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Decomposition Theorems for Partial Isometries

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PRELIMINARIES

N. Dunford in [1] and [2] introduced the concept of spectral operators on a complex Banach space, as an extension of the spectral theory to bounded[1] operators with resolution of the identity E.

A main result of the bounded case [2, Theorem 8] is that T is a bounded spectral operator iff

$$T = S + N,$$

where S is a bounded scalar type operator, $S = \int \lambda \, dE_\lambda$, and N is a quasi-nilpotent operator, i.e., $\lim_n \| N^n \|^{1/n} = 0$, commuting with S. Moreover, such decomposition is unique. The terms S and N are called the scalar and the radical part, respectively, of the spectral operator T.

C. Apostol [5] gave a condition for a bounded linear operator T on a Hilbert space H, satisfying condition

$$\lim_n \| T^*T^n - T^nT^* \|^{1/n} = 0, \tag{1}$$

to be spectral.

S.K. Berberian [6], defining a certain extension of a Hilbert space H to a Hilbert space K, reduced the problem of the approximate point spectrum of an operator T on H to the point spectrum problem of the corresponding oper-

[1] Unbounded spectral operators were considered by W. G. Bade [3] and J. Schwartz [4].

ator T' on K. The extension K of H is formed by the bounded sequences of vectors from H, and the inner product is defined with the help of a Banach limit.

Let H be a complex Hilbert space. Vectors $x \in H$ of unit norm, i.e., $\| x \| = 1$, will be referred to as unit vectors.

For an operator $T : H \to H$, $R(T)$, $\operatorname{Ker}(T)$, $\sigma(T)$, $\pi(T)$, $\alpha(T)$, and $\rho(T)$ denote the range, the kernel, the spectrum, the point spectrum, the approximate point spectrum, and the resolvent set, respectively. $\lambda \in \alpha(T)$ if there exists a sequence (x_n) of unit vectors, such that

$$\lim_n \|(\lambda - T)\, x_n \| = 0.$$

Let $B(H)$ represent the bounded linear operators on H. For each $x \in H$, $(\lambda - T)^{-1} x$ is analytic in λ defined on $\rho(T)$ with values in H. Dunford [2, Theorem 2] showed that, if T is spectral, the function $(\lambda - T)^{-1} x$ has a unique maximal single-valued analytic extension. The domain of this extension is denoted by $\rho(x)$ and its spectrum $\sigma(x)$ is defined as the complement of $\rho(x)$.

If a linear manifold $S \subset H$ decomposes T then $T \mid S$ stands for the restriction of T to S.

Let

$$H_0 = \{x : \lim_n \| T^n x \|^{1/n} = 0\}, \tag{2}$$

a linear manifold in H. Clearly, H_0 is invariant under T.

PROPOSITION 1 (Apostol). *Let $T \in B(H)$ satisfy condition* (1). *Then H_0 is invariant under T^*, i.e. $T^* H_0 \subset H_0$.*

PROPOSITION 2 (Apostol). *Let $T \in B(H)$ satisfy condition* (1). *Then $\overline{H}_0$ and $H_0^{\perp}$ reduce T and $T \mid H_0^{\perp}$ is normal.*

PROPOSITION 3 (Apostol). *Let $T \in B(H)$ satisfy condition* (1). *T is spectral with its scalar part normal if the manifold H_0 is closed.*

PROPOSITION 4 (Berberian). *There exist a Hilbert space K and linear mappings $\varphi : H \to K$, $\theta : B(H) \to B(K)$, such that:*

(a) $(\varphi x, \varphi y) = (x, y)$, *for every* $x, y \in H$;

(b) $\theta T = T'$ *is a *-algebra isometry;*

(c) $T' \varphi x = \varphi T x$, *for each* $x \in H$, *and* $T \in B(H)$;

(d) $\alpha(T) = \alpha(T') = \pi(T')$.

A defining condition for a partial isometry T on a Hilberts space H is

$$T = TT^*T. \tag{3}$$

The adjoint T^* of a partial isometry T is itself a partial isometry with the initial space and final space interchanged. Furthermore (e.g. [7]), the norm of a nonzero partial isometry is one and its spectrum is included in the closed unit disk.

In this paper we give a necessary and sufficient condition that H_0 be closed, and show that every partial isometry subject to condition (1) is spectral. Finally, we give a decomposition for a larger class of partial isometries.

Some Properties of The Resolvents of an Operator

An exposition of analytic functions with values in a complex Banach space is given in [8, Chapter 9].

Let X, $B(X)$, and $\mathbb{C}$ denote a complex Banach space, the bounded linear operators on X, and the complex plane, respectively.

For spectral operators, the linear manifold H_0, as defined by (2), is determined by the resolution of the identity E.

Theorem 1. *If $T \in B(X)$ is a spectral operator, then $H_0 = E((0))\,X$, and thus it is closed.*

Proof. By [2, Theorems 3 and 4],

$$E((0))\,X = \{x : x = 0 \text{ or } \sigma(x) = (0)\}.$$

If $0 \neq x \in H_0$, then

$$R_\lambda x = \sum_{n=1}^{\infty} \lambda^{-n} T^{n-1} x \tag{4}$$

is an analytic function defined on $\mathbb{C} - (0)$ such that

$$(\lambda - T)\,R_\lambda x = x, \tag{5}$$

and

$$R_\lambda x = (\lambda - T)^{-1}\,x, \qquad \lambda \in \rho(T). \tag{6}$$

Thus we have

$$x \in E((0))\,X.$$

Conversely, suppose $\sigma(x) = (0)$. Then there is an analytic function $R_\lambda x$

defined on $\mathbb{C} - (0)$ such that (5) and (6) hold. Since $\mathbb{C} - (0)$ is an annulus, the uniqueness of the Laurent expansion for $R_\lambda x$ shows that series (4) converges for every $\lambda \neq 0$. This implies that

$$\lim_n \| T^n x \|^{1/n} = 0,$$

and hence $x \in H_0$.

THEOREM 2. *Suppose that $T \in B(X)$, H_0 is dense in X, and* 0 *is not a limit point of $\sigma(T)$. Then T is quasinilpotent.*

Proof. Since 0 is not a limit point of $\sigma(T)$ there is an $r > 0$ such that $(\lambda - T)^{-1} = R_\lambda$ is analytic in the annulus $0 < |\lambda| < r$. Let

$$R_\lambda = \sum_{n=1}^{\infty} c_n \lambda^{-n} + \sum_{n=0}^{\infty} d_n \lambda^n$$

be its Laurent expansion.

Suppose that $x \in H_0$. Then

$$f(\lambda) = \sum_{n=1}^{\infty} \lambda^{-n} T^{n-1} x$$

is an analytic function on $\mathbb{C} - (0)$ such that

$$(\lambda - T) f(\lambda) = x.$$

Let $0 < |\lambda| < r$. Then

$$(\lambda - T) f(\lambda) = x \qquad \text{and} \qquad (\lambda - T) R_\lambda x = x$$

which implies that

$$f(\lambda) = R_\lambda x$$

since $(\lambda - T)^{-1}$ exists. Hence

$$f(\lambda) = R_\lambda x = \sum_{n=1}^{\infty} c_n(x) \lambda^{-n} + \sum_{n=0}^{\infty} d_n(x) \lambda^n$$

and by the uniqueness of Laurent series we get

$$d_n(x) = 0, \quad n \geqslant 0 \qquad \text{and} \qquad c_n(x) = T^{n-1} x, \quad n \geqslant 1.$$

Since this is true for each x in H_0 and H_0 is dense we get

$$d_n = 0 \qquad \text{and} \qquad c_n = T^{n-1}.$$

Hence

$$R_\lambda = \sum_{n=1}^{\infty} \frac{1}{\lambda^n} T^{n-1} = \frac{1}{\lambda} \sum_{n=1}^{\infty} \left(\frac{T}{\lambda}\right)^{n-1}, \qquad 0 < |\lambda| < r.$$

This implies that $\lim_n \| T^n \|^{1/n} = 0$, thus T is quasinilpotent.

COROLLARY 1. *Let H be a Hilbert space and T satisfy* (1). *If* 0 *is an isolated point of* $\sigma(T \mid \bar{H}_0)$, *then T is a spectral operator.*

Proof. By Proposition 2, $H_0^\perp$ and $\bar{H}_0$ decompose T, and $T \mid H_0^\perp$ is normal. The second term $T \mid \bar{H}_0$ satisfies the premises of foregoing Theorem 2.

A SPECTRAL DECOMPOSITION

Let $\mathscr{S}(H)$ denote all partial isometries on a Hilbert space H, which satisfy condition (1), i.e.,

$$\lim_n \| T^*T^n - T^nT^* \|^{1/n} = 0.$$

Some properties will be obtained for partial isometries which satisfy the weaker pointwise condition:

$$\lim_n \|(T^*T^n - T^nT^*)\, x \|^{1/n} = 0, \qquad \text{for every} \qquad x \in H. \tag{1'}$$

Let Γ denote the unit circle of the complex plane, i.e.,

$$\Gamma = \{\lambda : |\lambda| = 1\}.$$

THEOREM 3. *The point spectrum of $T \in \mathscr{S}(H)$ is a subset of the union of the unit circle Γ and the singleton* (0),

$$\pi(T) \subset \Gamma \cup (0). \tag{7}$$

Proof. The proof can be given under the more general condition (1'). Let

$$0 \neq \lambda \in \pi(T).$$

There exists a nonzero vector x, such that

$$Tx = \lambda x, \qquad x = \lambda^{-1} Tx,$$

$$T^n x = \lambda^n x, \qquad \text{for every natural number } n. \tag{8}$$

Since condition (1′) is satisfied for every $x \in H$, with the help of (3) and relations (8), we have successively:

$$0 = \lim_n \| T^*T^n x - T^n T^* x \|^{1/n} = \lim_n \| \lambda^n T^* x - \lambda^{-1} T^n T^* T x \|^{1/n}$$

$$= \lim_n \| \lambda^n T^* x - \lambda^{-1} T^n x \|^{1/n} = | \lambda | \lim_n \| T^* x - \lambda^{-1} x \|^{1/n}.$$

It follows

$$T^* x = \lambda^{-1} x, \qquad \text{thus} \qquad \lambda^{-1} \in \pi(T^*).$$

Since we must simultaneously have $| \lambda | \leqslant 1$, $| \lambda^{-1} | \leqslant 1$, it follows $| \lambda | = 1$, and this completes the proof.

From the ergodic theorem it follows that the eigenmanifold $\operatorname{Ker}(\lambda - T)$ of any nonzero eigenvalue λ of $T \in \mathscr{S}(H)$, splits off from the entire space as a direct summand,

$$H = \operatorname{Ker}(\lambda - T) \oplus \overline{R(\lambda - T)},$$

and $\operatorname{Ker}(\lambda - T)$ can be obtained as $R(T(\lambda))$, where

$$T(\lambda)\, x = \lim_n n^{-1} \sum_{m=1}^{n} \left(\frac{T}{\lambda} \right)^m x.$$

In particular, if

$$T^* T^m - T^m T^* = 0, \qquad \text{for some natural } m,$$

T^m is normal, and $H_0 = \operatorname{Ker}(T^m)$. Thus H_0 is closed and according to Apostol's results [5, 9], T is a (finite) m-type spectral operator:

$$T = U \oplus N \tag{9}$$

with U unitary and N nilpotent of index ($\leqslant m$).

THEOREM 4. *The approximate point spectrum $\alpha(T)$ of $T \in \mathscr{S}(H)$ is a subset of the union of the unit circle Γ and the zero,*

$$\alpha(T) \subset \Gamma \cup (0). \tag{10}$$

Proof. By Proposition 4, Berberian's transform T' and its adjoint T'^* satisfy the same operator equations, in particular (1) and (3), as T and T^* do. Thus $T' \in \mathscr{S}(K)$, and then by Theorem 3,

$$\pi(T') \subset \Gamma \cup (0).$$

Moreover, by property (d) of Proposition 4,

$$\alpha(T) = \pi(T'),$$

and this completes the proof.

For $T \in \mathscr{S}(H)$ we shall show that property (10) holds for the entire spectrum. An immediate consequence of (10) is the following

THEOREM 5. *Let $T \in B(H)$ and*

$$\alpha(T) \subset \Gamma \cup (0).$$

Then

$$\sigma(T) \subset \Gamma \cup (0) \qquad or \qquad \sigma(T) = \{\lambda : |\lambda| \leqslant 1\}.$$

Proof. Consider

$$M = \{\lambda : 0 < |\lambda| < 1\}$$

as a topological space. Then M is connected. Since $\sigma(T)$ is closed $\sigma(T) \cap M$ is closed in M. Let $x \in \sigma(T) \cap M$ and write ∂ for the boundary. Since $\partial\sigma(T) \subset \alpha(T)$ and $\alpha(T) \cap M = \phi$, x is an interior point of $\sigma(T)$. It follows that x is an interior point of $\sigma(T) \cap M$ in M. Thus $\sigma(T) \cap M$ is both open and closed in M and hence

$$\sigma(T) \cap M = \phi \qquad \text{or} \qquad \sigma(T) \cap M = M.$$

Let $T \in B(H)$, and for every nonzero scalar λ, define $R_\lambda : H_0 \to \bar{H}_0$, as in (4).

LEMMA 1. *T is quasinilpotent, i.e.,*

$$\lim_n \| T^n \|^{1/n} = 0, \tag{11}$$

iff $H_0 = H$.

Proof. Clearly, if (11) holds, then $H_0 = H$. Conversely, if $H_0 = H$, R_λ is an algebraic inverse of $(\lambda - T)$ and, by the closed graph theorem, it is in $B(H)$. Hence $\sigma(T) = (0)$. Since the left-hand side of (11) is the spectral radius, it is zero.

THEOREM 6. *If H_0 is dense, i.e., $\bar{H}_0 = H$, then*

$$\sigma(T) - (0) = \alpha(T) - (0). \tag{12}$$

Proof. Let $0 \neq \lambda \in \sigma(T)$. Then R_λ is not bounded for otherwise it could be extended to $(\lambda - T)^{-1}$. Thus there is a sequence of unit vectors $(x_n) \in H_0$ such that

$$\| R_\lambda x_n \| > n, \qquad \text{for every } n. \tag{13}$$

Define

$$y_n = \frac{R_\lambda x_n}{\| R_\lambda x_n \|}.$$

Then, by (5),

$$(\lambda - T) y_n = \frac{x_n}{\| R_\lambda x_n \|},$$

and by condition (13),

$$\|(\lambda - T) y_n \| < \frac{1}{n} \to 0$$

and hence $\lambda \in \alpha(T)$. It follows $\sigma(T) - (0) \subset \alpha(T) - (0)$, but the approximate point spectrum is a part of the entire spectrum, thus (12) follows.

Now we are in a position to state the following

THEOREM 7. *Every $T \in \mathscr{S}(H)$ is a spectral partial isometry.*

Proof. Let

$$R = T \mid \bar{H}_0 .$$

First observe that

$$R \in \mathscr{S}(\bar{H}_0),$$

since $\bar{H}_0$ reduces T and T^*.

Next, by Theorems 4 and 6, we have

$$\sigma(R) \subset \Gamma \cup (0).$$

Thus, 0 is an isolated point of the spectrum and then, by Corollary 1, T is a spectral operator.

In the particular case: $H_0 = H$, by Lemma 1, $R = T$ is quasinilpotent, yet still spectral.

An example of a quasinilpotent partial isometry which is not nilpotent is given by the operator matrix

$$T = \begin{bmatrix} AU & 0 \\ (I - A^2)^{1/2}\, U & 0 \end{bmatrix}$$

where U is a unilateral shift, i.e., $Ue_i = e_{i+1}$, (e_i), $(i > 0)$ being an orthonormal basis, and A is a diagonal contraction:

$$\| A \| \leqslant 1, \qquad Ae_i = (\tfrac{1}{2})^{i^2} e_i .$$

COROLLARY 2. *Every* $T \in \mathscr{S}(H)$ *is the direct sum of a unitary and a quasinilpotent operator:* $T = U \oplus N$.

Proof. According to Proposition 2, $U = T \mid H_0^{\perp}$ is a normal partial isometry and hence the direct sum of a unitary and a zero operator. But the zero term of U is included in the quasinilpotent part $N = T \mid H_0$.

There remains the class of spectral partial isometries which are not in $\mathscr{S}(H)$. We will consider a class of partial isometries for which a similar decomposition is attainable. Whether such class is or is not spectral depends on the nonunitary part of the decomposition.

Halmos and McLaughlin [7, Theorem 2], and Halmos [10] showed that any closed subset of the unit disk which contains 0 is the spectrum of a partial isometry, by considering operator matrices of the form:

$$T = \begin{bmatrix} A & (I - AA^*)^{1/2} \\ 0 & 0 \end{bmatrix}, \qquad \text{with} \qquad \| A \| \leqslant 1, \quad Ae_i = \xi_i e_i , \tag{14}$$

i.e., A is a diagonal contraction and (ξ_i) is a sequence of complex numbers in the unit disk.

If there is an $\epsilon > 0$, such that

$$\epsilon \leqslant | \xi_i | \leqslant 1, \qquad \text{for every } i,$$

then it can be shown that T is a scalar partial isometry, i.e.,

$$T = \int \lambda \, dE_\lambda$$

for some resolution of the identity E_λ.

Thus if σ is any closed subset of the unit disk containing 0 as an isolated point then it is the spectrum of a spectral partial isometry.

QUASICOMMUTING PARTIAL ISOMETRIES

We now weaken the basic condition (1).

DEFINITION. We call a partial isometry T on a Hilbert space H, quasicommuting, if it satisfies following condition

$$\lim_n \| T^*T^n - T^nT^* \| = 0. \tag{15}$$

Some properties will be obtained for quasicommuting partial isometries satisfying, instead of (15), the pointwise condition

$$\lim_n \|(T^*T^n - T^nT^*)\, x\| = 0, \qquad \text{for every} \qquad x \in H. \tag{15'}$$

For the sake of simplicity we shall denote by $Q(H)$ the class of quasicommuting partial isometries on H.

For any $T \in B(H)$, let

$$H_1 = \{x : \lim_n \| T^n x \| = 0\}$$

be a manifold in H. Some useful properties of H_1 now follow.

LEMMA 2. *If there is a number M such that $\| T^n \| < M$ for each n, then H_1 is closed.*

Proof. Choose (x_m) in H_1 such that $x_m \to x$. We have

$$| \| T^n x_m \| - \| T^n x \| | \leqslant \| T^n x_m - T^n x \| \leqslant M \| x_m - x \| .$$

Let $\epsilon > 0$. We can find an m such that

$$\| x_m - x \| < \frac{\epsilon}{2M},$$

and an N such that $n \geqslant N$ implies

$$\| T^n x_m \| < \frac{\epsilon}{2} .$$

Then it follows

$$\| T^n x \| < \epsilon, \qquad \text{for} \qquad n \geqslant N.$$

Under the conditions of Lemma 2, $\overline{H}_0 \subset H_1$.

THEOREM 8. *If T is any bounded operator satisfying condition* (15), *then H_1 enjoys the following properties:*

(i) $TH_1 \subset H_1$, $T^*H_1 \subset H_1$;

(ii) $(TT^* - T^*T)\, H \subset H_1$;

(iii) $T \mid H_1^{\perp}$ *is normal.*

Proof. We follow the outline of similar proofs given by C. Apostol [5].

(i) The invariance of H_1 under T^* holds under the more general condition (15′). We start from the identity

$$T^n T^* x = (T^n T^* - T^* T^n)\, x + T^* T^n x, \qquad x \in H_1,$$

next, we take norms

$$\| T^n T^* x \| \leqslant \|(T^n T^* - T^* T^n)\, x \| + \| T^* \| \cdot \| T^n x \|,$$

and when $n \to \infty$, by condition (15′), we obtain

$$\lim_n \| T^n(T^* x)\| \leqslant \lim_n \|(T^n T^* - T^* T^n)\, x \| + \| T^* \| \lim_n \| T^n x \| = 0.$$

(ii) Let $x \in H$,

$$T^n(TT^* - T^*T)\, x = (T^{n+1} T^* - T^* T^{n+1})\, x + (T^* T^n - T^n T^*)\, Tx,$$

take norms and, at limit, obtain

$$\lim_n \| T^n(TT^* - T^*T)\, x \|$$

$$\leqslant \lim_n \|(T^{n+1} T^* - T^* T^{n+1})\, x \| + \lim_n \|(T^* T^n - T^n T^*)\| \cdot \| Tx \|.$$

Finally, condition (15) gives

$$\lim_n \| T^n(TT^* - T^*T)\, x \| = 0.$$

(iii) $H_1^{\perp}$, invariant under both T and T^*, is invariant under $TT^* - T^*T$. For every $x \in H_1^{\perp}$ we have

$$(TT^* - T^*T)\, x \in H_1^{\perp}.$$

On the other hand, by (ii),

$$(TT^* - T^*T)\, x \in H_1,$$

and hence

$$(TT^* - T^*T)\, x \in H_1^{\perp} \cap H_1 = (0).$$

COROLLARY 3. *A quasicommuting partial isometry T is decomposed by $H_1^{\perp}$ and H_1 in the direct sum*

$$T = U \oplus V, \tag{16}$$

where U is unitary and V satisfies

$$\lim_n \| V^n x \| = 0, \qquad \text{for every } x \text{ in the domain of } V.$$

Proof. By Lemma 2, $\overline{H}_1 = H_1$, and then properties (i) and (iii), of foregoing Theorem 8, accomplish the decomposition of T. $U = T \mid H_1{}^\perp$ is a normal partial isometry with its zero part included in $V = T \mid H_1$, thus unitary.

We observe that T decomposed as in (16) is not necessarily spectral.

A partial isometry $T \in Q(H)$ such that $H_1 = H$, but $\| T^n \| = 1$ for every n is given by (14) if A is a diagonal contraction with multipliers (ξ_i), and $\lim_i \xi_i = 1$.

THEOREM 9. *Let T be a partial isometry satisfying condition* (15′). *If*

$$R(T) \subset R(T^*) \qquad or \qquad R(T^*) \subset R(T),$$

then T is normal.

Proof. We confine the proof to the case

$$R(T) \subset R(T^*). \tag{17}$$

Since by condition (17), T^n is a partial isometry with initial space $R(T^*)$, for any n (see [11]), we have

$$\| T^n x \| = \| Tx \|, \qquad \| T^n T^* x \| = \| T^* x \|,$$

and (18)

$$\| T^* T^n x \| = \| Tx \|, \qquad x \in H.$$

From the identities

$$T^* T^n x = (T^* T^n - T^n T^*)\, x + T^n T^* x,$$
$$T^n T^* x = (T^n T^* - T^* T^n)\, x + T^* T^n x,$$

by taking norms, we obtain

$$\| T^* T^n x \| \leqslant \|(T^* T^n - T^n T^*)\, x \| + \| T^n T^* x \|,$$
$$\| T^n T^* x \| \leqslant \|(T^* T^n - T^n T^*)\, x \| + \| T^* T^n x \|.$$

With the help of relations (18) and condition (15′) we obtain

$$\| Tx \| = \| T^* x \|, \qquad \text{for every} \qquad x \in H,$$

thus T is normal.

LEMMA 3. *Let T be a partial isometry.* $\| T^n \| = 1$ *iff there exists a sequence* (x_i) *of unit vectors such that*

$$\text{(I)} \qquad \lim_i \| T^{n-1}x_i \| = 1,$$

$$\text{(II)} \qquad \lim_i (I - T^*T)\, T^{n-1}x_i = 0.$$

Proof. If: Since T^* is isometric on $R(T)$,

$$\| T^n x_i \| = \| T^*T^n x_i \| , \tag{19}$$

and then by (II),

$$\big| \| T^{n-1}x_i \| - \| T^*T^n x_i \| \big| \leqslant \| T^{n-1}x_i - T^*T^n x_i \| \xrightarrow{i} 0. \tag{20}$$

In view of condition (I), with the help of (20), we obtain

$$\lim_i \| T^*T^n x_i \| = 1,$$

and then by (19),

$$\lim_i \| T^n x_i \| = 1.$$

Only if: If $\| T^n \| = 1$, there exists a sequence (x_i) of unit vectors such that

$$\lim_i \| T^n x_i \| = 1. \tag{21}$$

From

$$\| T^n x_i \| \leqslant \| T^{n-1} x_i \| \leqslant 1$$

condition (I) follows.

Furthermore, from

$$T^{n-1}x_i = (I - T^*T)\, T^{n-1}x_i + T^*T(T^{n-1}x_i),$$

we have

$$\| T^{n-1}x_i \|^2 = \|(I - T^*T)\, T^{n-1}x_i \|^2 + \| T^*T(T^{n-1}x_i)\|^2,$$

or equivalently,

$$\| T^{n-1}x_i \|^2 - \| T^*T(T^{n-1}x_i)\|^2 = \|(I - T^*T)\, T^{n-1}x_i \|^2.$$

Conditions (I) and (21) complete the proof:

$$0 = \lim_i \| T^{n-1}x_i \|^2 - \lim_i \| T^n x_i \|^2 = \lim_i \|(I - T^*T)\, T^{n-1}x_i \|^2.$$

THEOREM 10. *Let T be a partial isometry. If for some k and h, $1 \leqslant k \leqslant n-1$, $0 \leqslant h \leqslant n$,*

$$\overline{R(T^k)} + \overline{R(T^*T^h)} \text{ is closed,}$$

and

$$\overline{R(T^k)} \cap \overline{R(T^*T^h)} = (0),$$

then $\| T^n \| < 1$ *and* $T \in Q(H)$ *with* $U = T \mid H_1^{\perp} = 0$.

Proof. Suppose $\| T^n \| = 1$. Choose (x_i) of unit vectors such that conditions (I) and (II) of Lemma 3 be satisfied. Condition (II) gives

$$\lim_i (T^{n-1}x_i - T^*T^n x_i) = 0, \tag{22}$$

but by condition (I)

$$\lim_i T^{n-1}x_i \neq 0. \tag{23}$$

Since

$$T^{n-1}x_i \in \overline{R(T^k)}, \qquad T^*T^n x_i \in \overline{R(T^*T^h)},$$

(22) and (23) contradict the direct sum

$$\overline{R(T^k)} \oplus \overline{R(T^*T^h)}.$$

Thus $\| T^n \| < 1$, and then $\lim_n \| T^n \| = 0$, $H_1 = H$ and, subsequently, $T \in Q(H)$.

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REFERENCES

1. N. DUNFORD, Spectral theory II. Resolution of the identity, *Pacific J. Math.* **2** (1952), 559–614.
2. N. DUNFORD, Spectral operators, *Pacific J. Math.* **4** (1954), 321–354.
3. W. G. BADE, Unbounded spectral operators, *Pacific J. Math.* **4** (1954), 373–392.
4. J. SCHWARTZ, Perturbation of spectral operators, and applications, I. Bounded perturbations, *Pacific J. Math.* **4** (1954), 415–458.
5. C. APOSTOL, Propriétés de certains opérateurs bornés des espaces de Hilbert II, *Rev. Roum. Math. Pures Appl.* **12** (1967), 759–762.
6. S. K. BERBERIAN, Approximate proper vectors, *Proc. Amer. Math. Soc.* **13** (1962), 111–114.

7. P. R. HALMOS AND J. E. MCLAUGHLIN, Partial isometries, *Pacific J. Math.* **13** (1963), 585–596.
8. J. DIEUDONNÉ, "Foundations of Modern Analysis," Academic Press, New York/London, 1960.
9. C. APOSTOL, Propriétés de certains opérateurs bornés des espaces de Hilbert, *Rev. Roum. Math. Pures Appl.* **10** (1965), 643–644.
10. P. R. HALMOS, "A Hilbert Space Problem Book," Van Nostrand, Princeton, N. J., 1967.
11. I. ERDELYI, Partial isometries closed under multiplication on Hilbert spaces, *J. Math. Anal. Appl.* **22** (1968), 546–551.