
p^{-1}(U_j) \arrow{r}{\partial} & U_j \times I
\end{tikzcd}$

where $H_0 = p$, $H_1 = *$ (since $U_j$ is contractible in $B$ via the homotopy $\overline{H}$ and $H = \overline{H} \circ (p \times 1)$). Also, since $\{U_j\}$ covers $B$, $\{p^{-1}(U_j)\}$ covers $E$. Because $H_1 = *$ and $p \circ G_1 = H_1 = *$, then $G_1(p^{-1}(U_j)) \subseteq F$. Let $W_{ij} = G_1^{-1}(V_i) \subseteq p^{-1}(U_j)$ and note that $W_{1j}, \ldots, W_{(m+1)j}$ covers $p^{-1}(U_j)$ since the image of $G_1$ is in $F$. We can explicitly show that each $f\vert_{W_{ij}}$ is nullhomotopic by defining (for $\hat{u} \in W_{ij}$), $K : W_{ij} \times I \to Y$,

$$
K(\hat{u}, t) = \begin{cases} 
  f(G_1(\hat{u}, 2t)), & 0 \leq t \leq \frac{1}{2}, \\
  L(G_1(\hat{u}), 2t - 1), & \frac{1}{2} \leq t \leq 1,
\end{cases}
$$

where $H_0 = p$, $H_1 = *$. Also, since $\{U_j\}$ covers $B$, $\{p^{-1}(U_j)\}$ covers $E$. Because $H_1 = *$ and $p \circ G_1 = H_1 = *$, then $G_1(p^{-1}(U_j)) \subseteq F$. Let $W_{ij} = G_1^{-1}(V_i) \subseteq p^{-1}(U_j)$ and note that $W_{1j}, \ldots, W_{(m+1)j}$ covers $p^{-1}(U_j)$ since the image of $G_1$ is in $F$. We can explicitly show that each $f\vert_{W_{ij}}$ is nullhomotopic by defining (for $\hat{u} \in W_{ij}$), $K : W_{ij} \times I \to Y$, $K(\hat{u}, t) = \begin{cases} 
  f(G_1(\hat{u}, 2t)), & 0 \leq t \leq \frac{1}{2}, \\
  L(G_1(\hat{u}), 2t - 1), & \frac{1}{2} \leq t \leq 1,
\end{cases}$
where \( L : V_i \times I \to Y \) is a nullhomotopy from \( f|_{V_i} \); \( L(x, 0) = f(i(x)) \) and \( L(x, 1) = * \).

Now do this for each \( j = 1, \ldots, n + 1 \) to obtain a categorical cover of \( E \), \( \{ W_i j : i = 1, \ldots, m + 1, j = 1, \ldots, n + 1 \} \) having \((n + 1)(m + 1)\) members. Hence, \( \text{cat}(f) \leq (n + 1)(m + 1) - 1 \). \( \square \)

Cornea has proved that \( \text{cat}(E/F) \leq \text{cat}(B) \) for \( B \) normal with non-degenerate basepoint. Theorem 4.1 then follows from this result using Lemma 5.1 below as well as by the direct proof given above. If we specialize the proof to the case \( f = 1_E : E \to E \) and recall that \( \text{cat}1_E = \text{cat}E \), we obtain the following well-known result.

**Corollary 4.2.** Let \( F \to E \to B \) be a fibration. Then
\[
\text{cat}(E) \leq (\text{cat}(B) + 1)(\text{cat}(F) + 1) - 1.
\]

**Example 4.3.** Take the Hopf fibration \( S^3 \to S^7 \to S^4 \) and factor by the action of \( S^1 \) on the fibre and total space. We obtain the fibration \( S^2 \to \mathbb{C}P^3 \to S^4 \) with \( \text{cat}(S^2) = 1 = \text{cat}(S^4) \) and \( \text{cat}(\mathbb{C}P^3) = 3 \). Putting these values in the inequality of Corollary 4.1 gives
\[
3 = \text{cat}(\mathbb{C}P^3) \leq (\text{cat}(S^2) + 1)(\text{cat}(S^4) + 1) - 1 = 3
\]
so we see that the inequality of Corollary 4.2 is sharp.

**Corollary 4.4.** If \( \text{cat}_E(F) = 0 \), then \( \text{cat}E \leq \text{cat}B \). In particular, if \( \tilde{X} \) is a covering space of \( X \), then \( \text{cat} \tilde{X} \leq \text{cat}X \).

**Example 4.5.** Interesting cases of Theorem 4.1 arise when \( f \circ i \) is nullhomotopic (i.e., \( \text{cat}(f \circ i) = 0 \)). If the fibration \( F \to E \to B \) arises as a pullback,

\[
\begin{array}{ccc}
F & \xrightarrow{f} & X \\
\downarrow i & & \downarrow q \\
E & \xrightarrow{p} & Y \\
\downarrow g & & \downarrow r \\
B & \xrightarrow{s} & Z
\end{array}
\]

where \( j \) is nullhomotopic, then commutativity of the diagram implies \( f \circ i \) is nullhomotopic as well. Fibrations with fibre inclusion nullhomotopic are well known. For example, the complex Stiefel fibrations
\[
U(k) \to V_{k,n}(\mathbb{C}) \to G_{k,n}(\mathbb{C})
\]
have null fibre inclusions when \( 2k \leq n \). (Here, \( V_{k,n}(\mathbb{C}) \) denotes the space of \( k \)-frames in \( \mathbb{C}^n \).)

**Corollary 4.6.** With the hypotheses of Theorem 4.1, suppose further that \( n = 0 \) and \( Y = K(L, 1) \). If \( H^k(f) \neq 0 \) with any coefficients, then \( \text{cat} B \geq k \).
**Proof.** This follows as in the proof of Corollary 1.4. □

A result such as Corollary 4.6 may be applied most effectively to situations where the map $f$ can be shown to have large category. (See Section 3 for the definition of $\kappa_R$-manifold.)

**Theorem 4.7.** A $\kappa_R$-manifold $M^n$ with map $f : M \to K(L, 1)$ cannot be the total space of a nontrivial fibration $F \to M \to B$ with finite dimensional base $B$, positive dimensional connected fibre $F$ and $f \circ i$ nullhomotopic.

**Proof.** Let $F \to M \to B$ be a fibration with $f \circ i \simeq \ast$. Then, by Corollary 4.2, we have (using Remark 3.3),

$$\dim B \geq \text{cat } B \geq \text{cat } f = n = \dim M.$$  

But this is impossible unless $\dim F = 0$. □

Because symplectically aspherical manifolds with nontrivial $\pi_2$ are $\kappa_Q$-manifolds (see Example 3.5), we have the following.

**Corollary 4.8.** A symplectic manifold $(M, \omega)$ with $\omega|_{\pi_2 M} = 0$ can never be the total space of a nontrivial fibration with simply connected fibre.

Note that the simple connectivity of the fibre ensures that $f \circ i$ is nullhomotopic, where $f : M \to K(\pi_1 M, 1)$ classifies the fundamental group of $M$. We can also apply Theorem 4.7 to the following problem in symplectic geometry.

**Problem 4.9** (Contractible Orbit Problem [17]). Is it possible to have a free $S^1$ action on a closed symplectic manifold with orbits contractible in the manifold?

A free $S^1$-action with contractible orbits (thus, having $f \circ i$ nullhomotopic for all $f$) would give a principal bundle to which Theorem 4.7 may be applied. Thus we obtain

**Corollary 4.10.** The contractible orbit problem has a negative solution for a symplectic manifold $(M, \omega)$ with $\omega|_{\pi_2 M} = 0$.

5. **Category of orbit spaces**

Throughout this section, except where noted explicitly, we will consider continuous (effective) actions of compact Lie groups on closed oriented topological manifolds. Our goal here is to focus on understanding the relationship between category and group actions in the particular case of a manifold with a group action and a map into an Eilenberg–MacLane space. We will see that, in this case, we may refine our earlier results, replacing the dimension bound with, in general, the much better category bound.
Specifically, let $M$ denote such a manifold and $G$ such a Lie group acting continuously on $M$ with orbit map $i: G \to M, \ g \mapsto g \cdot m$ for fixed $m \in M$. Also, let $f: M \to K(L, 1)$ denote a mapping from $M$ to an Eilenberg–MacLane space. In [8], the following results are proved:

I. The orbit space $M/G$ has the homotopy type of a finite CW complex.

II. Then $\ker p_\# \subseteq \ker f_\#$ if and only if $f$ factors as $M \xrightarrow{f} p \xrightarrow{p_\#} K(L, 1) \xrightarrow{f_\#} M/G$ where $p_\#$ and $f_\#$ are the induced homomorphisms of fundamental groups.

III. The fundamental group of the orbit space is identified as

$$\pi_1(M/G) \cong \left[\pi_1(M)/i_#(\pi_1G)\right]/H,$$

where $H \subseteq [\pi_1(M)/i_#(\pi_1G)]$ is a subgroup generated by elements of finite order.

Moreover, we will use modifications of certain arguments in [8] to obtain some of the results that follow. First, however, we need a general fact about category.

**Lemma 5.1.** If $g: X \to Y$ and $h: Y \to Z$ are maps of spaces, then $\text{cat}(h \circ g) \leq \min\{\text{cat} g, \text{cat} Y\}$.

**Proof.** The inequality is clearly true for $\text{cat} g$ because, if $g$ is nullhomotopic on an open set $U$, then $h \circ g$ is null as well. For $\text{cat} h$, let $U_1, \ldots, U_{n+1}$ be a categorical cover of $Y$ for $h$; that is $h$ is null on each $U_i$ by a homotopy $H: U_i \times I \to Z$. Let $V_i = g^{-1}(U_i)$ and define $\tilde{H}: V_i \times I \to Z$ by

$$\tilde{H}(v, t) = h(H(g(v), t)).$$

Then $V_1, \ldots, V_{n+1}$ is a categorical cover for $h \circ g$. □

**Remark 5.2.** Note that it is always true for any map $s: X \to Y$ that $\text{cat} s \leq \min\{\text{cat} X, \, \text{cat} Y\}$. Hence, for the maps above, $\text{cat} h \leq \min\{\text{cat} Y, \, \text{cat} Z\}$ and $\text{cat}(h \circ g) \leq \min\{\text{cat} X, \, \text{cat} Y, \, \text{cat} Z\}$. In our later results, $Z = K(L, 1)$ and it is known that $\text{cat} K(L, 1) = \dim K(L, 1)$. Because this is a rather bad bound for the category of the maps involved, we omit $\text{cat} K(L, 1)$ from the list of numbers to be minimized.

**Theorem 5.3.** With the notation above, suppose $L$ is torsionfree. If $G$ is either semisimple or $G = S^1$ and $f \circ i$ is nullhomotopic, then $\text{cat} f \leq \min\{\text{cat} M, \, \text{cat} M/G\}$. In particular, if $f$ has $H^k(f) \neq 0$, then $k \leq \min\{\text{cat} M, \, \text{cat} M/G\}$.
Proof. By II above, \( f' : M/G \to K(L, 1) \) exists with \( f'p \simeq f \) if and only if \( \text{Ker } p_\# \subseteq \text{Ker } f_\# \). Now, the hypotheses ensure that \( f_\# \circ i_\# = 0 \) (as in the proof of Corollary 1.5), so we have a factorization

\[
\begin{array}{ccc}
\pi_1 M & \xrightarrow{f_\#} & L \\
& \downarrow{\phi} & \\
\pi_1 M/\text{i}\#(\pi_1 G) & & \\
\end{array}
\]

But \( L \) is torsionfree and \( H \), as in III above, is generated by elements of finite order, so we obtain a factorization

\[
\begin{array}{ccc}
\pi_1 M & \xrightarrow{f_\#} & L \\
& \downarrow{\phi} & \\
\pi_1 M/\text{i}\#(\pi_1 G) & \xrightarrow{[\pi_1 M/\text{i}\#(\pi_1 G)]} & H \\
\end{array}
\]

By III, the composition along the left and bottom of the square is exactly \( p_\# : \pi_1 M \to \pi_1(M/G) \), so commutativity of the diagram shows \( \text{Ker } p_\# \subseteq \text{Ker } f_\# \). Hence, we have

\[
\begin{array}{ccc}
M & \xrightarrow{f} & K(L, 1) \\
& \downarrow{p'} & \\
M/G & \xrightarrow{f'} & \\
\end{array}
\]

and, by Lemma 5.1, \( \text{cat } f \leq \min\{\text{cat } f', \text{ cat } p\} \leq \min\{\text{cat } M, \text{ cat } M/G\} \).

For the second part, suppose \( \min\{\text{cat } M, \text{ cat } M/G\} < k \). This implies that \( \text{cat } f < k \) by what we have shown above and the same argument as in the proof of Corollary 1.4 gives \( H^k(f) = 0 \), contradicting the hypotheses. Hence, we have \( k \leq \min\{\text{cat } M, \text{ cat } M/G\} \) as desired. \( \square \)

Remark 5.4. In [2], Berstein showed that \( \text{dim } M/G \geq k \) under the hypotheses above. Therefore, since it is generally true that \( \text{cat } M \leq \text{dim } M \), Theorem 5.3 is an improvement on his result.

Example 5.5. Suppose \( M \) is symplectically aspherical as in Example 3.5 and has \( \pi_1 M \) torsionfree. Further suppose that either \( S^1 \) acts on \( M \) with orbits contractible in \( M \) or \( G \) is semisimple. Then we may apply Theorem 5.3 to get

\[
2n = \text{cat } f \leq \text{cat } M/G \leq \text{dim } M/G < \text{dim } M = 2n
\]

which is a contradiction. Hence, such actions—even when they are not free actions—cannot exist. The \( S^1 \) part of this example extends Corollary 4.10 to the non-free case. Compare to the analogous results in [14]. Theorem 6.1 makes clear exactly why such actions cannot occur—even in the non-free case—for symplectically aspherical manifolds with nontrivial \( \pi_2 \) (and torsionfree \( \pi_1 \)).

Before we leave this section, we would like to note that \( \text{cat } M/G \) is not an easy number to estimate in general. For instance, \( \mathbb{R}P^{2n+1} = S^{2n+1}/(\mathbb{Z}/2) \) has category \( 2n + 1 \) while
$S^{2n+1}$ has category one. More generally, in [10] it is shown that if $M = \prod_k S^{2n+1}$ is acted on freely and homologically trivially (with $\mathbb{Z}/p$-coefficients) by $G = (\mathbb{Z}/p)^k$, then $\text{cat} M/G = \text{dim} M/G$. (It is also shown in [10] that a similar result holds for complex Stiefel manifolds acted on by subgroups of the appropriate unitary group.) Since $\text{cat} M = k$ and $\text{dim} M/G = k(2n + 1)$, we see that category of orbit spaces by finite groups can increase arbitrarily. On the other hand, reflection through a plane transforms $S^n$ into a disk $D^n = S^n/\langle \mathbb{Z}/2 \rangle$, so category does not always increase for orbit spaces. There is a case, however, where we know exactly what category does. This will lead us into the main result of the next section as well.

**Theorem 5.6.** If a manifold $M$ satisfies $\text{cat} M = \text{dim} M$ and $G$ is a finite group acting freely on $M$, then $\text{cat} M/G = \text{dim} M/G$.

**Proof.** Since $G$ is a finite group, the fibres of $M \to M/G$ are contractible in $M$. Using Corollary 4.4, we obtain $\text{cat} M \leq \text{cat} M/G \leq \text{dim} M/G = \text{dim} M$. But the hypothesis $\text{cat} M = \text{dim} M$ then implies that all the inequalities are in fact equalities. \(\square\)

6. Restricting group actions by category

There is a long history of studying how the topology of a space restricts its geometry; in particular, the types of symmetries it supports. It was shown in [7] that aspherical manifolds support only certain types of toral symmetries. Since an aspherical manifold $M$ satisfies the conditions $\text{cat} M = \text{dim} M$ and $\pi_1 M$ is torsionfree, it is natural to ask whether these topological conditions are enough to restrict the types of group actions a space may have. The following result answers this question.

**Theorem 6.1.** Suppose $M$ is a topological manifold with $\text{cat} M = \text{dim} M$ and such that the center of $\pi_1 M$ is torsionfree. If a compact connected Lie group $G$ acts on $M$ continuously and effectively, then $G$ is a torus and $\pi_1 (G) \to \pi_1 (M)$ is injective (so all isotropy groups are finite).

In order to prove Theorem 6.1, we argue as in [7] once we replace various covering action and fixed point arguments with Corollary 1.7. The following lemma is then essential and may be thought of as a counterpart to [7, Lemma 5.1]. Also, note that Remark 1.8 shows that some condition on $\pi_1 M$ is necessary in the result.

**Lemma 6.2.** Suppose $M$ is a topological manifold such that the center of $\pi_1 M$ is torsionfree. If $S^1$ acts effectively on $M$ with orbit map $i: S^1 \to M$, then $i_*: \pi_1 S^1 \to \pi_1 M$ is trivial if and only if any orbit is contractible in $M$.

**Proof.** Of course one implication is trivial. For the other, suppose that $i_*: \pi_1 S^1 \to \pi_1 M$ is trivial. Let $O$ denote an orbit of the action and let $j: O \hookrightarrow M$ denote the inclusion map of the orbit into $M$. We wish to show that $j$ is nullhomotopic. Now, if $O$ is not a
point (in which case it is certainly contractible), \( O \) is homeomorphic to \( S^1 \) since (isotropy) subgroups in \( S^1 \) are finite cyclic groups. Then, in order for \( j \) to be nullhomotopic, it is only necessary to show that \( j_*: \pi_1 O \to \pi_1 M \) is zero.

Recall that it is a general fact that orbit maps induce fundamental group homomorphisms which have images in the centers of the fundamental groups of the spaces acted upon. Therefore, letting \( Z\pi_1 M \) denote the center of \( \pi_1 M \), we have the diagram

\[
\begin{array}{c}
Z = \pi_1 S^1 \\
\downarrow q_\# \\
Z\pi_1 M
\end{array}
\]

The projection \( q:S^1 \to O \) induces \( q_\#: Z \to n\mathbb{Z} \subseteq \mathbb{Z} \) on fundamental groups for some integer \( n > 0 \). Then, since \( i_\# = 0 \), we must have \( j_*(n\mathbb{Z}) = 0 \) as well. Thus,

\[
\text{Image}(j_*) \subseteq \text{Image}(Z/n\mathbb{Z}) = 0
\]

because \( Z\pi_1 M \) has no torsion. Therefore, \( j_* = 0 \) and \( O \) is contractible in \( M \).

Proof of Theorem 6.1. First suppose that a torus \( T \) is acting on \( M \) with \( \pi_1 (T) \to \pi_1 (M) \) not injective. Now, if \( u \) is an element of the kernel of this homomorphism, then there is a homomorphism \( S^1 \to T \) representing \( u \). The restricted action of \( S^1 \) on \( M \) then has contractible orbits in \( M \) by Lemma 6.2 and we may apply Corollary 1.7 to the projection \( M \to M/S^1 \) to obtain

\[
\dim M = \text{cat} M \leq \dim M/S^1 < \dim M,
\]

a contradiction. Hence Ker\((\pi_1 (T) \to \pi_1 (M)) = 0 \).

Now let \( G \) be a compact connected Lie group acting on \( M \) and let \( T \subseteq G \) be a maximal torus. We have a diagram

\[
\begin{array}{c}
H^1 G = \pi_1 G \\
\downarrow \\
H^1 T = \pi_1 T
\end{array}
\]

where the injection \( \pi_1 (T) \to \pi_1 (M) \) follows by the argument above. But, this injection implies that \( H^1 T \to H^1 G \) is injective as well. We then have an injection \( H^1 (T; \mathbb{Q}) \to H^1 (G; \mathbb{Q}) \) and a dual surjection \( H^1 (G; \mathbb{Q}) \to H^1 (T; \mathbb{Q}) \). Because \( \text{H}^\ast (T; \mathbb{Q}) \) is generated by \( \text{H}^1 (T; \mathbb{Q}) \), we obtain a surjection \( \text{H}^\ast (G; \mathbb{Q}) \to \text{H}^\ast (T; \mathbb{Q}) \). Moreover, a vector space splitting \( \text{H}^1 (T; \mathbb{Q}) \to \text{H}^1 (G; \mathbb{Q}) \) induces an algebra splitting \( \text{H}^\ast (T; \mathbb{Q}) \to \text{H}^\ast (G; \mathbb{Q}) \) since \( \text{H}^1 (T; \mathbb{Q}) \) freely generates \( \text{H}^\ast (T; \mathbb{Q}) \). Then we see that the exterior algebra on \( \text{rank}(G) = \dim T \) generators, \( \text{H}^\ast (G; \mathbb{Q}) \) contains a sub-exterior algebra on \( \text{rank}(G) = \dim T \) generators, \( \text{H}^\ast (T; \mathbb{Q}) \). This can only happen if \( \text{H}^\ast (G; \mathbb{Q}) = \text{H}^\ast (T; \mathbb{Q}) \). Hence, \( G = T \).

Examples 6.3. Classes of manifolds on which only tori can act effectively (as in Theorem 6.1) have been studied (see [15] for a comprehensive listing). With the exception
of one class where we do not know the answer, all of these manifolds satisfy \( \text{cat} M = \dim M \). Very specific properties of these manifolds are used to show the non-existence of non-toral actions, so, in this light, our condition on the center of the fundamental group is akin to these. In order to place Theorem 6.1 in perspective, we discuss some of these types of manifolds now.

1. Aspherical manifolds are known to have \( \text{cat} M = \dim M \) by [9]. In [7], the conclusion of Theorem 6.1 was also proved using properties of covering actions and cohomology properties of fixed sets. Note that aspherical manifolds have torsionfree fundamental groups, so Theorem 6.1 is a generalization of the results of [7].

2. In [8], the conclusion of Theorem 6.1 was shown for hyperaspherical manifolds. By definition, such a manifold has a map of degree one \( f : M^n \to N^n = K(\pi, 1) \) to an aspherical manifold \( N \) of the same dimension \( n \). Hyperaspherical manifolds are a subclass of the class of \( \kappa \text{R} \)-manifolds (see Proposition 3.2) and so \( \text{cat} M = \dim M \). (Also see [12] for a generalization of a hyperaspherical manifold called a \( K \)-manifold where similar results are proved.) The argument in [8] may be interpreted to have Theorem 5.3 built in. That is, suppose \( f_\# \circ i_\# = 0 \), where \( i : S^1 \to M \) is the orbit map. Then by Remark 3.3 and Theorem 5.3,

\[
\text{cat} f = n \leq \text{cat} M/S^1 \leq \dim M/S^1 < \dim M = n,
\]

a contradiction. As in the proof of Theorem 6.1, this is really all that is required to derive the conclusion that tori are the only groups which act on such manifolds.

3. Let \( M^n, n \geq 4 \) be a closed orientable aspherical PL-manifold and let \( N^n \) be any closed simply connected manifold. By Corollary 3.9, \( \text{cat}(M \# N) = \dim(M \# N) \) and \( \pi_1 M \) is torsionfree (since \( M \) is an aspherical manifold). Hence, these manifolds satisfy the hypotheses of Theorem 6.1 and, so, the conclusions as well.

4. From Example 3.5 and Proposition 3.2, we see that a symplectic manifold \( (M^{2n}, \omega) \) with \( \omega|_{\pi_2 M} = 0 \) has \( \text{cat} M = \dim M \). If \( Z \pi_1 M \) is torsionfree besides, then Theorem 6.1 applies.

5. In [6] (the topological case) and [23] (the smooth case), it was shown that, if \( \dim M = n \) and the integral or rational cuplength of \( M \) is \( n \), then the only compact connected Lie groups which can act effectively on \( M \) are tori and, furthermore, all isotropy groups of any such action must be finite. But it is a general fact that

\[
\text{cup} M \leq \text{cat} M \leq \dim M,
\]

so the assumption on the cuplength is really an assumption on the category of \( M \); namely that \( \text{cat} M = \dim M \). Also, if the integral cuplength condition is satisfied, then the one-dimensional cohomology classes may be used to show that \( M \) is hyperaspherical with \( N = T^n \), a torus. Even in the rational cuplength case, we may obtain a map from \( M \) into a product of \( K(\mathbb{Q}, 1) \)'s, again with torsionfree fundamental group and nontrivial map on \( \dim M \)-cohomology.

6. In [3], it was shown that, for a closed manifold \( M^n \) with \( \pi_1 M = \mathbb{Z}/2 \), \( \text{cat} M = \dim M \) if and only if \( \iota^n \neq 0 \), where \( \iota \in H^1(M; \mathbb{Z}/2) \) is a generator. Therefore, such manifolds satisfy \( \text{cat} M = \dim M \), but we cannot apply Theorem 6.1 as the example of \( \mathbb{R}P^3 \).
shows. Specifically, $\mathbb{R}P^3 = \text{SO}(3)$ acts freely on itself by multiplication, so has a non-toral action.

7. In [15], it is shown that admissible manifolds satisfy the conclusions of Theorem 6.1. An admissible manifold $M$ is defined by the property that the only periodic self-homeomorphisms of the universal cover $\tilde{M}$ which commute with $\pi_1 M$ (thought of as covering transformations) are elements of the center $Z \pi_1 M$. We do not know if all admissible manifolds obey $\text{cat} M = \text{dim} M$.

We also have the following generalization of a well-known result (see [8], for example).

**Corollary 6.4.** Let $M^n$, $n \geq 4$, be a closed orientable PL-manifold with $\text{cat} M = \text{dim} M$ and let $N^n$ be any closed connected manifold with $\pi_1 N \neq 0$. Then there are no effective actions of positive dimensional Lie groups on $M \# N$.

**Proof.** Suppose $G$ is a positive dimensional Lie group acting effectively on $M \# N$. By Corollary 3.9, $\text{cat}(M \# N) = n = \text{dim}(M \# N)$. Also, $\pi_1 (M \# N) = \pi_1 M \ast \pi_1 N$ and this group has no center since it is a free product. Thus the hypotheses and therefore the conclusions of Theorem 6.1 hold, but because the map $\pi_1 G \to \pi_1 (M \# N)$ factors through the trivial center of $\pi_1 (M \# N)$, this contradicts its injectivity. Hence, no such $G$ act. \( \square \)

It is clear then that Theorem 6.1 fits into the body of results which pertain to restricting the types of Lie groups which can act on certain manifolds. We can say more, however, in the case of a smooth action on a symplectic manifold. First note that, for a smooth action of a compact Lie group $G$ on a manifold $M$,

$$\dim M/G = \dim M - \dim P,$$

where $P$ is a principal orbit [5, Theorem 3.8]. Then we have

**Proposition 6.5.** Suppose a compact Lie group $G$ acts smoothly on a manifold $M$ so that each orbit is contractible in $M$. Then, if $P$ denotes a principal orbit of the action,

$$\dim P \leq \dim M - \text{cat} M.$$

In particular, if there is an almost free orbit (i.e., finite isotropy), then

$$\dim G \leq \dim M - \text{cat} M.$$

**Proof.** By Corollary 1.7, $\text{cat} M \leq \dim M/G$, so combine this with $\dim M/G = \dim M - \dim P$ to obtain the result. For the second part, we note that an almost free orbit $P$ has $\dim P = \dim G$. \( \square \)

**Remark 6.6.** In fact, by using standard results on the degree of symmetry of a manifold and an argument from [8] (Theorem 3.5 there), it is possible to show that, no matter the orbit structure,

$$\dim G \leq \frac{(\dim M - \text{cat} M)(\dim M - \text{cat} M + 1)}{2},$$
when orbits are contractible in \( M \). On the other hand, if we are in the situation of Corollary 1.5 say, then we may replace cat \( M \) by \( k \) as noted by Berstein. These cases arise from taking specific instances of the map \( f \) in Theorem 1.1; namely, \( f = 1_M : M \to M \) and \( f : M \to K(L, 1) \) with \( H^k(f) \neq 0 \). The problem is then to find other instances of \( f : M \to N \) which are nullhomotopic on orbits in \( M \) and which allow computable approximations of cat \( f \).

**Corollary 6.7.** Suppose a torus \( T^k \) acts smoothly and effectively on \( M \) such that all \( T^k \)-orbits are contractible in \( M \). Then

\[
k \leq \dim M - \text{cat } M.
\]

**Proof.** Any effective toral action has a free orbit. This may be seen by considering a principal isotropy group \( K \). By the Principal Orbit Theorem (see [5, Theorem IV.3.1] or the discussion in [1, pp. 19–21]), \( K \) fixes an open dense set. Thus it fixes all of \( M \) and this contradicts effectiveness of the action unless \( K = \{1\} \). Hence, a principal orbit has dimension \( k \) and the result follows by Proposition 6.5.

**Corollary 6.8.** With the hypotheses of Corollary 6.7 there are the following consequences:

(i) If \( T^k \) acts on \( M^{2n} \) and there is a class \( \omega \in H^2(M; R) \) (for any coefficients \( R \)) with \( \omega^n \neq 0 \), then \( 2k \leq \dim M \).

(ii) If \( \text{cat } M = \dim M \), then there are no positive dimensional toral actions with contractible orbits on \( M \).

**Proof.** For (i), a well known cuplength argument (applied to the cohomology class \( \omega \)) gives \( 2 \text{cup } M = \dim M \). Then, using \( \text{cup } M \leq \text{cat } M \), we have \( \dim M \leq 2 \text{ cat } M \). Then (multiplying by 2 in the inequality of Corollary 6.7) we obtain

\[
2k \leq 2 \dim M - 2 \text{ cat } M \leq 2 \dim M - \dim M = \dim M.
\]

For (ii), \( \dim M = \text{cat } M \) implies \( \dim T^k = 0 \). Of course, when \( \pi_1 M \) is torsionfree, say, this also follows from Theorem 6.1 since the assumption of contractible orbits (in \( M \)) contradicts the conclusion that the induced orbit map homomorphism on fundamental groups is injective (or that isotropy groups must be finite).

The actions which are especially important in symplectic geometry are the Hamiltonian actions (see for instance [13,1,14] for an algebraic topological interpretation). A Hamiltonian action of a compact Lie group \( G \) on a symplectic manifold \((M, \omega)\) has a moment map \( \Phi : M \to g^* \) associated with it. (Here \( g^* \) is the dual of the Lie algebra \( g \) of \( G \).) Hamiltonian actions are well known to have fixed points so that orbit maps are always nullhomotopic. Exactly when contractible orbits in the manifold result from this is at present unknown it seems. Also, Hamiltonian actions of compact abelian Lie groups \( G \) have \( G \)-invariant open dense sets on which the action is free (see [13, Ch. 27] or [1, Corollary 3.4.3]). Therefore, for such actions (with orbits contractible in \( M \)), the results of Corollaries 6.7 and 6.8 hold. In particular, (i) of Corollary 6.8 is well-known in symplectic geometry for Hamiltonian torus actions. Also, from Example 3.5, we know that
there are symplectic manifolds $M$ with $\text{cat} M = \dim M$, so (ii) fits in this framework as well.

**Example 6.9.** It is necessary to point out that it is not always true that $\text{cat} M = \dim M$ for symplectic manifolds. As an example, take $M = S^2 \times T^2$ with the standard symplectic structures on the sphere and torus and the product structure on $M$. It is known that cuplength is a lower bound for category and that category obeys $\text{cat}(X \times Y) \leq \text{cat} X + \text{cat} Y$. Hence,

$$3 = \text{cup}(S^2 \times T^2) \leq \text{cat}(S^2 \times T^2) \leq \text{cat} S^2 + \text{cat} T^2 = 1 + 2 = 3$$

so all the inequalities are equalities and $\text{cat}(S^2 \times T^2) = 3$ while $\dim(S^2 \times T^2) = 4$. The same argument works to show that the $k$-tuple product $S^2 \times \cdots \times S^2$ has category equal to $k$—well below the dimension of $2k$—but this type of result is true in general for simply connected symplectic manifolds; namely, if $M$ is simply connected symplectic, then $2\text{cat} M = \dim M$. So we can ask, when is it true that a symplectic manifold has $\text{cat} M = \dim M$? More generally, we ask the

**Question 6.10.** Let $M^{2n}$ denote a closed symplectic manifold which is not simply connected and which is not decomposable as $M \simeq N \times T^{2k}$ where $N$ is simply connected symplectic. What is the category of $M$? Is $\text{cat} M = \dim M$?

Finally, by Corollary 6.7, we see that only $S^1$ can act effectively with contractible orbits on $S^2 \times T^2$. In fact, it does so in a Hamiltonian way as can be seen by taking the product of the rotation action on $S^2$ and the trivial action on $T^2$. Note that this is a better estimate than the usual symplectic geometry estimate $1/2 \cdot \dim(S^2 \times T^2) = 2$.

There is a more general version of the estimate $2 \dim G \leq \dim M$ for Hamiltonian actions of a compact abelian Lie group $G$. If any compact Lie group $G$ acts effectively on $(M, \omega)$, if the action is Hamiltonian and if the moment map $\Phi$ is a submersion for at least one $x \in M$, then $\dim G + \dim T \leq \dim M$, where $T$ is the maximal torus of $G$. The hypothesis on the moment map ensures the existence of a free orbit. We will use this as our hypothesis and improve this inequality in the case of contractible orbits also.

**Proposition 6.11.** Let a compact Lie group $G$ act on $M^{2n}$ effectively with orbits contractible in $M$ and at least one free orbit. Suppose that there is a class $\omega \in H^2(M; R)$ (for any coefficients $R$) with $\omega^n \neq 0$. Then

$$\dim G + \dim T \leq \frac{3}{2} \dim M - \text{cat} M,$$

where $T$ is the maximal torus of $G$.

**Proof.** The maximal torus $T$ acts effectively with contractible orbits on $M$, so Corollary 6.8 implies $2 \dim T \leq \dim M$. The assumption of a free orbit gives $\dim G \leq \dim M - \text{cat} M$ by Proposition 6.5. Hence,

$$\dim G + \dim T \leq \dim M - \text{cat} M + \frac{1}{2} \dim M = \frac{3}{2} \dim M - \text{cat} M. \quad \Box$$
Remark 6.12. Of course $\text{cat } M \geq \text{cup } M \geq 1/2 \dim M$, so the inequality above gives

$$\dim G + \dim T \leq \frac{1}{2} \dim M - \text{cat } M \leq \frac{1}{2} \dim M - \frac{1}{2} \dim M = \dim M$$

in accordance with the symplectic result. Note however, that the hypothesis of contractible orbits provides a refined estimate whenever $2 \text{cat } M > \dim M$.

Of course we can turn the inequality in Corollary 6.7 around to estimate $\text{cat } M$ by the dimension of groups acting on $M$. We then have the

Problem 6.13. For symplectic manifolds with $\dim M > \text{cat } M$, construct Hamiltonian compact Lie group actions of $G$ on $M$ with contractible orbits so that the moment map is submersive in at least one point to estimate $\text{cat } M$ by

$$\frac{1}{2} \dim M \leq \text{cup } M \leq \text{cat } M \leq \frac{1}{2} \dim M - \dim G - \dim T.$$

Example 6.14. The standard rotation action of $S^1$ on $S^2$ has contractible orbits, so we have

$$1 = \frac{1}{2} \dim M \leq \text{cat } S^2 \leq \frac{1}{2} \dim S^2 - \dim S^1 = 1$$

which verifies the well-known fact that the sphere has category one.

References