On the fine structure of the global attractor of a uniformly persistent flow

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Dedicated to José María Montesinos on the occasion of his 65th birthday

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ABSTRACT

We study the internal structure of the global attractor of a uniformly persistent flow. We show that the restriction of the flow to the global attractor has duality properties which can be expressed in terms of certain attractor-repeller decompositions. We also study some natural Morse decompositions of the flow and calculate their Morse equations. These equations provide necessary and sufficient conditions for the existence of attractors with the shape of $S^1$ or such that their suspension has spherical shape.

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1. Introduction

This paper is devoted to the study of some questions related to persistence of flows. This is a topic classically connected to population dynamics, the central issue being to determine whether some components of the population are over time driven to extinction or, on the contrary, they evolve towards some stable states where coexistence between all the components is achieved. The term persistence is generally applied to systems whose orbits do not approach the boundary of the nonnegative orthant $\mathbb{R}_+^n$ when $t \to \infty$, a situation that would imply the risk of extinction. One of the classical results is the Butler–Waltman theorem, which establishes sufficient conditions for the detection of a stronger form of persistence called uniform persistence and also cooperativeness, permanent coexistence or, shortly, permanence. It means that positive semitrajectories are eventually uniformly bounded away from the boundary (see [3,21,42]).
Robustness of uniform persistence has been investigated by several authors. Although uniform persistence is not a robust property, sufficient conditions for robustness have been found in the literature. In this paper we contribute some new results using the point of view of the Conley index theory [6,7], in particular Conley’s notion of continuation. We consider parametrized families of dissipative flows and prove that uniform persistence has weak continuation properties. A consequence of this is that small perturbations of the flow don’t drive to extinction populations within a certain range (which can be arbitrarily chosen). We also introduce a regularity condition which ensures full continuation. The main part of the paper is devoted to the study of the dynamics inside the global attractor of uniformly persistent flows. The existence of a repelling point, \( p \), implies the existence of a dual attractor \( A \) with spherical shape whose basin of attraction is \( \text{int} \mathbb{R}^n_+ - \{ p \} \). By shape we mean here the notion introduced and studied by K. Borsuk in [2] which has become a fundamental tool for the study of the global properties of compact invariant sets and attractors of dynamical systems. This result has some interesting topological nuances since the attractor \( A \) is not generally homeomorphic to a sphere. We also study some natural Morse decompositions of uniformly persistent flows, calculating their Morse equations and proving that these equations are sufficient to detect the existence of attractors with the shape of \( S^1 \) in the plane or attractors whose suspension has the shape of spheres in higher-dimensional cases.

In the sequel we fix some terminology and state a few results that will be used along the paper. An attractor of a flow \( \varphi : E \times \mathbb{R} \to E \), where \( E \) is a locally compact metrizable space, is in this paper an asymptotically stable invariant compactum. A repeller is a negatively asymptotically stable invariant compactum, i.e. an attractor for the reverse flow. We shall use the following characterization of repellers (see [34]): An invariant compactum \( K \) is a repeller if and only if there is a neighborhood \( U \) of \( K \) in \( E \) such that for every \( x \in U - K \) there is \( t > 0 \) such that \( xt \notin U \). This characterization can be dualized for attractors.

The flow \( \varphi \) is said to be dissipative if \( \omega(x) \neq \emptyset \) for every \( x \in E \) and \( \bigcup_{x \in E} \omega(x) \) has compact closure. If \( E \) is not compact we shall often consider the Alexandrov compactification \( E \cup \{ \infty \} \) and the extended flow (\( E \cup \{ \infty \} \) \( \times \mathbb{R} \to E \cup \{ \infty \} \) leaving fixed \( \infty \). Then dissipativeness is equivalent to \( \{ \infty \} \) being a repeller (see [10] and [16]).

In this paper \( E \) will often be a closed subset of \( X \), where \( X \) is a locally compact metric space, and we shall denote by \( \partial E \) the boundary of \( E \) in \( X \). We shall say that the dissipative flow \( \varphi : E \times \mathbb{R} \to E \) is uniformly persistent if there exists \( \beta > 0 \) such that for every \( x \in E \)

\[
\liminf \{d(\varphi(x, t), \partial E) \mid t \to \infty\} \geq \beta.
\]

If \( E \) is compact then \( \varphi \) is uniformly persistent if and only if \( \partial E \) is a repeller of \( \varphi \). If \( E \) is not compact then \( \varphi \) is uniformly persistent if and only if \( \partial E \cup \{ \infty \} \) is a repeller for the flow extended to \( E \cup \{ \infty \} \) (see [10] for proofs of these results).

Let \( K \) be an invariant compactum of \( \varphi \) and \( K_0 \subset K \) an invariant subcompactum. We say that \( K_0 \) is of repelling type if \( W^s(K_0) \subset K \) where \( W^s(K_0) \) is the stable manifold of \( K_0 \). This type of sets were introduced and studied by Wójcik (see [46]).

**Proposition 1.** Let \( K \) be an invariant compactum of \( \varphi : E \times \mathbb{R} \to E \) and \( K_0 \subset K \) an invariant subcompactum. Suppose that \( K_0 \) is isolated as a subset of \( E \) (i.e. there exists an isolating neighborhood of \( K_0 \) in \( E \)). Then \( K_0 \subset K \) is of repelling type if and only if there exists a neighborhood \( U \) of \( K_0 \) in \( E \) such that for every point \( x \in U - K \) there is \( t > 0 \) such that \( xt \notin U \).

**Proof.** If \( K_0 \) is not of repelling type then there exists a point \( y \in E - K \) such that \( \omega(y) \subset K_0 \). Hence, if \( U \) is an arbitrary neighborhood of \( K_0 \) there exists \( t_0 > 0 \) such that \( y[t_0, \infty) \subset U \). Therefore \( x = yt \) does not satisfy the condition in the statement. Conversely, if \( U \) is an isolating compact neighborhood such that there is \( x \in U - K \) with \( x[0, \infty) \subset U \) then \( \omega(x) \subset U \) and hence \( \omega(x) \subset K_0 \) and \( K_0 \) is not of repelling type. \( \square \)

On the topological side, we use classical algebraic topology, in particular homology, cohomology and duality theory together with some rudiments of shape theory. The notion of homotopy type is
too rigid for the study of the topological objects which appear in dynamics. For this reason many authors have used instead Borsuk’s shape theory as an essential tool which provides a geometric insight on the global structure of compacta, mainly on those with complicated topological structure as many attractors and invariant sets. For the benefit of the reader we present here a very short introduction, essentially based on the presentation of this subject given by Kapitanski and Rodnianski in [24].

A metrizable space \( M \) is said to be an absolute neighborhood retract (notation \( M \in ANR \)) if for every homeomorphism \( h \) mapping \( M \) onto a closed subset \( h(M) \) of a metrizable space \( X \) there is a neighborhood \( U \) of \( h(M) \) in \( X \) such that \( h(M) \) is a retract of \( U \). The following are two important characterizations of ANRs.

**Theorem 1.** A metrizable space \( M \) is an ANR if and only if for every map \( f : Y \to M \) of a closed subset \( Y \) of any metrizable space \( Y' \) there is a neighborhood \( U \) of \( Y \) in \( Y' \) and a map \( f' : U \to M \) being an extension of \( f \).

**Theorem 2.** A metrizable space \( M \) is an ANR if and only if it is homeomorphic to a retract of an open subset of a convex set lying in a Banach space.

In particular, open subsets of Euclidean spaces are ANRs.

All metric spaces can be viewed as subsets of ANRs. In fact by the Kuratowski–Wojdyłowski theorem [22] every metric space can be embedded into an ANR as a closed subspace.

Let \( X \) be a closed subset of an ANR \( M \) and \( Y \) a closed subset of an ANR \( N \). Denote by \( \mathbb{U}(X; M) \) (resp. \( \mathbb{U}(Y; N) \)) the set of all open neighborhoods of \( X \) in \( M \) (resp. \( Y \) in \( N \)).

Let \( f = \{f : U \to V\} \) be a collection of continuous maps from the neighborhoods \( U \in \mathbb{U}(X; M) \) to \( V \in \mathbb{U}(Y; N) \). We call \( f \) a mutation if the following conditions are fulfilled:

1. For every \( V \in \mathbb{U}(Y; N) \) there exists (at least) a map \( f : U \to V \) in \( f \).
2. If \( f : U \to V \) is in \( f \) then the restriction \( f|_{U_1} : U_1 \to V_1 \) is also in \( f \) for every neighborhood \( U_1 \subset U \) and every neighborhood \( V_1 \supset V \).
3. If the two maps \( f, f' : U \to V \) are in \( f \) then there exists a neighborhood \( U_1 \subset U \) such that the restrictions \( f|_{U_1} \) and \( f'|_{U_1} \) are homotopic.

An example of mutation is the identity mutation \( \text{id}_{\mathbb{U}(X; M)} \) consisting of the identity maps \( i : U \to U \). The notions of composition of mutations and homotopy of mutations can be defined in a straightforward way that the reader can easily guess (see [24] for details).

Two metric spaces \( X \) and \( Y \) have the same shape if they can be embedded as closed sets in ANRs \( M \) and \( N \) in such a way that there exist mutations \( f = \{f : U \to V\} \) and \( g = \{g : V \to U\} \) such that the compositions \( gf \) and \( fg \) are homotopic to the identity mutations \( \text{id}_{\mathbb{U}(X; M)} \) and \( \text{id}_{\mathbb{U}(Y; M)} \) respectively.

The notion of shape of sets depends neither on the ANRs they are embedded in nor on the embeddings.

Spaces belonging to the same homotopy type have the same shape.

ANRs have the same shape if and only if they have the same homotopy type. A consequence of the two former statements is that the notion of shape may be seen as a generalization of the notion of homotopy type.

For a complete treatment of shape theory we refer the reader to [2,8,9,27]. The use of shape in dynamics is illustrated by the papers [12–15,24,31,32,35,36,39,40]. For information about basic aspects of dynamical systems we recommend [1,33,45] and for algebraic topology, the books by Hatcher [18] and Spanier [43] are very useful.

Most of the results in this paper were obtained while the author was visiting the University of Manchester on leave from Universidad Complutense of Madrid. He wishes to express his gratitude to the School of Mathematics of Manchester University and, very specially, to Nigel Ray for their hospitality. The author is also grateful to Julián López-Gómez for stimulating conversations on the subject of permanence.
2. The Butler–Waltman theorem

The Butler–Waltman theorem [4] is one of the most relevant results in the theory of persistent flows. It provides a criterion for uniform persistence which in the more elementary applications may be reduced to readily testable hypotheses. This result shows that some questions of persistence may be addressed by appealing to suitable conditions on the boundary flow. Butler and Waltman stated their result in terms of isolated acyclic coverings but later Garay presented in [10] a reformulation in terms of Morse decompositions. Garay’s results are written in the spirit of the Conley index theory and, in particular, he makes use of notions related to chain recurrence. We believe that it might be of some interest to start this paper by giving a short new proof of the Butler–Waltman–Garay theorem, which has the advantage of using only the most elementary notions of topological dynamics and may be useful to readers not acquainted with the theory of chain recurrence. We first prove an abstract form of the theorem (as rephrased by Hofbauer [20]) and then the theorem itself.

Theorem 3. Let \( \varphi : Y \times \mathbb{R} \to Y \) be a flow on a compact metric space \( Y \). Suppose \( K \) is a compact invariant set and \( \{M_1, \ldots, M_n\} \) is a Morse decomposition of \( K \) where all \( M_i \) are of repelling type and isolated in \( Y \). Then \( K \) is a repeller.

Proof. Suppose first that \( n = 2 \). Since \( M_2 \) is of repelling type and it is a repeller for \( \varphi|_K \), then \( M_2 \) is a repeller for \( \varphi \). Consider \( U_1 \) and \( U_2 \) disjoint isolating neighborhoods of \( M_1 \) and \( M_2 \) respectively satisfying the repelling type property in Proposition 1. If \( K \) is not a repeller then there is a sequence \( x_n \) of points in \( Y - K, x_n \to K \) such that every neighborhood of \( K \) contains \( \omega(x_n) \) for almost every \( n \) (this is an immediate consequence of the characterization of repellers given above). By the connectedness of \( \omega(x_n) \), we have that \( \omega(x_n) \not\subseteq U_1 \cup U_2 \) for almost every \( n \) and thus there exists \( y \in K - \bigcup_i M_i \) which is limit of a sequence \( z_{n_k} \in \omega(x_{n_k}) \). Since \( y \) is repelled by \( M_2 \) then for every neighborhood \( V \) of \( M_2 \) and for almost every \( k \) there is \( t_{n_k} < 0 \) with \( z_{n_k} t_{n_k} \in V \cap \omega(x_{n_k}) \) (hence \( V \cap \omega(x_{n_k}) \neq \emptyset \)). Using this it is possible to select \( x'_{n_k} \in \gamma^+(x_{n_k}) \) with \( x'_{n_k} \to y \) and \( t'_{n_k} \to +\infty \) such that \( x'_{n_k} t'_{n_k} \to M_2 \), which contradicts the fact that \( M_2 \) is a repeller. Hence \( K \) is a repeller. Now the proof is easily completed using induction on the number of Morse sets. \( \square \)

As a consequence of Theorem 3 we prove now the Butler–Waltman theorem in the formulation given by Garay.

Theorem 4 (Butler–Waltman–Garay). Let \( X \) be a locally compact metric space and let \( E \) be a closed subset of \( X \). Suppose we are given a dissipative dynamical system \( \varphi \) on \( E \) for which \( \partial E \) is invariant. Let \( \mathcal{M} = \{M_1, M_2, \ldots, M_n\} \) be a Morse decomposition for \( \varphi|_M \), where \( M \) is the maximal compact invariant set in \( \partial E \). Further assume that for each \( i \in \{1, 2, \ldots, n\} \)

(a) there exists a \( \gamma > 0 \) such that the set \( \{x \in \hat{E} \mid d(x, M_i) < \gamma\} \) contains no entire trajectories, and
(b) \( \hat{E} \cap W^+(M_i) = \emptyset \).

Then \( \varphi \) is uniformly persistent.

Proof. If \( E \) is compact the proof is an immediate consequence of Theorem 3. If \( E \) is not compact, consider the Alexandrov compactification \( Y = E \cup \{\infty\} \) of \( E \). Then \( \{M_1, M_2, \ldots, M_n, \infty\} \) is a Morse decomposition of \( K = \partial E \cup \{\infty\} \) where all the Morse sets are of repelling type. Hence \( \partial E \cup \{\infty\} \) is a repeller by Theorem 3 and the flow is uniformly persistent. \( \square \)

3. Continuation properties of uniform persistence

A desirable feature of uniform persistence is robustness, i.e. preservation of uniform persistence after the flow is subjected to small perturbations. Some important contributions to this area of study are the following.
V. Hutson considered in [23] the system on \( \mathbb{R}^n_+ \)

\[
\dot{x} = f(x),
\]

where \( f = (f_1, \ldots, f_n), x = (x_1, \ldots, x_n), \) and \( \dot{x}_i = dx_i/dt, \) and the following restrictions are placed on \( f: \)

1. \( f \) is locally Lipschitz,
2. for any \( x(0) = x, \) the solution \( x(t) \) through \( x \) exists for all \( t \geq 0, \)
3. there exists \( b_0 > 0 \) such that for any \( x \in \mathbb{R}^n_+, x(t) \in B(b_0) \) for some \( t > 0, \) where \( B(b_0) \) is the open ball with centre the origin and radius \( b_0, \)
4. \( f_i(x) = 0 \) if \( x_i = 0, \) for each \( i = 1, \ldots, n, \)

together with a perturbed system

\[
\dot{x} = f(x) + g(x, t),
\]

where \( g \) is well behaved enough to ensure the existence and uniqueness of solutions. He assumed that the solutions of the perturbed system are uniformly bounded, that is for any \( \alpha > 0 \) there exists \( \beta(\alpha) \) such that if \( t \geq t_0 \geq 0 \) and \( x_0 \in B(\alpha) \) (the closed ball) then \( x(t, t_0, x_0) \in B(\beta(\alpha)), \) where \( x(t, t_0, x_0) \) is the solution with \( x(t_0, t_0, x_0) = 0. \)

Hutson introduced and studied a notion of repulsion under perturbations for the boundary \( \partial \mathbb{R}^n_+. \) His main result is that if the original system

\[
\dot{x} = f(x),
\]

is uniformly persistent then the boundary \( \partial \mathbb{R}^n_+ \) is repulsive under perturbations. We refer the reader to Hutson’s paper [23] for details.

Schreiber studied in [41] \( C^r \)-vector fields \( x_i^\lambda = x_i f_i(x) \) \( (i = 1, \ldots, n) \) that generate dissipative flows on \( \mathbb{R}^n_+. \) He derived a necessary condition and a sufficient condition for \( C^r \)-robust permanence involving the average per-capita growth rates \( \int f_i \, d\mu \) with respect to invariant measures \( \mu. \)

Schreiber’s results were improved by Hirsch, Smith and Zhao in [19] where they proved, among other things, that uniform persistence is stable to perturbation by a \( C^0 \)-small Lipschitz vector field. They also found sufficient conditions for a parametrized family \( \varphi_\lambda, \) of discrete semiflows to be robust. They basically proved that uniform persistence of \( \varphi_0 \) and weak uniform persistence of all \( \varphi_\lambda, \) uniform in parameters, together with some asymptotic requirements are enough for robustness.

Garay and Hofbauer proved more general results in [11], giving sufficient conditions for robust permanence along a more classical approach using topological dynamics, in particular “good” average Liapunov functions (GALF), the Zubov–Ura–Kimura theorem, and Morse decompositions.

It is easy to see that, in a general context, uniform persistence is not a robust property. For instance, the illustration in Fig. 1 shows that small perturbations of a uniformly persistent flow may destruct this property.

We shall discuss in this section some related matters using the point of view of continuation, a central notion in the Conley index theory. Roughly speaking, we say that a certain property continues if whenever we have a parametrized family of flows \( \varphi_\lambda, \lambda \in [0, 1], \) and \( \varphi_0 \) has this property then \( \varphi_\lambda, \) also has this property for small values of \( \lambda. \) In this section, we basically see that all uniformly persistent flows have weak continuation properties, meaning by this that small perturbations of the flow never drive to extinction populations within a certain range (which can be arbitrarily chosen). Moreover, if a regularity condition is placed on the original flow we achieve full continuation. We need a definition to introduce the notion of regularity.

**Definition 1.** Let \( X \) be a locally compact metric space and let \( E \) be a closed subset of \( X. \) Suppose we are given a (continuous) parametrized family of dissipative dynamical systems \( \varphi_\lambda, \lambda \in I, \) on \( E, \) for
which $\partial E$ is invariant. We say that the family is regular at $\lambda_0$ provided that there exists a compact set $K \subset \bar{E}$, $K \neq \emptyset$, and $\epsilon > 0$ such that for every $x \in \bar{E}$ and for every $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$, the trajectory $\varphi_\lambda(x, \cdot)$ visits $K$.

We express our continuation results first in a general form and then we consider the particular case of flows defined in the nonnegative orthant.

**Theorem 5 (Weak continuation of uniform persistence).** Let $X$ be a locally compact metric space and let $E$ be a closed subset of $X$. Suppose we are given a (continuous) parametrized family of dissipative dynamical systems $\varphi_\lambda$, $\lambda \in I$, on $E$, for which $\partial E$ is invariant. Further, assume that $\varphi_0$ is uniformly persistent. Then there exists $\beta > 0$ such that for every compact set $K \subset \bar{E}$ there exists $\lambda_0 > 0$ such that

$$
\liminf\left\{d(\varphi_\lambda(x, t), \partial E) \mid t \to \infty \right\} \geq \beta
$$

for every $\lambda \leq \lambda_0$ and for every $x \in K$.

**Proof.** We discuss the case when $E$ is not compact, the compact case being easier. Since $E$ is locally compact we may consider its Alexandrov compactification $\bar{E} \cup \{\infty\}$ and extend all flows $\varphi_\lambda$ to it in such a way that $\varphi_\lambda(\infty) = \infty$ (we still use the same notation, $\varphi_\lambda$, to denote such extension). It can be easily proved that we obtain in this way a continuous parametrized family of flows $\varphi_\lambda : (\bar{E} \cup \{\infty\}) \times \mathbb{R} \to \bar{E} \cup \{\infty\}$, with $\lambda \in I$.

Since $\varphi_0$ is uniformly persistent then $R = \partial E \cup \{\infty\}$ is a repeller and we denote by $A$ its dual attractor. Now, a basic fact in Conley’s index theory is that the pair $(A, R)$ continues to a family of attractor-repeller pairs $(A_\lambda, R_\lambda)$ for the flows $\varphi_\lambda$ for $\lambda$ sufficiently small. This means, in particular, that if $U$ is a neighborhood of $A$ and $V$ is a neighborhood of $R$ (which can be assumed to be isolating) with $U \cap V = \emptyset$ then $A_\lambda \subset U$ and $R_\lambda \subset V$ and $A_\lambda$ and $R_\lambda$ are the maximal invariant sets for $\varphi_\lambda$ in $U$ and $V$ respectively for $\lambda$ sufficiently small. Moreover, since $R = \partial E \cup \{\infty\}$ is also invariant for $\varphi_\lambda$, we necessarily have that $R \subset R_\lambda$.

Since $(A_\lambda, R_\lambda)$ is an attractor-repeller pair for $\varphi_\lambda$ there exists a $T_\lambda > 0$ such that $\varphi_\lambda(x, t) \in U$ for every $x \in E - V$ and every $t \geq T_\lambda$ (since $E - V$ is a compact set contained in the basin of attraction of $A_\lambda$). Now, given a compactum $K \subset \bar{E}$ and in order to establish the theorem, we choose the previously mentioned pair $(U, V)$ satisfying the additional condition that $V$ is a neighborhood of $R$ contained in the complement in $E \cup \{\infty\}$ of $\bar{U} \cup K$. Hence, if we define $\beta$ as the distance from the compact set $\bar{U}$ to $\partial E$ we immediately deduce that

$$
\liminf\{d(\varphi_\lambda(x, t), \partial E) \mid t \to \infty\} \geq \beta
$$
for every $x \in E - V$ (in particular for $x \in K$) and for every $\lambda$ less than or equal to a certain positive $\lambda_0$. This proves the theorem. \)

**Corollary 1.** Let $\varphi_\lambda$, $\lambda \in I$, be a (continuous) parametrized family of dissipative flows on the nonnegative orthant $\mathbb{R}_+^n$. Further, assume that $\varphi_0$ is uniformly persistent. Then there exists $\alpha > 0$ such that for every $\epsilon$ and every $M$ with $0 < \epsilon < M$ there exists $\lambda_0 > 0$ such that (a) $\liminf d(\varphi_\lambda(x, t), \partial E) \mid t \to \infty > \alpha$ for every $x$ with $d(x, \partial \mathbb{R}_+^n) \geq \epsilon$ and $\|x\| \leq M$ and for every $\lambda \leq \lambda_0$ and (b) the set

$$W_\lambda = \{ x \in \mathbb{R}_+^n \mid \liminf \{ d(\varphi_\lambda(x, t), \partial E) \mid t \to \infty \} > \alpha \}$$

is contractible and the set

$$R_\lambda = \{ x \in \mathbb{R}_+^n \mid \liminf \{ d(\varphi_\lambda(x, t), \partial E) \mid t \to \infty \} \leq \alpha \} \cup \{ \infty \}$$

has the shape of $\mathbb{S}^{n-1}$ for every $\lambda \leq \lambda_0$.

**Proof.** The first part is a consequence of the fact that, given $\epsilon$ and $M$ with $0 < \epsilon < M$, the set $K = \{ x \in \mathbb{R}_+^n \mid d(x, \partial \mathbb{R}_+^n) \geq \epsilon$ and $\|x\| \leq M \}$ is a compactum contained in the interior of the nonnegative orthant $\mathbb{R}_+^n$. Then we take any $\alpha > 0$ such that $\alpha < \beta$ for a $\beta$ as stated in Theorem 5. For the second part we remark that the set

$$W_\lambda = \{ x \in \mathbb{R}_+^n \mid \liminf \{ d(\varphi_\lambda(x, t), \partial E) \mid t \to \infty \} > \alpha \}$$

is the basin of attraction of the continuation $A_\lambda$ of the global attractor $A$. By [38] all $A_\lambda$ are attractors with trivial shape and their basins of attraction are contractible. Similarly, the sets

$$R_\lambda = \{ x \in \mathbb{R}_+^n \mid \liminf \{ d(\varphi_\lambda(x, t), \partial E) \mid t \to \infty \} \leq \alpha \} \cup \{ \infty \}$$

are continuations of the repeller $\partial \mathbb{R}_+^n \cup \{ \infty \}$ (for the flow extended to $\mathbb{R}_+^n \cup \{ \infty \}$). Now, the shape of repellers is preserved by continuation (again by [38]). Therefore $\text{Sh}(R_\lambda) = \text{Sh}(\mathbb{S}^{n-1})$. \)

**Theorem 6** (Continuation of uniform persistence for regular families of dissipative systems). Let $X$ be a locally compact metric space and let $E$ be a closed subset $X$. Suppose we are given a family of dissipative dynamical systems $\varphi_\lambda$ on $E$ for which $\partial E$ is invariant. Further, assume that the family is regular at $\lambda = 0$ and $\varphi_0$ is uniformly persistent. Then there exists $\lambda_0 > 0$ such that $\varphi_\lambda$ is uniformly persistent for every $\lambda \leq \lambda_0$. Moreover the continuation is uniform in parameters, in the sense that there exists $\beta > 0$ such that $\liminf(\varphi_\lambda(x, t), \partial E) \mid t \to \infty \geq \beta$ for every $\lambda \leq \lambda_0$ and for every $x \in E$.

**Proof.** We discuss again the noncompact case. If the family $(\varphi_\lambda)$ is regular at $\lambda = 0$ then its extension to the Alexandrov compactification $\varphi_\lambda : (E \cup \{ \infty \}) \times \mathbb{R} \to E \cup \{ \infty \}$ has the same property. Let $K$ be the compact set in the definition of regularity at $\lambda = 0$. Using the same notation as in the proof of Theorem 5 we consider an isolating neighborhood $(U, V)$ of $(A, R)$ for $\varphi_0$ with $U \cap V = \emptyset$ and $V \subset (E \cup \{ \infty \}) - K$. Then $R \subset R_\lambda$ for $\lambda$ sufficiently small and, since $R$ and $R_\lambda$ are invariant for $\varphi_\lambda$, and there are no full orbits in $V - (\partial E \cup \{ \infty \})$ (since every trajectory visits $K$ and $K \cap V = \emptyset$), we necessarily have that $\lambda = 0$. Hence $\partial E \cup \{ \infty \} = R$ is a repeller for $\varphi_\lambda$ and $\varphi_\lambda$ is uniformly persistent (always for $\lambda$ sufficiently small). Uniformity in parameters of the continuation is a consequence of the fact that all attractors $A_\lambda$ (dual of the repellers $R_\lambda = R$) are contained in $U$ for $\lambda$ sufficiently small. Hence, if we define $\beta$ as the distance from the compact set $U$ to $\partial E$ we immediately deduce that

$$\liminf \{ d(\varphi_\lambda(x, t), \partial E) \mid t \to \infty \} \geq \beta$$

for every $x \in \hat{E}$ and for every $\lambda$ less than or equal to a certain positive $\lambda_0$. \)
Corollary 2. Let \( \varphi_\lambda, \lambda \in I, \) be a (continuous) parametrized family of dissipative flows on the nonnegative orthant \( \mathbb{R}^n_+ \). Further, assume (a) there exist \( \epsilon \) and \( M, \) with \( 0 < \epsilon < M, \) and \( \lambda_0 > 0 \) such that for every \( x \) in the interior of \( \mathbb{R}^n_+ \) and for every \( \lambda \leq \lambda_0 \) there exists a \( t \in \mathbb{R} \) with \( d(\varphi_\lambda(x, t), \partial \mathbb{R}^n_+) \geq \epsilon \) and \( \|\varphi_\lambda(x, t)\| \leq M \) and (b) \( \varphi_0 \) is uniformly persistent. Then \( \varphi_\lambda \) is uniformly persistent for every \( \lambda \leq \lambda_0. \)

Proof. As in the previous corollary, it is a consequence of the fact that, given \( \epsilon \) and \( M \) with \( 0 < \epsilon < M, \) the set \( K = \{x \in \mathbb{R}^n_+ | d(x, \partial \mathbb{R}^n_+) \geq \epsilon \) and \( \|x\| \leq M \} \) is a compactum contained in the interior of the nonnegative orthant \( \mathbb{R}^n_+ \). \( \square \)

It would be of interest to study the implications of these results in some particular situations. Theorem 5 suggests that permanence does not vanish completely in an abrupt way. Even if it does not continue, permanence still remains when we limit ourselves to populations within a certain range. As an interesting case, S. Cano-Casanova and J. López-Gómez prove in [5] (see also [26]) that permanence of two species is possible under strong mutual aggression. In other words, they prove that if the birth rates are high enough then the species are permanent irrespective of the competition strength in the regions where competition occurs. They actually measure how large the birth rate must be. An interesting problem would be to study to what extent permanence remains for populations within a certain range in spite of their reproduction rate being below the limit threshold.

4. The global attractor

We are interested now in discussing some features of the dynamics inside the global attractor of a uniformly persistent flow. As a motivation consider the Holling-type interaction, associated to several phenomena in ecology and chemical kinetics (see [21]). It is modelled by the equations

\[
\begin{align*}
\dot{x} &= rx \left( 1 - \frac{x}{k} \right) - y \frac{cx}{a + x}, \\
\dot{y} &= y \left( -d + \frac{bx}{a + x} \right)
\end{align*}
\]

in \( \mathbb{R}^2_+ \), where all parameters are positive.

If \( b > d \) and \( k > \frac{ad}{b - d} \) then the system admits a unique interior fixed point \( F = (\bar{x}, \bar{y}) \) with

\[
\bar{x} = \frac{ad}{b - d}.
\]

This equilibrium is a global attractor iff \( k < a + 2\bar{x} \). If \( k > a + 2\bar{x} \) then \( F \) becomes a repeller and a Hopf bifurcation takes place inside the global attractor \( K \) and an attracting cycle \( C \) is created which starts its evolution from \( F \). Therefore the repelling point \( F \) and the attracting cycle \( C \) induce an attractor-repeller decomposition of the global attractor \( K \). We would like to study this type of decompositions in a general form. This situation has a rich topological meaning and the description of the involved duality properties requires the use of shape theory. Only at this level the results are satisfactorily expressed since, generally speaking, the results do not hold if we replace shape by topological type or even homotopy type.

We introduce first some notions that are used in the sequel. We say that a continuum \( K \) is point-like in \( \mathbb{R}^n \) provided \( \mathbb{R}^n - K \) is homeomorphic to \( \mathbb{R}^n - \{p\} \), where \( p \) is a point. An important example of point-like continua are global attractors in Euclidean spaces (see [1]). It is well known that point-like continua have trivial shape (i.e. the shape of a point), although the converse statement is not, in general, true.
Theorem 7. Let $\varphi : \mathbb{R}^n_+ \times \mathbb{R} \to \mathbb{R}^n_+$ be a dissipative flow. If $\varphi$ is uniformly persistent then:

(a) Suppose $L$ is a point-like repeller (in particular a repelling point) in the interior of $\mathbb{R}^n_+$, then there exists an attractor $K_0$ with the shape of $S^{n-1}$ contained in the global attractor $K$ and whose basin of attraction is $\text{int} \mathbb{R}^n_+ - L$.

(b) Suppose $L$ is a repeller with the shape of $S^{n-1}$ in the interior of $\mathbb{R}^n_+$. Then $L$ decomposes $\text{int} \mathbb{R}^n_+$ into two connected components. Moreover, if the bounded component is simply connected then there exists an attractor with the shape of a point contained (together with its basin of attraction) in the interior of the global attractor $K$.

Proof. Since $L \subset \text{int} \mathbb{R}^n_+$ is a repeller and $K$ attracts $\text{int} \mathbb{R}^n_+$, then the basin of repulsion $\mathcal{R}$ of $L$ (which is an open set) is necessarily contained in $K$. Moreover, the restriction $\varphi|_K$ of the flow to $K$ induces an attractor-repeller decomposition of $K$ where the attractor is $K_0 = K - \mathcal{R}$. We prove now that $K_0$ is an attractor (we remind that only asymptotically stable attractors are considered) for $\varphi$ whose basin of attraction is $\text{int} \mathbb{R}^n_+ - L$. Suppose $C$ is a compact set in $\text{int} \mathbb{R}^n_+ - L$. Then $C \cap K \subset K - L$, which is the region of attraction of $K_0$ (for the restricted flow $\varphi|_K$). Moreover the compact set $\overline{C} - K$ is contained in the invariant set $(\text{int} \mathbb{R}^n_+ - K) \cup K_0$ and also in the region of attraction of $K$. This implies that $C$ is attracted by $K_0$ and from this it is clear that the basin of attraction of $K_0$ is $\text{int} \mathbb{R}^n_+ - L$. On the other hand, it was proved by Kapitanski and Rodnianski in [24] that the inclusion of an attractor in its basin of attraction is a shape equivalence. Then $\text{Sh}(K_0) = \text{Sh}(\text{int} \mathbb{R}^n_+ - L)$ and, since $L$ is point-like, we have that $\text{int} \mathbb{R}^n_+ - L \approx \text{int} \mathbb{R}^n_+ - \{p\}$, where $p$ is a point in $\text{int} \mathbb{R}^n_+$. On the other hand $\text{int} \mathbb{R}^n_+ - \{p\}$ is homotopy equivalent to $S^{n-1}$, which establishes part (a) of the theorem.

Suppose now that $L$ is a repeller with the shape of $S^{n-1}$ in $\text{int} \mathbb{R}^n_+$. Then, by Alexander duality, $\tilde{H}_0(\text{int} \mathbb{R}^n_+ - L) \approx H_1(\text{int} \mathbb{R}^n_+ \cup \{\infty\}, \text{int} \mathbb{R}^n_+ - L) \approx \tilde{H}^{n-1}(L) \approx \mathbb{Z}$ (where the first isomorphism is easily derived from the long homology sequence of the pair $(\text{int} \mathbb{R}^n_+, \text{int} \mathbb{R}^n_+ - L)$ and the last one is a consequence of the shape invariance of Čech cohomology $\tilde{H}^*$). As a consequence of this $\text{int} \mathbb{R}^n_+ - L$ is composed of two open components, where the bounded one, $B$, is contained in the global attractor $K$. Using Alexander duality again we have that $H_k(\text{int} \mathbb{R}^n_+ - L) \approx H^{n-k-1}(L) \approx \{0\}$ for $k \neq 0, n - 1$, which implies that $H_k(B) \approx \{0\}$ for $k \neq 0, n - 1$. For $k = n - 1$, we consider the Alexander compactification $\text{int} \mathbb{R}^n_+ \cup \{\infty\}$ of $\text{int} \mathbb{R}^n_+$, which is decomposed by $L$ into two connected components, one of them being $B$. Alexander duality for the pair $(\text{int} \mathbb{R}^n_+ \cup \{\infty\}, L)$ implies in this case that $H_{n-1}(\text{int} \mathbb{R}^n_+ \cup \{\infty\} - L)$ is trivial, which, in turn, implies that $H_{n-1}(B) \approx \{0\}$. Then, since $B$ is simply connected, we get from the homological version of Whitehead theorem that $B$ is contractible. Now, if we consider the flow $\varphi$ restricted to $B \cup L$ then by an argument similar to the one in part (a), we see that there is an attractor $K_0 \subset B \subset K$ dual to the repeller $L$ for the flow $\varphi|_{B \cup L}$, whose basin of attraction is $B$ and, since $B$ is contractible and the inclusion $K_0 \to B$ is a shape equivalence (by Kapitanski and Rodnianski result again), we get that $K_0$ has the shape of a point. □

Remark 1. We don’t know whether the simple-connectedness hypothesis is necessary in part (b) of Theorem 7. As a fact of fact there exist wild embeddings of the sphere in the 3-dimensional Euclidean space such that the bounded connected component of the complement is not simply connected (for instance, some variants of the Alexander horned sphere). However, Sánchez-Gabites has proved in [37] that they can be neither attractors nor repellers of flows in $\mathbb{R}^3$. In fact, Sánchez-Gabites results can be used to prove that if a repeller in $\mathbb{R}^3$ (or $\text{int} \mathbb{R}^3_+$) has the shape of $S^2$ then the bounded component of the complement is a topological open ball (then, as a consequence, the argument in our theorem proves that the region of attraction of the dual attractor is homeomorphic to $\mathbb{R}^3$ and the attractor itself is point-like). In the 2-dimensional case the bounded component of the complement of a continuum with the shape of $S^1$ is always a topological open disc (see [29]) and hence the dual attractor of $L$ is point-like. Therefore in the lower-dimensional cases the hypothesis of simple connectedness is unnecessary.

On the other hand, it is easy to construct a uniformly persistent flow in $\mathbb{R}^2_+$ with a repelling point and such that the complementary attractor is the Warsaw circle (see Hastings [17]). This shows that,
In fact, it is not even possible to prove that it is homotopically equivalent to $S^{n-1}$.

In our next result we see that the Morse theory of uniformly persistent flows with an attracting cycle is quite straightforward, irrespective of the complexity of the flow in the boundary. Suppose $\varphi : \mathbb{R}_+^n \times \mathbb{R} \to \mathbb{R}_+^n$ is a uniformly persistent flow. We say that $\mathcal{M} = \{M_1, M_2, \ldots, M_k\}$ is a natural Morse decomposition of the flow if (a) $M_1$ is an attracting cycle or, more generally, an attractor with the shape of $S^1$ then the Morse equation of $\mathcal{M}$ takes a simple form. On the opposite direction we see that using this equation we can recognize the existence of attractors with the shape of $S^1$ in the plane or attractors whose suspension has the shape of $S^2$ for higher dimensions.

**Theorem 8.** Let $\varphi : \mathbb{R}_+^n \times \mathbb{R} \to \mathbb{R}_+^n$ be a dissipative flow. Suppose $\varphi$ is uniformly persistent and $\mathcal{M} = \{M_1, M_2, \ldots, M_k\}$ is a natural Morse decomposition of $\mathbb{R}_+^n$ for $\varphi$. Then:

1. If $M_1$ has the shape of $S^1$ then the Morse equation of the decomposition $\mathcal{M}$ with coefficients in $\mathbb{Z}$ or a field is

$$1 + t + t^2 = 1 + (1 + t)t.$$  

2. Conversely, if the Morse equation of $\mathcal{M}$ is (1) then $\text{Sh}(M_1) = \text{Sh}(S^1)$ for $n = 2$ and $\text{Sh}(\Sigma M_1) = \text{Sh}(S^2)$ for $n \geq 2$, where $\Sigma M_1$ is the suspension of $M_1$.

**Proof.** Since the Morse sets $M_1 \subset \partial \mathbb{R}_+^n$ are of repelling type we have, by Wójcik’s Theorem 2 in [46], that their homological Conley index $CH_i(M_1)$ is trivial. On the other hand the index of $\{\infty\}$ is also trivial since it is a repeller in $\partial \mathbb{R}_+^n \cup \{\infty\}$. We analyze now the exact sequence of the attractor repeller decomposition $(M_1, M_2)$ of the global attractor $K$:

$$\cdots \xrightarrow{\partial} CH_i(M_1) \to CH_i(K) \to CH_i(M_2) \xrightarrow{\partial} \cdots.$$  

Since $K$ is a global attractor, its reduced Čech homology and cohomology are those of a point. Now, attractors have the property that their Čech homology and cohomology agree with that of their compact positively invariant neighborhoods contained in their region of attraction. So, we can choose a neighborhood of $K$ satisfying the additional property of being an ENR (Euclidean Neighborhood Retract) and such that its Čech (and hence its singular) reduced homology are those of a point. It follows from this that $CH_i(K)$ is $\{0\}$ for every $i > 0$ and $CH_0(K) = \mathbb{Z}$. Moreover, since $\text{Sh}(M_1) = \text{Sh}(S^1)$ and Čech homology and cohomology are shape invariants, similar considerations show that $CH_i(M_1) = \{0\}$ for $i \geq 2$ and $CH_i(M_1) = \mathbb{Z}$ for $i = 0, 1$. On the other hand, from the exact sequence considered above we immediately obtain that $CH_i(M_2) = \{0\}$ for $i \neq 2$ and $CH_2(M_2) = \mathbb{Z}$. It is clear from this that the Morse equation of the decomposition is

$$1 + t + t^2 = 1 + (1 + t)t.$$  

The same equation is obtained if we use any field $\mathbb{K}$ of coefficients instead of $\mathbb{Z}$.

Conversely, if the equation of a natural Morse decomposition is (1) then, given that its second member is $1 + (1 + t)t$, we deduce that the ranks of $CH_1(M_1)$ and $CH_2(M_2)$ are $\neq 0$. Moreover, since the first member of the equation is $1 + t + t^2$, then $\text{rank} CH_0(M_1) = \text{rank} CH_1(M_1) = \text{rank} CH_2(M_2) = 1$ and equal to 0 otherwise. Since this holds for homology with coefficients in an arbitrary field $\mathbb{K}$ or in $\mathbb{Z}$, then by an easy application of the universal coefficients theorem for homology we get that $CH_i(M_1)$ has no torsion in any dimension. Since $M_1$ is an attractor, a consequence of the above
remarks is that ̃H_s(M_1) agrees with the homology of S^1. An analogous statement holds for Čech cohomology by the universal coefficients theorem [28]. By Spież theorem [44] this is enough to conclude that Sh(M_1) = Sh(S^1) when n = 2.

We consider now the suspension ΣM_1. In the sequel we make use of some shape-theoretic invariants such as movability, approximative n-connectedness and fundamental dimension (see [2,9,27] for the basic definitions). It is well known that ΣM_1 is approximatively 1-connected. Moreover, since the Čech cohomology of M_1 is that of S^1 then the Čech cohomology of ΣM_1 is that of S^2. By Nowak’s theorem [30], the fundamental dimension Fd(ΣM_1) = c(ΣM_1), where c(ΣM_1) is the cyclic coefficient of ΣM_1 (the greatest r such that ̃H^r(ΣM_1) ̸= {0}). Hence Fd(ΣM_1) = 2.

Now by Spież’s theorem, the shape of a pointed compactum (C, c_0) is equal to the shape of (S^n, a) if and only if C is a movable, approximatively 1-connected continuum with Fd(C) = n and with its Čech cohomology groups isomorphic to the corresponding cohomology groups of S^n. Being an attractor, M_1 has the shape of a polyhedron and, as a consequence, it is movable, as is ΣM_1, too. Hence the shape of ΣM_1 is that of S^2. This completes the proof of the theorem. □

An analogous result can be proved for higher dimensions. We give here the statement without proof, which can be obtained by a simple modification in the proof of Theorem 8.

Theorem 9. Let ϕ : R^n x R → R^n be a dissipative flow. Suppose ϕ is uniformly persistent and M = {M_1, M_2, ..., M_k} is a natural Morse decomposition of R^n for ϕ. Then:

(1) If M_1 has the shape of S^1 then the Morse equation of the decomposition M with coefficients in Z or a field is

\[ 1 + t^r + t^{r+1} = 1 + (1 + t)t^r. \] (1)

(2) Conversely, if the Morse equation of M is (1) then Sh(ΣM_1) = Sh(S^{r+1}) for n ≥ 1, where ΣM_1 is the suspension of M_1.

Theorems 8 and 9 do not hold if we replace the suspension ΣM_1 of the attractor by the attractor M_1 itself. It is easy to define a uniformly persistent flow on R^n_+ where the global attractor K is a ball with an attractor-repeller decomposition such that the repeller R is a knotted arc with extremes in ∂K and the dual attractor A is an ANR. Then A has the homology of S^1 but fundamental group different from Z. Therefore Sh(A) ̸= Sh(S^1) in spite of the fact that the Morse equation of any natural Morse decomposition with M_1 = A, M_2 = R is 1 + t + t^2 = 1 + (1 + t)t.

In his paper [25] D. Li has developed new ideas for the study of Morse decompositions from the point of view of Morse–Lyapunov functions. It would be interesting to apply Li’s ideas to the present context.

References


