Available online at www.sciencedirect.com
ScienceDirect
International Journal of Approximate Reasoning
47 (2008) 368–386
INTERNATIONAL JOURNAL OF APPROXIMATE REASONING
www.elsevier.com/locate/ijar

Generalized real analysis and its applications ☆

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Received 9 August 2006; received in revised form 1 March 2007; accepted 29 May 2007
Available online 4 July 2007

Abstract

In this paper there are stressed some of the advantages of a generalized real analysis (called pseudo-analysis) based on some real operations which are taken instead of the usual addition and product of reals. Namely, there are covered with one theory and so with unified methods many problems (usually nonlinear) from many fields (system theory, optimization, control theory, differential equations, difference equations, etc.). There are presented some important real aggregation functions as triangular norms and triangular conorms and a real semiring with pseudo-operations. First there is presented how these operations occur as basic operations in the theory of fuzzy logics and fuzzy sets and there is shown a generalization of the utility theory represented by hybrid probabilistic–possibilistic measure. The real semirings serve as a base for pseudo-additive measures, pseudo-integrals, pseudo-convolutions which form the pseudo-analysis. There are presented some of the applications by large deviation principle, nonlinear Hamilton–Jacobi equation, cumulative prospect theory.

Keywords: Aggregation function; Triangular norm; Pseudo-analysis; Large deviation principle; Hamilton–Jacobi equation; Cumulative prospect theory

1. Introduction

Building models in practice often we are facing with *nonlinearity and uncertainty* and we are trying to find *the optimal model*. For that purpose usually we have to use *different mathematical tools*. We will present here some results on a generalization of the real mathematical analysis, the so called *pseudo-analysis*. For the range of functions and measures instead of the field of real numbers it is taken a semiring (see [20]) on a real interval $[a,b] \subset [-\infty,+\infty]$, denoting the corresponding operations as $\oplus$ (*pseudo-addition*) and $\odot$ (*pseudo-multiplication*), see [30,31,43]. There are many different applications of this theory by modeling nonlinearity, uncertainty in many optimization problems, nonlinear partial differential equations, nonlinear difference equations, optimal control, fuzzy systems, decision making, game theory.

☆ Dedicated to Prof. Scozzafava in the occasion of his 70. The work on this contribution was supported by grant MNTRS 144012, French-Serbian project "Pavle Savic", grant of MTA HTMT and PSNTR 00629.
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doi:10.1016/j.ijar.2007.05.015

Starting from the semiring structure, it is developed by Maslov and his group [19,21–23], the so called *idempotent analysis* (with some traces before in some applications, see [5]), see also [2,3,15,21,32,39] and then in a more general setting [15, 29–31,43,35], the so called *pseudo-analysis* in an analogous way as the classical analysis, introducing pseudo additive measures, pseudo-integral, pseudo-convolution, pseudo-Laplace transform, etc. This structure is applied for solving nonlinear equations (ODE, PDE, difference equations) using the pseudo-linear superposition principle, which means that a pseudo-linear combination of solutions of the considered nonlinear equation is also a solution. The advantage of the pseudo-analysis is that there are covered with one theory, and so with unified methods, problems (usually nonlinear and uncertain) from many different fields. Important fact is that this approach gives also solutions in the form which are not achieved by other theories, e.g., Bellman difference equation, Hamilton–Jacobi equation with non-smooth Hamiltonians. It is also important to investigate the basic real operations with different properties as *aggregation functions*, see [13] and more general non-additive measures [1,14,30–32].

In the next section we present first general aggregation functions and then some special important real operations as triangular norms and triangular conorms and finally a real semiring with pseudo-operations. In Section 3 we briefly show how previously presented operations occurs as basic operations in the theory of fuzzy logics and fuzzy sets and we present a generalization of the utility theory represented by hybrid probabilistic–possibilistic measure. In the fourth section we present some basic notions of the pseudo-analysis as pseudo-additive measure, pseudo-integral and pseudo-convolution taking special attention on the idempotent analysis with the basic representation theorem and its achievement through limit procedure by the generated case. In the fifth section we present some of the applications of pseudo-analysis in the theory of large deviation principle, nonlinear Hamilton–Jacobi equation, cumulative prospect theory.

2. Real operations

2.1. Aggregation functions

We shall start with the operations on the real interval $[0,1]$ and a general class of operations important in many applications, see [13]. Aggregation of several input values into a single output value is an important tool of mathematics, and many applications in physics, engineering, economical, social and other sciences.

Definition 1. An n-ary *aggregation function* is a function $\mathbf{A}^{(n)} : [0,1]^n \to \mathbb{R}$ that is nondecreasing in each place and fulfills the following boundary conditions

$$\inf_{(x_1,\ldots,x_n)\in[0,1]^n} \mathbf{A}^{(n)}(x_1,\ldots,x_n) = 0 \quad \text{and} \quad \sup_{(x_1,\ldots,x_n)\in[0,1]^n} \mathbf{A}^{(n)}(x_1,\ldots,x_n) = 1.$$

Definition 2. An *extended aggregation function* is a sequence $(\mathbf{A}^{(n)})_{n\geqslant 1}$, whose nth element is an n-ary aggregation function $\mathbf{A}^{(n)} : [0,1]^n \to \mathbb{R}$.

We remark that in general, a general extended aggregation function, for different n and m the functions $\mathbf{A}^{(n)}$ and $\mathbf{A}^{(m)}$ need not be related. However, some properties, such as associativity or decomposability, force such relationships. Usually in many applications we are requiring some additional properties of the aggregation operator which enables, on one side easier mathematical calculation and on the other side better fitting the model requirements.

2.2. Triangular norms and conorms

We present a special important conjunctive aggregation function, i.e., with the property $\mathbf{A} \leqslant \mathbf{M}\mathrm{in}$. The result of applications of such functions can be high only if all the inputs values are high.

Definition 3. A *triangular norm* T (t-norm *briefly*) is a function $\mathsf{T} : [0,1]^2 \to [0,1]$ such that

(T1)	$\mathsf{T}(x,y) = \mathsf{T}(y,x)$	(commutativity)
(T2)	$\mathsf{T}(x, T(y,z)) = \mathsf{T}(\mathsf{T}(x,y),z)$	(associativity)
(T3)	$\mathsf{T}(x,y) \leqslant \mathsf{T}(x,z)$ for $y \leqslant z$	(monotonicity)
(T4)	$\mathsf{T}(x,1) = x$	(boundary condition)

The following are the most important t-norms

$$\mathsf{T}_{\mathbf{M}}(x,y) = \min(x,y), \quad \mathsf{T}_{\mathbf{P}}(x,y) = xy,$$
$$\mathsf{T}_{\mathbf{L}}(x,y) = \max(0, x+y-1), \quad \mathsf{T}_{\mathbf{D}}(x,y) = \begin{cases} \min(x,y) & \text{if } \max(x,y) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding dual operation, which is a disjunctive aggregation function, i.e., $\mathsf{A} \leqslant \mathsf{M}$ax, is given by

Definition 4. A *triangular conorm* S (*t-conorm briefly*) is a function $\mathsf{S} : [0,1]^2 \to [0,1]$ such that

(S1)	$\mathsf{S}(x,y) = \mathsf{S}(y,x)$	(commutativity)
(S2)	$\mathsf{S}(x, \mathsf{S}(y, z)) = \mathsf{S}(\mathsf{S}(x,y),z)$	(associativity)
(S3)	$\mathsf{S}(x,y) \leqslant \mathsf{S}(x,z)$ for $y \leqslant z$	(monotonicity)
(S4)	$\mathsf{S}(x,0) = x$	(boundary condition)

The following are the most important t-conorms:

$$\mathsf{S}_{\mathbf{M}}(x,y) = \max(x,y) \;, \quad \mathsf{S}_{\mathbf{P}}(x,y) = x + y - xy,$$
$$\mathsf{S}_{\mathbf{L}}(x,y) = \min(1, x+y), \quad \mathsf{S}_{\mathbf{D}}(x,y) = \begin{cases} \max(x,y) & \text{if } \min(x,y) = 0 \\ 1 & \text{otherwise.} \end{cases}$$

Many other important t-norms and t-conorms can be found in [16].

2.3. A real semiring

We extend now the unit interval for the considered operations. Let $[a,b]$ be a closed (in some cases semiclosed) subinterval of $[-\infty,+\infty]$. We consider here a total order $\leqslant$ on $[a,b]$ (although it can be taken in the general case a partial order). The operation $\oplus$ (pseudo-addition) is a function $\oplus : [a,b] \times [a,b] \to [a,b]$ which is commutative, nondecreasing, associative and has a zero element, denoted by $\mathbb{0}$. Let $[a,b]_+ = \{x \mid x \in [a,b], x \geqslant \mathbb{0}\}$. The operation $\odot$ (*pseudo-multiplication*) is a function $\odot : [a,b] \times [a,b] \to [a,b]$ which is commutative, positively nondecreasing, i.e. $x \leqslant y$ implies $x \odot z \leqslant y \odot z, z \in [a,b]_+$, associative and for which there exist a unit element $\mathbb{1} \in [a,b]$, i.e., for each $x \in [a,b], \mathbb{1} \odot x = x$. We suppose, further, $\mathbb{0} \odot x = \mathbb{0}$ and that $\odot$ is a distributive pseudo-multiplication with respect to $\oplus$, i.e., $x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$. The structure $([a,b], \oplus, \odot)$ is a *semiring* (we can consider semirings in general settings on an arbitrary set endowed with two operations satisfying the previously mentioned properties, see [20]). Here we will take as basic the following special real semirings (using the equality $\mathbb{0} \odot x = \mathbb{0}$ we can consider also closed intervals in the following examples, since this equality gives the desired conventions, e.g. for ∞–∞):

Case I: Pseudo-addition is idempotent and pseudo-multiplication is not idempotent, e.g., $x \oplus y = \min(x,y), x \odot y = x + y$, on the interval $]-\infty, +\infty]$. We have $\mathbb{0} = +\infty$ and $\mathbb{1} = 0$.

Case II: Semirings with pseudo-operations defined by monotone and continuous generator. In this case we will consider only strict pseudo-addition, i.e., such that the function $\oplus$ is continuous and strictly increasing in $]a,b[\times]a,b[$ and therefore there exists a monotone function g (generator for $\oplus$), $g : [a,b] \to [-\infty,\infty]$ (or with values in $[0,\infty]$) such $g(\mathbb{0}) = 0$ and

$$u \oplus v = g^{-1}(g(u) + g(v)), \quad u \odot v = g^{-1}(g(u)g(v)).$$

The continuity at the border points a and b are reached using limits.

Case III: Both pseudo-addition and pseudo-multiplications are idempotent, e.g., let $\oplus = \max$, $\odot = \min$ on the interval $[-\infty, +\infty]$. We have $\mathbb{0} = -\infty$ and $\mathbb{1} = +\infty$.

A generalization of the presented semiring is related to a relaxation of the distributivity law, which enables a generalization of the classical von Neumann–Morgenstern utility theory to hybrid probability-possibilistic utility theory, see Section 3.2.

3. Applications of *t*-norms and *t*-conorms

3.1. t-Norms in fuzzy logics and fuzzy sets

We shall show very briefly where in the fuzzy logics and fuzzy sets occur the preceding operations of t-norms and t-conorms. For more details see [4,16], and for applications of t-norms and t-conorms different from min and max, respectively, see [17]. Taking a t-norm $\mathbf{T}$, the Zadeh strong negation c given by $c(x) = 1 - x$ and, implicitly, with the t-conorm $\mathbf{S}$ dual to $\mathbf{T}$ given by $\mathbf{S}(x, y) = c(\mathbf{T}(c(x), c(y)))$, we can introduce the basic connectives in a $[0, 1]$-valued logic as follows:

$$\text{conjunction}: \quad x \wedge_{\mathbf{T}} y = \mathbf{T}(x, y), \qquad \text{disjunction}: \quad x \vee_{\mathbf{T}} y = \mathbf{S}(x, y).$$

If x and y are the truth values of two propositions A and B, respectively, then $x \wedge_{\mathbf{T}} y$ is the truth value of 'A AND B', $x \vee_{\mathbf{T}} y$ is the truth value of 'A OR B', and $c(x)$ is the truth value of 'NOT A'. When restricting ourselves to Boolean (i.e., two-valued) logic with truth values 0 and 1 only, then we obtain the classical logical connectives. However, $([0, 1], \mathbf{T}, \mathbf{S}, c, 0, 1)$ never yields a Boolean algebra. As in classical logic, it is possible to construct implication, bi-implication and so on by means of negation, conjunction and disjunction. Taking into account that in Boolean logic 'NOT A OR B' is equivalent to 'IF A THEN B', one possibility of modelling the implication in a $[0, 1]$-valued logic (based on $\mathbf{T}$, c and $\mathbf{S}$) is to define the function $I_{\mathbf{T}} : [0, 1]^2 \to [0, 1]$ by

$$I_{\mathbf{T}}(x, y) = \mathbf{S}(c(x), y) = c(\mathbf{T}(x, c(y))).$$

It is clear that in this case the *law of contraposition* $I_{\mathbf{T}}(x, y) = I_{\mathbf{T}}(c(y), c(x))$ is always valid.

For the two basic t-norms $\mathbf{T_M}, \mathbf{T_L}$ we obtain the following implications:

$$I_{\mathbf{T_M}}(x, y) = \begin{cases} y & \text{if } x + y \geqslant 1, \\ 1 - x & \text{otherwise,} \end{cases} \qquad I_{\mathbf{T_L}}(x, y) = \begin{cases} 1 & \text{if } x \leqslant y, \\ 1 - x + y & \text{otherwise.} \end{cases}$$

Another way of extending the classical binary implication operator (acting on $\{0, 1\}$) to the unit interval $[0, 1]$ uses the *residuation* (see [38,10,16])

$$R_{\mathbf{T}}(x, y) = \sup\{z \in [0, 1] \mid \mathbf{T}(x, z) \leqslant y\}.$$

For the two previous important t-norms $\mathbf{T_M}$, $\mathbf{T_L}$ we obtain the following residuations:

$$R_{\mathbf{T_M}}(x, y) = \begin{cases} 1 & \text{if } x \leqslant y, \\ y & \text{otherwise;} \end{cases} \qquad R_{\mathbf{T_L}}(x, y) = \begin{cases} 1 & \text{if } x \leqslant y, \\ 1 - x + y & \text{otherwise.} \end{cases}$$

In general, $I_{\mathbf{T}}$ and $R_{\mathbf{T}}$ are different (although both are extensions of the Boolean implication), but that $I_{\mathbf{T_L}} = R_{\mathbf{T_L}}$.

Given a (crisp) universe of discourse X, as it is well-known a *fuzzy subset* A of X is characterized by its *membership function* $\mu_A : X \to [0, 1]$, where for $x \in X$ the number $\mu_A(x)$ is interpreted as the *degree of membership* of x in the fuzzy set A or, equivalently, as the *truth value* of the statement '*x is element of A*'. The membership function μ_A of a fuzzy subset A of X is a quite natural generalization of the *characteristic function* $\mathbf{1}_B : X \to \{0, 1\}$ of a crisp subset B of X, assigning the value 1 to all elements of X which belong to B, and the value 0 to all remaining elements of X. In order to generalize the Boolean set–theoretical operations like *intersection* and *union* (or, equivalently, the corresponding logical operations conjunction and disjunction, respectively), it is quite natural to use triangular norms and conorms. Given a t-norm $\mathbf{T}$ and a t-conorm $\mathbf{S}$, for any fuzzy subsets A and B of the universe X, the membership functions of the *intersection* $A \cap B$, the *union* $A \cup B$ and the *complement* A^c are given by

$$\mu_{A\cap B}(x) = \mathbf{T}(\mu_A(x), \mu_B(x)), \quad \mu_{A\cup B}(x) = \mathbf{S}(\mu_A(x), \mu_B(x)),$$
$$\mu_{A^c}(x) = 1 - \mu_A(x).$$

The values $\mu_{A\cap B}(x)$, $\mu_{A\cup B}(x)$ and $\mu_{A^c}(x)$ describe the truth values of the statements '*x is element of A* AND *x is element of B*', '*x is element of A* OR *x is element of B*', and '*x is* NOT *element of A*', respectively.

Let $X_1 \times X_2$ is the crisp Cartesian product of two crisp sets X_1 and X_2. Then a *fuzzy relation R* on $X_1 \times X_2$ is a fuzzy set on $X_1 \times X_2$. The notion of *fuzzy relation equation*, first introduced in [41] for $\mathbf{T_M}$, is important in many applications, as for example in automatic control by fuzzy controllers. For the results on the solution of a special type of fuzzy relation equation see [41,7,10].

3.2. Hybrid utility function

$\mathbf{T}$ is conditionally distributive over $\mathbf{S}$, i.e., they satisfy the property (CD):

$$\mathbf{T}(x, \mathbf{S}(y,z)) = \mathbf{S}(\mathbf{T}(x,y), \mathbf{T}(x,z))$$

for all $x, y, z \in [0,1]$ such that $\mathbf{S}(y,z) < 1$. We shall call $([0,1], \mathbf{S}, \mathbf{T})$ a *conditionally distributive semiring*, see [16]. There was obtained in [16] the following characterization of continuous conditionally distributive semiring (see Fig. 1).

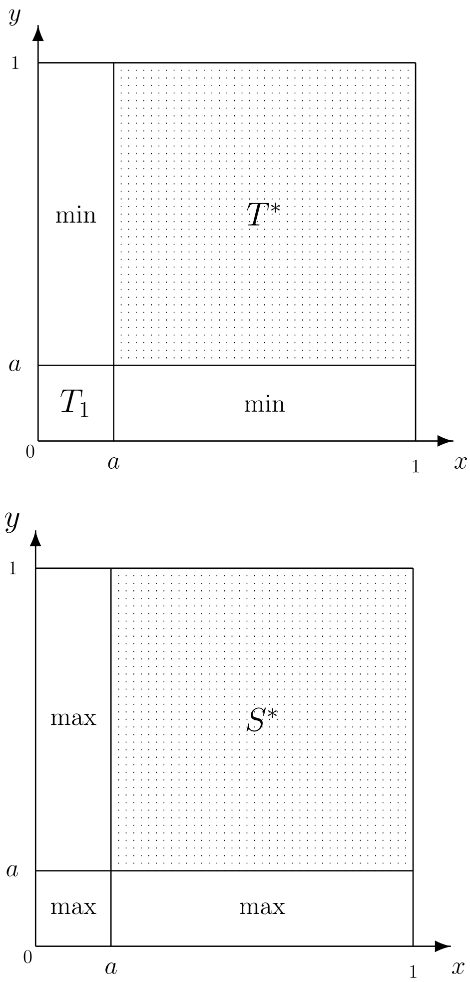

Fig. 1. A pair $(\mathbf{T}, \mathbf{S})$ from Theorem 1 for $0 < a < 1$.

Theorem 1. *A continuous t-norm* $\mathbf{T}$ *is conditionally distributive over a continuous t-conorm* $\mathbf{S}$ *if and only if there exists a value* $a \in [0,1]$, *a strict t-norm* $\mathbf{T}^*$ *and a nilpotent t-conorm* $\mathbf{S}^*$ *such that the additive generator* s^* *of* $\mathbf{S}^*$ *satisfying* $s^*(1) = 1$ *is also a multiplicative generator of* $\mathbf{T}^*$ *such that*

$$\mathbf{T} = (\langle 0, a, \mathbf{T}_1\rangle, \langle a, 1, \mathbf{T}^*\rangle),$$

where $\mathbf{T}_1$ *is an arbitrary continuous t-norm and*

$$\mathbf{S} = (\langle a, 1, \mathbf{S}^*\rangle).$$

Since every strict t-norm is isomorphic to $\mathbf{T_P}$ and every nilpotent t-conorm is isomorphic to $\mathbf{S_L}$ we shall use the following notation

$$(\langle \mathbf{S_M}, \mathbf{S_L}\rangle, \langle \mathbf{T}_1, \mathbf{T_P}\rangle)_a$$

for this canonical representative.

In order to generalize decision theory to non-probabilistic uncertainty, one approach is to generalize mixture sets. In the paper [8] there are characterized the families of operations involved in generalized mixtures, due to a previous result on the characterization of the pairs of continuous t-norm and t-conorm such that the former is conditionally distributive over the latter.

A basic notion in probability theory is independence. The main issue in probabilistic independence is the existence of special events $A_1, \ldots, A_n$ such that $p(A_1 \cap \cdots \cap A_n) = \prod_{i=1}^n p(A_i)$. Such events are called independent events. In order to preserve the computational advantages of independence, any operation $*$ for which it could be established that $p(A_1 \cap \cdots \cap A_n) = *_{i=1}^n p(A_i)$, would do. However the Boolean structure of sets of events and the additivity of the probability measure, impose considerable constraint on the choice of operation $*$. In the paper [8] is studied the possible operations $*$ when changing p for a pseudo-additive (decomposable) measure based on a t-conorm $\mathbf{S}$. A first remark is that it is natural to require that $*$ be a continuous triangular norm. If $A = X$ is a sure event, then A and X are independent, and it follows that $m(A \cap X) = m(A) * m(X) = m(A) * 1 = m(A)$. Commutativity and associativity of $*$ reflect the corresponding properties for conjunctions. It is also very natural that $*$ be non-decreasing in each place and continuous. We try to find which triangular norms can be used for extending the notion of independence for pseudo-additive measures in the sense of a prescribed triangular conorm. Since the term independence has a precise meaning in probability theory, we shall speak of separability in the framework of $\mathbf{S}$-measures. Two events A and B are said to be $*$-*separable* if $m(A \cap B) = m(A) * m(B)$ for a triangular norm $*$. It turns out by [8] that the only reasonable pseudo-additive measures admitting of an independence-like concept, are based on conditionally distributive pairs ($\mathbf{S}$,$\mathbf{T}$) of t-conorms and t-conorms in the form $(\langle \mathbf{S_M}, \mathbf{S_L}\rangle, \langle \mathbf{T}_1, \mathbf{T_P}\rangle)_a$, namely:

(a) probability measures (and $* =$ product);
(b) possibility measures (and $*$ is any t-norm);
(c) suitably normalized hybrid set-functions m such that there is $a \in]0,1[$ which gives for A and B disjoint

$$m(A \cup B) = \begin{cases} m(A) + m(B) - a & \text{if } m(A) > a, m(B) > a, \\ \max(m(A), m(B)) & \text{otherwise} \end{cases}$$

and for separability:

$$m(A \cap B) = \begin{cases} a + \frac{(m(A)-a)(m(B)-a)}{1-a} & \text{if } m(A) > a, \ \ m(B) > a, \\ a \cdot \mathbf{T}_1\left(\frac{m(A)}{a}, \frac{m(B)}{a}\right) & \text{if } m(A) \leqslant a, m(B) \leqslant a, \\ \min(m(A), m(B)) & \text{otherwise.} \end{cases}$$

Any probability distribution on a finite set X can be represented as a sequence of binary lotteries. A binary lottery is 4-uple (A, α, x, y) where $A \subset X$ and $\alpha \in [0,1]$ such that $p(A) = \alpha$, and it represents the random event that yields x if A occurs and y otherwise. Let p be a probability on X such that $p_i = p(x_i), x_i \in X$. Assume $X = \{x_1, x_2, x_3\}$ then p can be described by the following. The binary tree is obtained as follows: First partition X into $\{x_1\}$ and $\{x_2, x_3\}$ with probabilities p_1 and $p_2 + p_3$, respectively, then partition $X \backslash \{x_1\}$ into $\{x_2\}$ and

$\{x_3\}$ with probabilities $p_2/(p_2+p_3)$ and $p_3/(p_2+p_3)$, respectively. The two trees are equivalent, provided that the probability of x_i is calculated by performing the product of weights on the path from the root of the tree until the leaf x.

More generally, suppose m is a $\mathbf{S}$-measure on $X = \{x_1, x_2, x_3\}$ and $m_i = m(\{x_i\})$. Suppose we want to decompose the ternary tree into the binary tree so that they are equivalent. Then the reduction of lottery property enforces the following equations

$$\mathbf{S}(v_1, v_2) = 1, \quad \mathbf{T}(\mu, v_1) = m_2, \quad \mathbf{T}(\mu, v_2) = m_3,$$

where $\mathbf{T}$ is the triangular norm that expresses separability for $\mathbf{S}$-measures. The first condition expresses normalization (with no truncating effect for t-conorm $\mathbf{S}$ allowed). If these equations have unique solutions, then by iterating this construction, any distribution of a $\mathbf{S}$-measure can be decomposed into a sequence of binary lotteries. Turning the $\mathbf{S}$-measure into a sequence of binary trees leads to the necessity of solving the following system of equations

$$\alpha_1 = \mathbf{T}(\mu, v_1), \quad \alpha_2 = \mathbf{T}(\mu, v_2), \quad \mathbf{S}(v_1, v_2) = 1 \tag{1}$$

for given α_1 and α_2. Assuming that $\mathbf{T}_1 = \min$ the system of Eqs. (1) was completely solved in [8] and exhibited the analytical forms of (μ, v_1, v_2). We define the set $\Phi_{\mathbf{S},a}$ of ordered pairs (α, β) in the following way

$$\Phi_{\mathbf{S},a} = \{(\alpha,\beta) \mid (\alpha,\beta) \in (a,1)^2, \alpha+\beta = 1+a\} \bigcup \{(\alpha,\beta) \mid min(\alpha,\beta) \leqslant a, \max(\alpha,\beta) = 1\}.$$

A *hybrid mixture set* is a quadruple $(\mathcal{G}, M, \mathbf{T}, \mathbf{S})$ where $\mathcal{G}$ is a set, $(\mathbf{S}, \mathbf{T})$ is a pair of continuous t-conorm and t-norm, respectively, which satisfy the condition (CD) and $M : \mathcal{G}^2 \times \Phi_{\mathbf{S},a} \to \mathcal{G}$ is a function (hybrid mixture operation) given by

$$M(x, y; \alpha, \beta) = \mathbf{S}(\mathbf{T}(\alpha, x), \mathbf{T}(\beta, y)).$$

It is enough to restrict to the case $(\langle \mathbf{S_M}, \mathbf{S_L}\rangle, \langle \mathbf{T}_1, \mathbf{T_P}\rangle)_a$. Then it is easy to verify that M satisfies the axioms **M1-M5** on $\Phi_{S,a}$, where

M1. $M(x, y; 1, 0) = x$;
M2. $M(x, y; \alpha, \beta) = M(y, x; \beta, \alpha)$;
M3. $M(M(x, y; \alpha, \beta), y; \gamma, \delta) = M(x, y; T(\alpha, \gamma), \mathbf{S}(\mathbf{T}(\beta, \gamma), \mathbf{T}(\delta, 1)))$.
M4. $M(x, x; \alpha, \beta) = x$.
M5. $M(M(x, y; \alpha, \beta), M(x, y; \gamma, \delta); \lambda, \mu) = M(x, y; \mathbf{S}(\mathbf{T}(\alpha, \lambda), \mathbf{T}(\gamma, \mu)), \mathbf{S}(\mathbf{T}(\beta, \lambda), \mathbf{T}(\delta, \mu)))$ holds for all $x, y \in \mathcal{G}$ and all $(\alpha, \beta), (\gamma, \delta), (\lambda, \mu) \in \Phi_{\mathbf{S},a}$.

This kind of mixtures exhausts the possible solutions to **M1–M5**.

Let us show the main appeal of **M1**–**M5**. Let $(\mathbf{S}, \mathbf{T})$ be a pair of continuous t-conorm and t-norm, respectively, of the form $(\langle \mathbf{S_M}, \mathbf{S_L}\rangle, \langle \mathbf{T}_1, \mathbf{T_P}\rangle)_a$. Let u_1, u_2 be two utilities taking values in the unit interval $[0, 1]$ and let μ_1, μ_2 be two degrees of plausibility from $\Phi_{S,a}$. Then we define *the optimistic hybrid utility function* by means of the hybrid mixture as

$$U(u_1, u_2; \mu_1, \mu_2) = \mathbf{S}(\mathbf{T}(u_1, \mu_1), \mathbf{T}(u_2, \mu_2)).$$

In the paper Dubois et al. [8] it is examined in details this utility function. Although the above description of optimistic hybrid utility is rather complex, it can be easily explained, including the name optimistic, see [8]. Putting together the results of this paper, the utility of a n-ary lottery can be computed by decomposing the $\mathbf{S}$-measure into a sequence of binary trees and applying the above computation scheme for hybrid utility recursively from the bottom to the top of the binary tree expansion. More details and proofs of theorems stated in this paper can be found in [8].

4. Pseudo-analysis and generalizations

4.1. Pseudo-operations, pseudo-additive measures, pseudo-integrals

We use the notations from Section 2.3. Let X be a non-empty set. Let $\mathscr{A}$ be a σ-algebra of subsets of X. A set function $m : \mathscr{A} \to [a,b]_+$ (or semiclosed interval) is a $\oplus$-decomposable measure if there hold $m(\emptyset) = 0$; and $m(A \cup B) = m(A) \oplus m(B)$ for $A, B \in \mathscr{A}$ such that $A \cap B = \emptyset$. A $\oplus$-decomposable measure m is *$\oplus$-measure* if

$$m\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigoplus_{i=1}^{\infty} m(A_i)$$

hold for any sequence $(A_i)_{i\in\mathbb{N}}$ of pairwise disjoint sets from $\mathscr{A}$.

The construction of *pseudo-integral* based on $\oplus$-measure, denoted by

$$\int_X^{\oplus} f \odot \mathrm{d}m,$$

is similar to the construction of the Lebesgue integral, see [30]. Then it can be introduced *pseudo-convolution* and *pseudo-Laplace transform* with many different applications, see [23,30,31,35]. Let G be subset of $\mathbb{R}^n$ and $*$ a commutative binary operation on G such that $(G,*)$ is a cancellative semigroup with unit element e and

$$G_+ = \{x \mid x \in G, x \geqslant e\}$$

is a subsemigroup of G. We shall consider functions whose domain will be G.

Definition 5. The *pseudo-convolution of the first type* of two functions $f : G \to [a,b]$ and $h : G \to [a,b]$ with respect to a $\oplus$-measure m and $x \in G_+$ is given in the following way

$$f \star h(x) = \int_{G_+^x}^{\oplus} f(u) \odot \mathrm{d}m_h(v),$$

where $G_+^x = \{(u,v) | u * v = x, v \in G_+, u \in G_+\}$, $m_h = m$ in the case of sup-measure $m(A) = sup_{x\in A} h(x)$, in the case of inf-measure $m(A) = \inf_{x\in A} h(x)$, and $\mathrm{d}m_h = h \odot \mathrm{d}m$ in the case of $\oplus$-measure m, where $\oplus$ has an additive generator g and $g \circ m$ is the Lebesgue measure.

We consider also *the second type of pseudo-convolution* when $(G,*)$ is a group and the pseudo-integral is taken over whole set G:

$$f \star h(x) = \int_G^{\oplus} f(x * (-t)) \odot \mathrm{d}m_h(t),$$

where $(-t)$ is unique inverse element for t and $x \in G$.

Remark 1. When $*$ is the usual addition on $\mathbb{R}$ and $G = \mathbb{R}$, pseudo-convolutions of the first and the second type, for $x \in \mathbb{R}^+$, are

$$(f \star h)(x) = \int_{[0,x]}^{\oplus} f(x-t) \odot \mathrm{d}m_h(t), \qquad (f \star h)(x) = \int_G^{\oplus} f(x-t) \odot \mathrm{d}m_h(t),$$

respectively, see [36].

Pseudo-delta function is given by

$$\delta_e^{\oplus,\odot}(x) = \begin{cases} \mathbf{1} & \text{for } x = e, \\ \mathbf{0} & \text{for } x \neq e, \end{cases}$$

where $\mathbf{0}$ is zero element for $\oplus$, $\mathbf{1}$ is unit element for $\odot$ and e is zero element for $*$.

Example 1. Let for $* = +$ and $G = \mathbb{R}$. For the semiring $([-\infty,\infty[, \max, +)$ from Section 2.3, the pseudo-integral, with respect to sup-measure $m, m(A) = \sup_{x\in A} h(x)$, is given by

$$\int_{\mathbb{R}}^{\oplus} f \odot \mathrm{d}m = \sup_{\mathbb{R}}(f(x) + h(x))$$

and the pseudo-convolution of the first and second type of the functions f and h will be

$$(f \star h)(x) = \sup_{0 \leqslant t \leqslant x}(f(x-t) + h(t)), \qquad (f \star h)(x) = \sup_{t \in \mathbb{R}}(f(x-t) + h(t)),$$

respectively. Unit element for this pseudo-convolutions is the following pseudo-delta function

$$\delta_0^{\max,+}(x) = \begin{cases} \mathbf{1}(=0) & \text{if } x = 0, \\ \mathbf{0}(=-\infty) & \text{if } x \neq 0. \end{cases}$$

We mention here only that the pseudo-convolution covers many important basic notions in different fields, see [36]:

(1) The basic notion of the theory of probabilistic metric spaces, the triangle function, is based on the pseudo-convolution of the first type.
(2) The arithmetical operations with fuzzy numbers are based on Zadeh's extension principle (see [16]): Let T be an arbitrary but fixed t-norm and $*$ a binary operation on $\mathbb{R}$. Then the operation $*$ is extended to fuzzy numbers A and B by

$$A *_{\mathsf{T}} B(z) = \sup_{x*y=z} \mathsf{T}(A(x), B(y))$$

for $z \in \mathbb{R}$.
(3) Pseudo-convolutions are as important tools in the system theory as the classical convolution was, see [23].

There are further generalizations related the pseudo-operations in the theory of pseudo-analysis. First generalization of the real semiring structure is to the case when the operations $\oplus$ and $\odot$ are noncommutative and non-associative, and the (left) right distributivity of $\odot$ over $\oplus$ plays a crucial rule, see [37]. There is obtained a representation theorem for such operations and there is given a complete characterization for generalized pseudo-addition and pseudo-multiplication. Another generalization consists in some symmetrization of maximum and minimum, used in an integral representation of the utility functional, see [11,12] and Section 5.3.

4.2. A representation theorem in idempotent analysis

An idempotent semigroup (semiring, e.g., cases I and III from Section 2.3) P is called an *idempotent metric semigroup (semiring)* if it is endowed with a metric $d : P \times P \to \mathbb{R}$ such that the operation $\oplus$ is (respectively, the operations $\oplus$ and $\odot$ are) uniformly continuous on any order-bounded set in the topology induced by d and any order-bounded set is bounded in the metric, see [23]. Let X be a set, and let $P = (P, \oplus, d)$ be an idempotent metric semigroup. The set $B(X, P)$ of bounded mappings $X \to P$, i.e., mappings with order-bounded range, is an idempotent metric semigroup with respect to the pointwise addition $(\varphi \oplus \psi)(x) = \varphi(x) \oplus \psi(x)$, the corresponding partial order, and the uniform metric $\mathrm{d}(\varphi, \psi) = \sup_x \mathrm{d}(\varphi(x), \psi(x))$. If $P = (P, \oplus, \odot, \rho)$ is a semiring, then $B(X, P)$ has the structure of an *idempotent semimodule P-semimodule*, i.e., the multiplication by elements of P is defined on $B(X, P)$ by $(a \odot \varphi)(x) = a \odot \varphi(x)$. This P-semimodule will also be referred to as the space of (bounded) P-valued functions on X. If X is a topological space, then by $C(X, P)$ we denote the subsemimodule of continuous functions in $B(X, P)$. If X is finite, $X = \{x_1, \ldots, x_n\}$, $n \in \mathbb{N}$, then the semimodules $C(X, P)$ and $B(X, P)$ coincide and can be identified with the semimodule $P^n = \{(a_1, \ldots, a_n) \mid a_j \in P\}$. Any vector $a \in P^n$ can be uniquely represented as a pseudo linear combination

$$a = \bigoplus_{j=1}^{n} a_j \odot e_j,$$

where $\{e_j \mid j = 1, \ldots, n\}$ is the standard basis of P^n (the jth coordinate of e_j is equal to $\mathbb{1}$, and the other coordinates are equal to $\mathbb{0}$). As in the classical linear algebra, we can readily prove that the semimodule of continuous homomorphisms h: $P^n \to P$ (in what follows such homomorphisms are called *pseudo linear functionals* on P^n) is isomorphic to P^n itself. Similarly, any endomorphism $H : P^n \to P^n$ (a linear operator on P^n) is determined by an P-valued $n \times n$ matrix.

We present here the simplest versions of some general facts of idempotent analysis, in particular, restricting our consideration to the case of standard $(\min, +)$ semiring P. All idempotent measures are absolutely continuous; i.e., any such measure can be represented as the idempotent integral of a density function with respect to some standard measure. Let us formulate this fact more precisely. Let $C_{\mathbb{0}}(X, P)$ denote the space of continuous functions $f : X \to P$ on a locally compact normal space X vanishing at infinity (i.e. such that for any $\varepsilon > 0$ there exists a compact set $K \subset X$ such that $\mathrm{d}(\mathbb{0}, f(x)) < \varepsilon$ for all $x \in X \backslash K$). The topology on $C_{\mathbb{0}}(X, P)$ is defined by the uniform metric $d(f, g) = \sup_X d(f(x), g(x))$. The space $C_{\mathbb{0}}(X, P)$ is an idempotent semimodule. If X is a compact set, then the semimodule $C_{\mathbb{0}}(X, P)$ coincides with the semimodule $C(X, P)$ of all continuous functions from X to P. The homomorphisms $C_{\mathbb{0}}(X, P) \to P$ will be called *pseudo linear functionals* on $C_{\mathbb{0}}(X, P)$. The set of linear functionals will be denoted by $C^*_{\mathbb{0}}(X, P)$ and called the *dual semimodule* of $C_{\mathbb{0}}(X, P)$.

Theorem 2. *For any $m \in C^*_{\mathbb{0}}(X, P)$ there exists a unique lower semicontinuous and bounded below function f: $X \to P$ such that for every $h \in C_{\mathbb{0}}(X, P)$*

$$m(h) = \inf_x f(x) \odot h(x). \tag{2}$$

Conversely, any function $f : X \to P$ bounded below defines an element $m \in C^*_{\mathbb{0}}(X, P)$ by formula (2). At last, the functionals m_{f_1} and m_{f_2} coincide if and only if the functions f_1 and f_2 have the same lower semicontinuous closures; that is, $Clf_1 = Clf_2$, where

$$(\mathrm{Cl}f)(x) = \sup\{\psi(x) \mid \psi \leqslant f, \psi \in C(X, P)\}.$$

The Riesz–Markov theorem in functional analysis establishes a one-to-one correspondence between continuous linear functionals on the space of continuous real functions on a locally compact space X vanishing at infinity and regular finite Borel measures on X. Similar correspondence exists in idempotent analysis. We can define an idempotent measure μ_f on the subsets of X by the formulas $\mu_f(A) = \inf\{x | x \in A\}$. This is a inf-measure. Eq. (2) specifies a continuation of m_f to the set of P-valued functions bounded below. On analogy with conventional analysis, we say that such functions are integrable with respect to the measure μ_f and denote the values taken by m_f on these functions by the *idempotent integral*

$$m_f(h) = \int_X^{\oplus} h(x)\mathrm{d}\mu_f(x) = \inf_x f(x) \odot h(x).$$

Theorem 2 is equivalent to the statement that all idempotent measures are absolutely continuous with respect to the standard idempotent measure $m_{\mathbb{1}}$.

4.3. Idempotent integral as limit of generated integrals

Despite of the different behavior of the idempotent case there is a close connection with the generated case.

Let $([a, b], \oplus, \odot)$ be a semiring of type II from Section 2.3 with a generator $g : [a, b] \to [0, \infty]$. As it is shown in [25], for $\lambda \in (0, \infty)$ the function g^λ is a generator for the semiring $([a, b], \oplus_\lambda, \odot_\lambda)$ with $x \oplus_\lambda y = (g^\lambda)^{-1}(g^\lambda(x) + g^\lambda(y))$ and $x \odot_\lambda y = (g^\lambda)^{-1}(g^\lambda(x) \cdot g^\lambda(y)) = x \odot y$. Hence $([a, b], \oplus_\lambda, \odot_\lambda) = ([a, b], \oplus_\lambda, \odot)$.

The following three theorems were proved in [25].

Theorem 3. *Let $g : [a, b] \to [0, \infty]$ be a strictly decreasing generator of the semiring $([a, b], \oplus, \odot)$ of the type II and g^λ the function g on the power $\lambda \in (0, \infty)$. Then g^λ is a generator of the semiring $([a, b], \oplus_\lambda, \odot)$ and for every $\varepsilon > 0$ and every $(x, y) \in [a, b]^2$ there exists λ_0 such that $|x \oplus_\lambda y - \inf(x, y)| < \varepsilon$ for all $\lambda \geqslant \lambda_0$. For g increasing, the same result holds for* sup.

Theorem 4. *Let m be a* sup-*measure on* $([0, \infty], \mathscr{B}_{[0,\infty]})$, *where*

$$m(A) = \operatorname*{ess\,sup}_{\mu}\{\varphi(x) \mid x \in A\},$$

where $\varphi : [0,\infty] \to [0,\infty]$ is a continuous density. Then for any generator g there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measures on $([0,\infty],\mathscr{B}_{[0,\infty]})$, where $\oplus_\lambda$ is generated by g^λ, $\lambda \in]0,\infty[$, such that $\lim_{\lambda\to\infty} m_\lambda = m$ (i.e. $\lim_{\lambda\to\infty} m_\lambda(A) = m(A)$, for all $A \in \mathscr{B}_{[0,\infty]}$).

Theorem 5. *Let $([0,\infty],\sup,\odot)$ be a semiring with $\odot$ generated by generator g. Let m be the same as in the Theorem 4. Then there exists a family $\{m_\lambda\}$ of $\oplus_\lambda$-measures, where $\oplus_\lambda$ is generated by g^λ, $\lambda \in]0,\infty[$, such that for every continuous function $f : [0,\infty] \to [0,\infty]$*

$$\int^{\sup} f \odot \mathrm{d}m = \lim_{\lambda\to\infty} \int^{\oplus_\lambda} f \odot \mathrm{d}m_\lambda = \lim_{\lambda\to\infty} (g^{-1})^\lambda \left(\int (g^\lambda \circ f) \odot \mathrm{d}x \right).$$

5. Applications of the pseudo-analysis

5.1. Large deviation principle

The theory of large deviations concerned with *the asymptotic estimation of probabilities of rare events,* and typically provide exponential bound on probability of such events and characterize them. This theory has found many applications in information theory, coding theory, image processing, statistical mechanics, various kind of random processes (certain types of finite state Markov chains, Brownian motion, Wiener process), stochastic differential equations, etc. Contemporary large deviation theory uses various approaches, [6,24].

Let Ω be a topological space and $\mathscr{A}$ be the algebra of its Borel sets. A family of probabilities (P_ε), $\varepsilon > 0$, on $(\Omega,\mathscr{A})$ obeys the *large deviation principle* if there exists a *rate function* $I : \Omega \to [0,\infty]$ such that

(1) I is lower semi-continuous and $\Omega_a = \{\omega \in \Omega | I(\omega) \leqslant a\}$ is a compact set for any $a < \infty$,
(2) $-\limsup_{\varepsilon\to 0} \varepsilon \log P_\varepsilon(C) \geqslant \inf_{\omega\in C} I(\omega)$ for each closed set $C \subset \Omega$,
(3) $-\liminf_{\varepsilon\to 0} \varepsilon \log P_\varepsilon(U) \leqslant \inf_{\omega\in U} I(\omega)$ for each open set $U \subset \Omega$.

Then $m(A) = \inf_{\omega\in A} I(\omega)$ is a positive idempotent measure on $\mathscr{A}$. Therefore, it is naturally to generalize the previous definition in the following way [2,39]. For any Borel set A let

$$P^{\text{out}}(A) = \limsup_{\varepsilon\to 0} \varepsilon \log P_\varepsilon(A),$$
$$P^{in}(A) = \liminf_{\varepsilon\to 0} \varepsilon \log P_\varepsilon(A).$$

One says that (P_ε) obeys *the weak large deviation principle*, if there exists a positive idempotent measure m on $(\Omega,\mathscr{A})$ such that

(1) There exists a sequence (Ω_n) of compact subsets of Ω such that

$$m(\Omega_n^c) \to \mathbb{0} = +\infty \quad \text{as } n \to \infty,$$

where C^c stands for the complimentary set of C,
(2) $m(C) \leqslant -P^{\text{out}}(C)$ for each closed $C \subset \Omega$,
(3) $m(U) \geqslant -P^{\text{in}}(U)$ for each open $U \subset \Omega$.

Using Theorem 2 and its generalizations one can prove that the large deviation principle and its weak version are actually equivalent for some (rather general) "good" spaces Ω. One can obtain also an interesting correspondence between the tightness conditions for probability and idempotent measures and for further generalizations on random sets.

Further generalization with respect to pseudo-additive measures are obtained, see [39,27]. We shortly present the result from [27]. Motivated by Theorems 4 and 5, there was considered the convergence in the sense of large deviation principle of $\oplus_{r_n}$-measures m_n on $[0,\infty]$ with the property $m_n([0,\infty]) = \mathbf{1}$ to the limit sup-measure m on $[0,\infty]$ with the property $m([0,\infty]) = \mathbf{1}$. We denote by $\mathscr{B}_{[0,\infty]}$ the Borel σ-algebra of subsets of $[0,\infty]$

(with the usual topology on $[0,\infty]$). Let $\mathscr{O}$ and $\mathscr{F}$ denote the families of open and closed sets in $[0,\infty]$, respectively.

Let $([0,\infty],\oplus,\odot)$ be a semiring of type II, see Section 2.3, with $\oplus$ and $\odot$ generated by a continuous strictly increasing function $g:[0,\infty]\to[0,\infty]$, such that $g(0)=0$ and $g(\mathbf{1})=1$, where 0 and $\mathbf{1}$ are neutral elements for $\oplus$ and $\odot$, respectively. Let $([0,\infty],\sup,\odot)$ be a semiring with the same operation $\odot$ as in semiring $([0,\infty],\oplus,\odot)$ (and 0 is also the neutral element for sup). Let $m:\mathscr{B}_{[0,\infty]}\to[0,\infty]$ be a completely maxitive, $\mathscr{F}$-smooth sup-measure on $[0,\infty]$ with property $m([0,\infty])=\mathbf{1}$. Let $(r_n)_{n\in\mathbb{N}}$ be a sequence of real numbers greater than *1* satisfying $\lim_{n\to\infty}r_n=\infty$. According to Theorem 3 g^{r_n} is a generator of the semiring $([0,\infty],\oplus_{r_n},\odot)$. Let $(m_n)_{n\in\mathbb{N}}$ be a sequence of $\oplus_{r_n}$-measures defined on $([0,\infty],\mathscr{B}_{[0,\infty]})$ with the property $m_n([0,\infty])=\mathbf{1}$.

Definition 6. The sequence $(m_n)_{n\in\mathbb{N}}$ *large deviation converges* at rate (g,r_n) to m (LD *converge*, for short) if for all continuous and bounded functions $f:X\to\mathbb{R}^+$ holds

$$\lim_{n\to\infty}\int_{[0,\infty]}^{\oplus_{r_n}} f\odot \mathrm{d}m_n=\int_{[0,\infty]}^{\sup} f\odot \mathrm{d}m. \tag{3}$$

If the limit value (3) exists, it is uniquely determined.

The following theorem generalize a part of Portmanteau theorem, see [27].

Theorem 6. *If the sequence $(m_n)_{n\in\mathbb{N}}$ of $\oplus_{r_n}$- measures LD converges at rate (g,r_n) to sup -measure m, then*:

(*a*) *for arbitrary open set $O\subseteq[0,\infty]$ holds*

$$\liminf_{n\to\infty} m_n(O)\geqslant m(O),$$

(*b*) *for arbitrary closed set $F\subseteq[0,\infty]$ holds*

$$\limsup_{n\to\infty} m_n(F)\leqslant m(F).$$

Observe that in Definition 6 we have integral based on sup-measure as the limit value and the second operation for the integration is a pseudo-multiplication. For applications on convergence theorems of random sets see [28,33].

5.2. Hamilton–Jacobi equation

The introduced generalized analysis is applied for solving nonlinear equations (ODE, PDE, difference equations, etc.) using now the pseudo-linear principle, which means that if u_1 and u_2 are solutions of the considered nonlinear equation, then also $a_1\odot u_1\oplus a_2\odot u_2$ is a solution for any constants a_1 and a_2 from $[a,b]$. This section is devoted to a short presentation, see [19,22,23,35], of the theory of generalized solutions to the Cauchy problem

$$\begin{cases}\frac{\partial w}{\partial t}+H\left(x,\frac{\partial w}{\partial x}\right)=0,\\ w|_{t=0}=w_0(x)\end{cases} \tag{4}$$

of a Hamilton–Jacobi equation (HJ) with Hamiltonian H being convex in second variable. Many important applications, e.g., control theory, require nonsmooth Hamiltonian, e.g., absolute value or maximum. Since any convex function can be written in the form

$$H(x,p)=\max_{u\in U}(pf(x,u)-g(x,u))$$

with some functions *f*, *g*, equation in (4) can be written in the equivalent form

$$\frac{\partial w}{\partial t}+\max_{u\in U}\left(\left\langle\frac{\partial w}{\partial x},f(x,u)\right\rangle-g(x,u)\right)=0,$$

which is called the *nonstationary Bellman differential equation* (HJB).

Let us discuss first, what difficulties occur when one tries to give a reasonable definition of the solutions to problem (4)?

First, as simple examples shows, the classical (i.e., smooth) solution of the Cauchy problem (4) does not exist for large time even for smooth H and w_0. Therefore one cannot hope to obtain smooth solutions for nonsmooth H and w_0. On the other hand, in contrast with the theory of linear equations, where generalized solutions can be defined in the standard way as functionals on the space of test functions, there is no such approach in the theory of nonlinear equations.

The most popular approach to the theory of generalized solutions of the HJB equation is the vanishing viscosity method. This means that one defines a generalized solution to problem (4) as the limit as $h \to 0$ of solutions of the Cauchy problem

$$\frac{\partial w}{\partial t} + H\left(x, \frac{\partial w}{\partial x}\right) - \frac{h}{2}\frac{\partial^2 w}{\partial x^2} = 0, \qquad (5)$$
$$w|_{t=0} = w_0(x) = S_0(x).$$

For continuous initial data and under some reasonable restrictions on the growth of H and w_0, one can prove that there exists a unique smooth solution $w(t,x,h)$ of problem (5) and that the limit $w(t,x) = \lim_{h\to 0} w(t,x,h)$ exists and is continuous. Furthermore, it turns out that the solutions thus obtained are selected from the set of continuous functions satisfying the Hamilton–Jacobi equation almost everywhere by some conditions on the discontinuities of the derivatives. In some cases, these conditions have a natural physical interpretation. They also can be used as a definition of a generalized solution. However, *this method cannot be used to construct a reasonable theory of generalized solutions to (4) for discontinuous initial functions.* Furthermore, it is highly desirable to devise a theory of problems (4) on the basis of only intrinsic properties of HJB equations (i.e., regardless of the way in which the set of HJB equations is embedded in the set of higher-order equations). Such a theory, including solutions with discontinuous initial data, can be constructed for equations with convex Hamiltonians on the basis of idempotent analysis, using the new superposition principle for the solutions of (4) (which was first noted in [22]) and the idempotent analogue of the inner product

$$\langle f, g\rangle_P = \inf_x f(x) \odot g(x), \qquad (6)$$

replacing the usual L_2-product.

We discuss it now in more detail, defining generalized solutions for the case of a smooth function H. For nonsmooth H, a limit procedure can be easily applied.

Let H satisfy the following conditions.

(1) H is C^2 and the second derivatives of H are uniformly bounded:

$$\max\left(\sup_{x,p}\left\|\frac{\partial^2 H}{\partial x^2}\right\|, \sup_{x,p}\left\|\frac{\partial^2 H}{\partial x \partial p}\right\|, \sup_{x,p}\left\|\frac{\partial^2 H}{\partial p^2}\right\|\right) \leqslant c = \text{const}.$$

(2) H is strongly convex; that is, there exists a constant $\delta > 0$ such that the least eigenvalue of the matrix $\partial^2 H/\partial p^2$ is not less than δ for all (x,p).

By $(y(\tau,\xi,p_0), p(\tau,\xi,p_0))$ we denote the solution of Hamilton's system

$$\dot{y} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial x}$$

with the initial conditions $y(0) = \xi$, $p(0) = p_0$. Let $w(t,x,\xi)$ denote the greatest lower bound of the action functional $\int_0^t L(z(\tau), \dot{z}(\tau))\mathrm{d}\tau$ over all continuous piecewise smooth curves joining ξ with x in time t ($z(0) = \xi$ and $z(t) = x$). Here $L(x,v)$ is the Lagrangian of the variational problem associated with H, that is, the Legendre transform of H with respect to the variable p. One can show that under the given assumptions on H the two-point function $w(t,x,\xi)$ is smooth for all x, ξ and $t \in]0, t_0[$ with some t_0, and strictly convex in both ξ and x. It follows that if the initial function $w_0(x)$ in the Cauchy problem (4) is convex, then the function

$$(R_t^* w_0)(x) = w(t,x) = \min_{\xi}(w_0(\xi) + w(t,x,\xi)) \tag{7}$$

is continuously differentiable for all $t \leqslant t_0$ and $x \in \mathbb{R}^n$. Indeed, the minimum in (7) is obviously attained at a unique point $\xi(t,x)$. It follows then from the calculus of variation that (7) specifies the unique classical solution of the Cauchy problem (4).

To define generalized solution of this Cauchy problem with arbitrary initial data we proceed as follows. Smooth convex functions form a "complete" subset in $C_{\mathbb{0}}(\mathbb{R}^{2n})$, since they approximate the idempotent "δ-function"

$$\delta_{\xi}^{\min,+}(x) = \lim_{n\to\infty} n(x-\xi)^2 = \begin{cases} \mathbb{1} = 0, & x = \xi, \\ \mathbb{0} = +\infty, & x \neq \xi. \end{cases}$$

Consequently, each functional $\varphi \in (C_{\mathbb{0}}(\mathbb{R}^n))^*$ is uniquely determined by its values on this set of functions. The Cauchy problem

$$\frac{\partial w}{\partial t} + H\left(x, -\frac{\partial w}{\partial x}\right) = 0, \qquad w|_{t=0} = w_0(x) \tag{8}$$

with Hamiltonian $\widetilde{H}(x,p) = H(x,-p)$ will be called the *adjoint problem* to the Cauchy problem (4). This terminology is due to a simple observation that the classical resolving operator R_t^* of the Cauchy problem (8) is determined on smooth convex functions by the formula

$$(R_t^* w_0)(x) = \min_{\xi}(w_0(\xi) + w(t,\xi,x)) \tag{9}$$

is pseudo linear (with respect to the operations $\oplus = \min$ and $\odot = +$) on this set of functions, and is the adjoint of the resolving operator R_t (7) of the initial Cauchy problem with respect to the inner product (6). We are now in a position to define weak solutions of problem (4) by analogy with the theory of linear equations; we also take into account the fact that, by Theorem 2, the functionals $\varphi \in (C_{\mathbb{0}}(\mathbb{R}^n))^*$ are given by usual functions bounded below.

Let $w_0 : \mathbb{R}^n \to A = \mathbb{R} \cup \{+\infty\}$ be a function bounded below, and let $m_{w_0} \in (C_{\mathbb{0}}(\mathbb{R}^n))^*$ be the corresponding functional. Let us define the *generalized weak solution* of the Cauchy problem (4) as the function $(R_t w_0)(x)$ determined by the equation $m_{R_t w_0}(\psi) = m_{w_0}(R_t^*\psi)$, or equivalently $\langle R_t w_0, \psi\rangle_A = \langle w_0, R_t^*\psi\rangle_A$, for all smooth strictly convex functions ψ. The following theorem is a direct consequence of this definition, Theorem 2 and formulas (6), (7), (9).

Theorem 7. *Suppose that the Hamiltonian H satisfies the above-stated conditions* (1) *and* (2)*. For an arbitrary function $w_0(x)$ bounded below, the generalized weak solution of the Cauchy problem* (4) *exists and can be found according to the formula $(R_t w_0)(x) = \inf_{\xi}(w_0(\xi) + w(t,x,\xi))$. Various solutions have the same lower semicontinuous closure Cl, so the solution in the class of semicontinuous functions is unique and is given by the formula $Cl(R_t w_0) = R_t \mathrm{Cl} w_0$.*

There are many important nonlinear partial differential equations treated by pseudo-analysis, see [19,22,32,35,37]. One of the main problem in the application on nonlinear PDE is the identification of operations $\oplus$ and $\odot$. Goard and Broadbridge [9] have obtained a close connection with Lie symmetry algebras. There are a number of computer algorithms for finding Lie symmetry algebra and this can be used for finding $\oplus$ and $\odot$.

5.3. Cumulative prospect theory

In this section we shall consider general set functions $m : \mathscr{A} \to [0,1]$ which are monotone, called usually fuzzy measures. We suppose that $m(X) = 1$.

In the field of decision theory the cumulative prospect theory (CPT), introduced by Tverski and Kahneman [40], combines cumulative utility and a generalization of expected utility, so called sign dependent expected utility. CPT holds if there exist two fuzzy measures, m^+ and m^-, which ensure that the utility functional L, model for preference representation, can be represented by the difference of two Choquet integrals, i.e.,

$$L(f) = (C)\int f^{+}\,\mathrm{d}m^{+} - (C)\int f^{-}\,\mathrm{d}m^{-},$$

where $f^{+} = f \vee 0$ and $f^{-} = (-f) \vee 0$. It was proved by Narukawa [26] that comonotone-additive and monotone functional can be represented as a difference of two Choquet integrals and gave the conditions for which it can be represented by one Choquet integral, see [42].

It is well known that the Sugeno integral is one of the non-linear functional on the class of measurable functions which is comonotone-maxitive, monotone and $\wedge$-homogeneous [14,30]. An extension of the Sugeno integral in the spirit of the symmetric extension of Choquet integral proposed by Grabisch [11] is useful as a framework for cumulative prospect theory in an ordinal context. In the paper [34] we considered representation by two Sugeno integrals of the functional L defined on the class of functions $f : X \to [-1,1]$ on a finite set X. In the case of infinitely countable set X we obtain as a consequence of results on general fuzzy rank and sign dependent functionals that the symmetric Sugeno integral is comonotone-$\varovee$-additive functional on the class of functions with finite support. Let $f : X \to [-1,1]$ be a function on X with finite support. Consider the class of functions with finite support denoted by $\mathscr{K}_1(X)$:

$$\mathscr{K}_1(X) = \{f | f : X \to [-1,1], \mathrm{card}(\mathrm{supp}(f)) < \infty\},$$

where the support is given by $\mathrm{supp}(f) = \{x—f(x) \neq 0\}$. $\mathscr{K}_1^{+}(X)$ and $\mathscr{K}_1^{-}(X)$ denote the class of non-negative and non-positive functions with finite support, respectively.

Recall that two measurable functions f and g on X are called *comonotone* if they are measurable with respect to the same chain $\mathscr{C}$ in $\mathscr{A}$. Recall that equivalently, comonotonicity of the functions f and g can be expressed as follows: $f(x) < f(x_1) \Rightarrow g(x) \leqslant g(x_1)$ for all $x, x_1 \in X$.

The *symmetric maximum* $\varovee$: $[-1,1]^2 \to [-1, 1]$, originally introduced by Grabisch [12], is defined by

$$a \varovee b = \begin{cases} -(|a| \vee |b|), & b \neq -a \text{ and } |a| \vee |b| = -a \text{ or } = -b, \\ 0, & b = -a, \\ |a| \vee |b|, & \text{otherwise.} \end{cases}$$

The *symmetric minimum* $\varowedge$: $[-1,1]^2 \to [-1,1]$, introduced also by Grabisch [12], is defined by

$$a \varowedge b = \begin{cases} -(|a| \wedge |b|), & \text{sign } a \neq \text{sign } b, \\ |a| \wedge |b|, & \text{otherwise.} \end{cases}$$

We have

$$a \varovee b = (|a| \vee |b|)\mathrm{sign}(a + b), \quad a \varowedge b = (|a| \wedge |b|)\mathrm{sign}(a \cdot b).$$

Let f and g be two functions defined on X with values in $[-1,1]$. Then, we define functions $f \varovee g$ and $f \varowedge g$ for any $x \in X$ by $(f \varovee g)(x) = f(x) \varovee g(x)$, $(f \varowedge g)(x) = f(x) \varowedge g(x)$, and for any $a \in [0,1]$ we have $(a \varowedge f)(x) = a \varowedge f(x)$.

Due to non-associativity of the operation $\varovee$ on $[-1,1]$, it cannot be used directly as n-ary operator. The expression $\varovee_{i=1}^{n} a_i$ is unambiguously defined iff $\vee_{i=1}^{n} a_i \neq -\wedge_{i=1}^{n} a_i$. If equality occurs, several *rules of computation* can ensure uniqueness ([12]):

(1) Put $\varovee_{i=1}^{n} a_i = 0$. This rule is defined by

$$\left\lfloor \mathop{\varovee}_{i=1}^{n} a_i \right\rfloor = \left(\mathop{\varovee}_{a_i \geqslant 0} a_i\right) \varovee \left(\mathop{\varovee}_{a_i < 0} a_i\right) = \left(\bigvee_{a_i \geqslant 0} a_i\right) \varovee \left(\bigwedge_{a_i < 0} a_i\right).$$

(2) Discard all occurrences of $\vee_{i=1}^{n} a_i$ and $-\wedge_{i=1}^{n} a_i$ and continue with the reduced list of inputs, until the condition $\vee_{i=1}^{n} a_i \neq -\wedge_{i=1}^{n} a_i$ is satisfied. We denote this rule by $\langle \varovee_{i=1}^{n} a_i \rangle$.

We refer the reader to [12] for a detailed study of the properties of the introduced rules.

Definition 7 ([11,30]). Let m be a fuzzy measure on the measurable space $(X, \mathscr{A})$.

(i) *The Sugeno integral* of $f \in \mathscr{K}_1^+(X)$ with respect to m is defined by

$$(S)\int f\,\mathrm{d}m = \bigvee_{\alpha\in[0,1]} \alpha \wedge m(\{x|f(x) \geqslant \alpha\}).$$

(ii) *The symmetric Sugeno integral* of $f \in \mathscr{K}_1(X)$ with respect to m is defined by

$$Ⓢ\int f\,\mathrm{d}m = \left((S)\int f^+\,\mathrm{d}m\right) \circledvee \left(-(S)\int f^-\,\mathrm{d}m\right),$$

where $f^+ = f \vee 0$ and $f^- = (-f) \vee 0 = -(f \wedge 0)$.

Further, when X is a finite set, i.e., $X = \{x_1, \cdots, x_n\}$, the Sugeno integral of a function $f \in \mathscr{K}_1^+(X)$ with respect to m can be written as

$$(S)\int f\,\mathrm{d}m = \bigvee_{i=1}^{n} f_{\alpha(i)} \wedge m(A_{\alpha(i)}),$$

where f has a comonotone maxitive representation $f = \bigvee_{i=1}^n f_{\alpha(i)} \wedge \mathbf{1}_{A_{\alpha(i)}}$ for $\alpha = (\alpha(1), \alpha(2), \ldots, \alpha(n))$ a permutation of index set $\{1, 2, \ldots, n\}$ such that $0 \leqslant f_{\alpha(1)} \leqslant \cdots \leqslant f_{\alpha(n)} \leqslant 1$ and $A_{\alpha(i)} = \{x_{\alpha(i)}, \ldots, x_{\alpha(n)}\}$, $f_i = f(x_i)$ and $\mathbf{1}_A$ denotes characteristic function of the crisp subset A of X. The symmetric Sugeno integral of a function $f \in \mathscr{K}_1(X)$ can be considered as it is proposed in Grabisch [11]

$$Ⓢ^1\int f\,\mathrm{d}m = \left\langle \left(\mathop{\circledvee}_{i=1}^{s} f_{\alpha(i)} \owedge m\left(\{x_{\alpha(1)}, \ldots, x_{\alpha(i)}\}\right)\right) \circledvee \left(\mathop{\circledvee}_{i=s+1}^{n} f_{\alpha(i)} \owedge m\left(\{x_{\alpha(i)}, \ldots, x_{\alpha(n)}\}\right)\right)\right\rangle, \tag{10}$$

where α is a permutation of index set such that $-1 \leqslant f_{\alpha(1)} \leqslant \cdots \leqslant f_{\alpha(s)} < 0$ and $0 \leqslant f_{\alpha(s+1)} \leqslant \cdots \leqslant f_{\alpha(n)} \leqslant 1$. More details about the symmetric Sugeno integral can be found in [11,12].

Distributivity of the operation $\owedge$ w.r.t $\circledvee$ does not hold in general. If we expect that distributivity is satisfied for $a, b \geqslant 0$ and $c \leqslant 0$, we have to suppose some additional conditions as in the next result, see [34].

Proposition 1. *Let $a, b \geqslant 0$ and $c \leqslant 0$. If $a \owedge b \neq a \owedge (-c)$ then*

$$a \owedge (b \circledvee c) = (a \owedge b) \circledvee (a \owedge c).$$

Further discussion of the distributivity can be found in [11].

Note that any function $f : X \to [-1, 1]$ can be represented by symmetric maximum of two comonotone functions $f^+ \geqslant 0$ and $-f^- \leqslant 0$, i.e., $f = f^+ \circledvee (-f^-)$.

Now we extend the notion of the symmetric Sugeno integral, see [34].

Definition 8. A functional $L : \mathscr{K}_1(X) \to [-1, 1]$ is a *fuzzy rank and sign dependent functional* (*f.r.s.d.*) on $\mathscr{K}_1(X)$ if there exist two fuzzy measures m^+ and m^- such that for all $f \in \mathscr{K}_1(X)$

$$L(f) = \left((S)\int f^+\,\mathrm{d}m^+\right) \circledvee \left(-(S)\int f^-\,\mathrm{d}m^-\right).$$

Note that in the case when $m^+ = m^-$ the fuzzy rank and sign dependent functional (f.r.s.d. functional for short) is exactly the symmetric Sugeno integral. If a f.r.s.d. functional L is the symmetric Sugeno integral then we have

$$L(-f) = -L(f).$$

Definition 9. Let $L : \mathscr{K}_1(X) \to [-1, 1]$, be a functional on $\mathscr{K}_1(X)$.

(i) L is *comonotone-$\circledvee$-additive* if $L(f \circledvee g) = L(f) \circledvee L(g)$ for all comonotone functions $f, g \in \mathscr{K}_1(X)$.
(ii) L is *monotone* if $f \leqslant g \Rightarrow L(f) \leqslant L(g)$ for all functions $f, g \in \mathscr{K}_1(X)$.

(iii) L is positive $\textcircled{\wedge}$-*homogeneous* if $L(a \textcircled{\wedge} f) = a \textcircled{\wedge} L(f)$ for all $f \in \mathscr{K}_1(X)$ and $a \in [0,1]$.

(iv) L is weak $\textcircled{\wedge}$-*homogeneous* if

$$L(a \textcircled{\wedge} \mathbf{1}_A) = a \textcircled{\wedge} L(\mathbf{1}_A) \quad \text{and} \quad L(a \textcircled{\wedge} (-\mathbf{1}_A)) = a \textcircled{\wedge} L(-\mathbf{1}_A)$$

for all $a \in [0,1]$ and $A \subseteq X$.

Weak $\textcircled{\wedge}$-homogeneity does not imply positive $\textcircled{\wedge}$-homogeneity in general.

Example 2. Let $X = \{1,2\}$ and $f : X \to [-1,1]$. Let L be a functional on $\mathscr{K}_1(X)$ defined by

$$L(f) = f(1) \textcircled{\vee} f(2).$$

For all $a \in [0,1]$, and $\emptyset \neq A \subseteq X$ we have $L(a \textcircled{\wedge} (\mathbf{1}_A)) = a = a \textcircled{\wedge} L(\mathbf{1}_A)$ and $L(a \textcircled{\wedge} (-\mathbf{1}_A)) = -a = a \textcircled{\wedge} L(-\mathbf{1}_A)$. Therefore L is weak $\textcircled{\wedge}$-homogeneous functional on $\mathscr{K}_1(X)$. It is not positive $\textcircled{\wedge}$-homogeneous, e.g., for f defined by $f(1) = 0.5$ and $f(2) = -0.8$ and $a = 0.3$ we have $L(0.3 \textcircled{\wedge} f) = 0.3 \textcircled{\vee} (-0.3) = 0$ and $0.3 \textcircled{\wedge} L(f) = 0.3 \textcircled{\wedge} (0.5 \textcircled{\vee} (-0.8)) = -0.3$.

Remark 2. Note that the Sugeno integral with respect to a fuzzy measure μ is a *comonotone-$\textcircled{\vee}$-additive* functional which maps $\mathscr{K}_1^+(X)$ into $[0,1]$. This implies that for all comonotone functions $f, g \in \mathscr{K}_1(X)$ we have

$$(S)\int (f^+ \textcircled{\vee} g^+)\mathrm{d}m = \left((S)\int f^+\,\mathrm{d}m\right) \textcircled{\vee} \left((S)\int g^+\,\mathrm{d}m\right)$$

and an analogous equality holds for f^- and g^-.

Theorem 8. *Let L be a comonotone-$\textcircled{\vee}$-additive, weak $\textcircled{\wedge}$-homogeneous and monotone functional, such that for all $f \in \mathscr{K}_1(X)$, we have for $f \neq 0$ that $L(f) \neq 0$, then L is positive $\textcircled{\wedge}$-homogeneous on $\mathscr{K}_1(X)$.*

In the case of finite set X and $\mathscr{K}_1(X)$ class of functions $f: X \to [-1,1]$ we have the next result.

Theorem 9. *Let X be a finite set. If $L : \mathscr{K}_1(X) \to [-1,1]$ is a comonotone-$\textcircled{\vee}$-additive, weak $\textcircled{\wedge}$-homogeneous and monotone functional on $\mathscr{K}_1(X)$, then L is a f.r.s.d functional, i.e., there exist two fuzzy measures m_L^+ and m_L^- such that*

$$L(f) = \left((S)\int f^+\,\mathrm{d}m_L^+\right) \textcircled{\vee} \left(-(S)\int f^-\,\mathrm{d}m_L^-\right).$$

A f.r.s.d. functional on $\mathscr{K}_1(X)$, where X is a finite set, is not always comonotone-$\textcircled{\vee}$-additive.

Theorem 10. *Let X be an infinitely countable set. If $L : \mathscr{K}_1(X) \to [-1,1]$ is a f.r.s.d functional such that $L(f) \neq 0$, for all $f \in \mathscr{K}_1(X), f \neq 0$, then it is a comonotone-$\textcircled{\vee}$-additive functional on the set $\mathscr{K}_1(X)$.*

Corollary 1. *Let X be an infinitely countable set. The symmetric Sugeno integral is comonotone-$\textcircled{\vee}$-additive functional on the class of functions*

$$\left\{f \in \mathscr{K}_1(X) | \textcircled{S}\int f\,\mathrm{d}m \neq 0\right\}.$$

The symmetric Sugeno integral is not comonotone-$\textcircled{\vee}$-additive on the whole class $\mathscr{K}_1(X)$.

Example 3. Let $X = \{x_1, x_2, \ldots\}$. We take a fuzzy measure m defined by

$$m(A) = \begin{cases} 1 & \text{if } A \neq \emptyset, \\ 0 & \text{if } A = \emptyset. \end{cases}$$

We consider a f.r.s.d. functional L defined by

$$L(f) = \left((S)\int f^+\,\mathrm{d}m\right) \textcircled{\vee} \left(-(S)\int f^-\,\mathrm{d}m\right).$$

If we take comonotone functions $f, g \in \mathscr{K}_1(X)$ with supports $\text{supp}(f) = \{x_1\}$ and $\text{supp}(g) = \{x_1, x_2\}$, defined by $f(x_1) = 0.5$ and $g(x_1) = 0.5$, $g(x_2) = -0.5$, respectively, then $L(f) = 0.5$ and $L(g) = 0$, but $L(f \bigcirc\!\!\!\!\vee\, g) = 0 \neq 0.5 = L(f) \bigcirc\!\!\!\!\vee\, L(g)$.

6. Conclusions

We have presented a part of the theory of generalized real analysis, called pseudo-analysis. There is given a short overview of some important applications in different fields. Still there are many other important fields of applications on which this new approach shades quite different lights.

We mention here only the famous Black–Sholes (BS) and Cox–Ross–Rubinstein (CRR) formulas are basic results in the modern theory of option pricing in financial mathematics. They are usually deduced by means of stochastic analysis; various generalizations of these formulas were proposed using more sophisticated stochastic models for common stocks pricing evolution. The systematic deterministic approach to the option pricing leads to a different type of *generalizations* of BS and CRR formulas characterized by more rough assumptions on common stocks evolution (which are therefore easier to verify). This approach reduces the analysis of the option pricing to the study of certain *pseudo homogeneous* (with respect to $\odot = +$) *nonexpansive maps,* which however are "strongly" infinite dimensional: they act on the spaces of functions defined on sets, which are not (even locally) compact. In the paper [18] there were obtained generalizations of the standard CRR and BS formulas can be obtained using the deterministic (actually game-theoretic) approach to option pricing. A class of pseudo homogeneous nonexpansive maps appear in these formulas, considering first a simplest model of financial market with only two securities in discrete time, then its generalization to the case of several common stocks, and then the continuous limit.

Therefore it is important to examine the infinite dimensional generalization of the theory of pseudo homogeneous nonexpansive maps (which does not exists at the moment), which would have direct applications to the analysis of derivative securities pricing. On the other hand, this approach, which uses neither martingales nor stochastic equations, makes the whole apparatus of the standard game theory appropriate for the study of option pricing.

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