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An analogue of Dirac's theorem on circular super-critical graphs[☆]

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Abstract

A graph G is called circular super-critical if $\chi_c(G \setminus u) < \chi_c(G) - 1$ for every vertex u of G. In this paper, analogous to a result of Dirac on chromatic critical graphs, a sharp lower bound on the vertex degree of circular super-critical graphs is proved. This lower bound provides a partial answer to a question of X. Zhu [The circular chromatic number of induced subgraphs, J. Combin. Theory Ser. B 92 (2004) 177–181]. Some other structural properties of circular super-critical graphs are also presented.

1. Introduction

An n-coloring of G is a mapping $\phi : V(G) \longmapsto \{1, 2, \ldots, n\}$ such that $\phi(u) \neq \phi(v)$ for every $uv \in E(G)$. The least integer k such that G admits a k-coloring is called the *chromatic number* of G, and is denoted by $\chi(G)$. A graph is called a *chromatic critical graph* if $\chi(G \setminus u) < \chi(G)$ for each vertex u of G. A k-chromatic critical graph is a chromatic critical graph G with $\chi(G) = k$.

Let $r \geq 1$ be a real number. A circular r-coloring of a graph G is a mapping $\psi : V(G) \longmapsto [0, r)$ such that for every edge u of G, $1 \leq |\psi(u) - \psi(v)| \leq r - 1$ [11]. A graph is called circular r-colorable if it admits a circular r-coloring. The *circular chromatic number* of G, denoted by $\chi_c(G)$, is the least r such that G is circular r-colorable.

It was proved elsewhere [2,7] that $\chi_c(G)$ is always attained at rational number and

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G) \text{ for any finite graph } G. \tag{1}$$

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If $\chi_c(G) = \frac{k}{d}$, then a circular $\frac{k}{d}$-coloring of G is equivalent to a mapping $\psi : V(G) \longmapsto \{0, 1, 2, \ldots, k-1\}$ with $d \leq |\psi(u) - \psi(v)| \leq k - d$ for every $uv \in E(G)$.

It is well known that the chromatic number decreases by at most 1 when one vertex is removed from the graph. But this is not valid for the circular chromatic number. From (1), it is easy to check that after removing a vertex from a graph, the decrease of the circular chromatic number must be less than 2. But there are infinitely many examples showing that the decrease could be arbitrarily close to 2.

If G is a chromatic critical graph, then the chromatic number decreases by exactly 1 on removing an arbitrary vertex, i.e., a k-chromatic critical graph G is a graph with $\chi(G) = k$ and $\chi(G \setminus u) = k - 1$ for every vertex u of G. The study of chromatic critical graphs was started by Dirac [3] who proved that for every k-chromatic critical graph G, $\delta(G) \geq k - 1$ and the subgraph induced by an arbitrary cut set of G is not complete. Since K_k, the complete graph of order k, is k-chromatic critical, the bound $\delta(G) \geq k - 1$ is sharp.

We may define, analogous to the concept of chromatic critical graphs, a *circular critical graph* of a graph G with $\chi_c(G \setminus u) < \chi_c(G)$ for each vertex u of G. Here, instead of considering the circular critical graphs, we focus on a family of graphs that demands more. A graph G is called *circular super-critical* if $\chi_c(G \setminus u) < \chi_c(G) - 1$ for each vertex u of G. A $\frac{k}{d}$-circular super-critical graph is a circular super-critical graph with circular chromatic $\frac{k}{d}$. It is clear that there is no r-circular super-critical graph for rational $r \leq 3$.

Does there exist a circular super-critical graph? In [8], Zhu conjectured a negative answer to this question. In [10], Zhu disproved his conjecture by constructing an infinite family of 4-regular 4-circular super-critical graphs G with $\chi_c(G - x) = \frac{8}{3}$ for every vertex x of G, and proposed several new questions as follows.

Question 1 ([10]). *Given integer $n \geq 5$, is there a circular super-critical graph G with $\chi(G) = n$?*

Question 2 ([10]). *Is there a circular super-critical graph G such that for every vertex u of G, $\chi_c(G) - \chi_c(G \setminus u) \geq 2 - \varepsilon$ for some $\varepsilon < \frac{2}{3}$? Or even for any $\varepsilon > 0$?*

Question 3 ([10]). *Is there a graph G for which $\chi_c(G) \neq \chi(G)$ and yet there is a vertex u of G such that $\chi_c(G) - \chi_c(G \setminus u) > 1$?*

Since $\chi_c(G \setminus u) \leq \chi_c(G) - 1$ implies $\chi(G \setminus u) = \chi(G) - 1$, every $\frac{k}{d}$-circular super-critical graph is $\lceil \frac{k}{d} \rceil$-chromatic critical. Therefore, every circular super-critical graph is 2-connected. We will show that for every circular super-critical graph G, G is 3-connected and the complement of G is 2-connected. Analogous to Dirac's theorem that says $\delta(G) \geq k - 1$ for every k-chromatic critical graph G, a sharp lower bound on the vertex degree of circular super-critical graphs is proved.

2. Main results

Theorem 1. *If G is a circular super-critical graph, then G is 3-connected and the complement of G is 2-connected. Furthermore, for any cut set S of G of cardinality at least 3, the subgraph induced by S contains at most $\frac{|S|(|S|-1)}{2} - 2$ edges.*

Theorem 2. *If u is a vertex of a graph G with $\chi_c(G) - \chi_c(G \setminus u) > 1$, then $d(u) \geq \frac{\chi_c(G \setminus u)}{\chi_c(G \setminus u) + 2 - \chi_c(G)}$.*

Theorem 2 provides us with a necessary condition for a graph to be circular super-critical. If a graph G is circular super-critical, then for every vertex u, $\chi_c(G) - \chi_c(G \setminus u) > 1$ and hence $d(u) \geq \frac{\chi_c(G\setminus u)}{\chi_c(G\setminus u)+2-\chi_c(G)}$.

Since the function $\frac{x}{x-(\chi_c(G)-2)}$ is monotone decreasing on $x \in (\chi_c(G) - 2, \chi_c(G) - 1]$, and since $\chi_c(G \setminus u) < \chi_c(G) - 1$, $\delta(G) > \lceil \frac{\chi_c(G)-1}{(\chi_c(G)-1)+2-\chi_c(G)} \rceil = \lceil \chi_c(G) - 1 \rceil = \chi(G) - 1$ for every circular super-critical graph, where $\chi(G) - 1$ is Dirac's bound on the minimum degree of chromatic critical graphs.

The bound of Theorem 2 is best possible in the sense that for any integer $n \geq 4$, there exists a graph G and a vertex u of G such that $\chi_c(G) = n$, $\chi_c(G) - \chi_c(G \setminus u) > 1$ and $d(u) = \frac{\chi_c(G\setminus u)}{\chi_c(G\setminus u)+2-\chi_c(G)}$.

When $n = 4$, the graph G constructed by Zhu in [10] is a 4-circular super-critical graph with $d(x) = 4 = \frac{\frac{8}{3}}{\frac{8}{3}+2-4} = \frac{\chi_c(G\setminus x)}{\chi_c(G\setminus x)+2-\chi_c(G)}$ for each vertex x. The other 4-chromatic example is as follows. Given integer $l \geq 2$, let G be the graph obtained from the odd circuit C_{2l+1} of length $2l + 1$ by adding a new vertex u and joining u to every vertex of C_{2l+1}. Then, $\chi_c(G) = 4$, $\chi_c(C_{2l+1}) = \frac{2l+1}{l}$, and $d(u) = 2l + 1 = \frac{\frac{2l+1}{l}}{\frac{1}{l}} = \frac{\chi_c(G\setminus u)}{\chi_c(G\setminus u)+2-\chi_c(G)}$.

For $n \geq 4$, let G be the graph obtained from the complement of the odd circuit C_{2n-1} by adding a new vertex u and joining u to each of the other vertices. Then, $\chi_c(G) = \chi(G) = n + 1$, $\chi_c(G \setminus u) = n - \frac{1}{2}$ and $d(u) = 2n - 1 = \frac{\chi_c(G\setminus u)}{\chi_c(G\setminus u)+2-\chi_c(G)}$.

The converse of Theorem 2 may not be true. To see this, let us consider the Petersen graph P. It is known that $\chi_c(P) = \chi(P) = 3$ (see [9]). Since for an arbitrary vertex x of P, $P \setminus x$ contains C_5 as an induced subgraph, $\chi_c(P \setminus x) \geq \frac{5}{2}$ (in fact, it is easy to check that $\chi_c(P \setminus x) = 3$). Hence $\frac{\chi_c(P\setminus x)}{\chi_c(P\setminus x)+2-\chi_c(P)} = \frac{\chi_c(P\setminus x)}{\chi_c(P\setminus x)-1} \leq \frac{\chi_c(P\setminus x)}{1.5} \leq 2$. Therefore, $3 = d(x) \geq \frac{\chi_c(P\setminus x)}{\chi_c(P\setminus x)+2-\chi_c(P)}$ for every vertex x of P. But P is not circular super-critical.

Circular super-critical graphs are all chromatic critical. We may restrict our focus to chromatic critical graphs. Theorem 3 below shows that if G is chromatic critical, then $d(u) = \frac{\chi_c(G\setminus u)}{\chi_c(G\setminus u)+2-\chi_c(G)}$ for every vertex u is almost sufficient for G being circular super-critical.

Theorem 3. *Let G be a k-chromatic critical graph ($k \geq 3$) with $d(u) = \frac{\chi_c(G\setminus u)}{\chi_c(G\setminus u)+2-\chi_c(G)}$ for every vertex u. Then, $\chi_c(G \setminus u) \leq \chi_c(G) - 1$ and the equality holds iff $d(u) = k - 1$ and $\chi_c(G) = k$.*

As a corollary of Theorem 2, we get an upper bound on the circular chromatic number of circular super-critical graphs that yields a partial answer to Question 2.

Corollary 1. *If G is a circular super-critical graph such that for some constant $\varepsilon > 0$, $\chi_c(G) - \chi_c(G \setminus u) \geq 2 - \varepsilon$ for every vertex u of G, then $\chi_c(G) \leq \varepsilon(\delta(G) - 1) + 2$.*

Proof. Since G is circular super-critical, by Theorem 2, for every vertex u, $d(u) \geq \frac{\chi_c(G\setminus u)}{\chi_c(G\setminus u)+2-\chi_c(G)}$, i.e.,

$$\chi_c(G \setminus u) \geq \frac{d(u)(\chi_c(G) - 2)}{d(u) - 1}$$

because $d(u) > 1$. Therefore, $\chi_c(G \setminus u_0) \geq \frac{\delta(G)(\chi_c(G)-2)}{\delta(G)-1}$ for some vertex u_0 of degree $\delta(G)$.

Since $\chi_c(G) - \chi_c(G \setminus u_0) \geq 2 - \varepsilon$, $\chi_c(G) - 2 + \varepsilon \geq \chi_c(G \setminus u_0) \geq \frac{\delta(G)(\chi_c(G)-2)}{\delta(G)-1}$, and hence $\chi_c(G) \leq \varepsilon(\delta(G) - 1) + 2$ as desired. □

Let G be a graph of order n. A vertex u of G is called a *major vertex* if $d(u) = n - 1$. If G contains a vertex, say v, of degree at least $n - 2$, then there must exist a vertex, say w, in G such that v is a major vertex in $G \setminus w$. Since a graph with a major vertex has the same circular chromatic number as chromatic number, $\chi_c(G \setminus w) = \chi(G \setminus w) \geq \chi(G) - 1 \geq \chi_c(G) - 1$.

Therefore, if $0 < \varepsilon < 1$ and $\chi_c(G) - \chi_c(G \setminus u) \geq 2 - \varepsilon$ for every vertex u, then G is a circular super-critical graph and $\delta(G) \leq |V(G)| - 3$, and hence $\chi_c(G) \leq \varepsilon(\delta(G) - 1) + 2 \leq \varepsilon(|V(G)| - 4) + 2 \leq \varepsilon|V(G)| + 2 - 4\varepsilon < \varepsilon|V(G)| + 2$ for some $\varepsilon > 0$. In another words, Corollary 1 says that for any constant $\varepsilon > 0$, Question 2 has no solutions in graphs with $\chi(G) \geq \varepsilon|V(G)| + 3$.

3. Proofs of the theorems

The following two lemmas will be used in the proof of Theorem 1.

Lemma 1 (*[5]*). *Let G be a graph such that the complement G^c of G is non-Hamilton. Then $\chi_c(G) = \chi(G)$.*

Lemma 2 (*[6]*). *Let G be a graph and u and v be two nonadjacent vertices of G such that in any $\chi(G)$-coloring of G, u and v always receive the same color. Then, $\chi_c(G) = \chi(G)$.*

To prove Theorem 1, we still need a lemma on the chromatic critical graphs. Let S be a cut set of a graph G, G_1 and G_2 be two induced subgraphs of G such that $V(G_1) \cap V(G_2) = S$ and $E(G_1) \cup E(G_2) = E(G)$. We call G_1 and G_2 a *pair of S-components* of G.

Lemma 3. *Let G be a k-chromatic critical graph, and S be a cut set of G that induces a subgraph in which all vertices are pairwise adjacent except u and v. Then, there are two induced subgraphs G_1 and G_2 of G such that*

(1) *in any $(k - 1)$-coloring of G_1, u and v always receive the same color, and*
(2) *in any $(k - 1)$-coloring of G_2, u and v always receive distinct colors.*

The proof is easy and we omit it.

Proof of Theorem 1. Let $\chi_c(G) = r$ and $\chi(G) = \lceil r \rceil = k$. Since the circular super-criticality implies chromatic criticality, G must be 2-connected. Suppose that S is a cut set of G, and let G_1 and G_2 be a pair of S-components. Then, $\chi_c(G_1) < r - 1$ and $\chi_c(G_2) < r - 1$.

If $|S| = 2$, we let $S = \{u, v\}$, then $uv \notin E(G)$. By a theorem of Dirac [4] (see also Theorem 8.3 of [1]), we may suppose that in any $(k - 1)$-coloring of G_1, u and v always receive the same color. Therefore, $r - 1 > \chi_c(G_1) = \chi(G_1) = k - 1 \geq r - 1$ by Lemma 2. This contradiction shows that G is 3-connected.

If $|S| \geq 3$ and the number of edges in the subgraph induced by S is no less than $\frac{|S|(|S|-1)}{2} - 1$, we may assume that all but two of the vertices, say u and v, of S are adjacent to each other. By Lemma 3, G has a $(k - 1)$-chromatic subgraph, say G_1, such that in any $(k - 1)$-coloring of G_1, u and v always receive the same color. A contradiction immediately follows from Lemma 2.

Let H be the complement of G. If H is separable, then H is non-Hamilton and $H \setminus x$ is non-Hamilton for every cut vertex x of H. By Lemma 1, $\chi_c(G) = \chi(G)$ and $\chi_c(G \setminus x) = \chi(G \setminus x)$ for every cut vertex x. This contradiction shows that H is 2-connected. □

For a real number $r > 0$ and two real numbers $p, q \in [0, r)$, we define the modulo r interval from p to q, denoted by $[p, q]_r$, to be $[p, q]$ if $p \leq q$, and to be $[p, r) \cup [0, q]$ if $p > q$.

Proof of Theorem 2. Let $\chi_c(G) = r$, $\chi_c(G \setminus u) = s$ and let $r - s = t > 1$. Let ϕ be a circular s-coloring of $G \setminus u$.

First we claim that for each $x \in N(u)$, there exists a vertex $y \in N(u)$ such that $|[\phi(x), \phi(y)]_r| \le 2 - t$. Otherwise, assume that there exists an $x \in N(u)$ such that

$$\text{for every } y \in N(u) \setminus \{x\}, |[\phi(x), \phi(y)]_r| > 2 - t. \tag{2}$$

Without loss of generality, we assume that $\phi(x) = 0$. Let $A = \{y \mid y \in N(u) \text{ and } 0 < \phi(y) < 1\}$. If $A \ne \emptyset$, then let $\alpha = \max_{y \in A}\{1 - \phi(y)\}$. Otherwise, let $\alpha = 0$. By (2),

$$\alpha < 1 - (2 - t) = t - 1. \tag{3}$$

Let $r' = s + \alpha + 1$, and let $\psi : V(G) \longmapsto [0, r')$ be defined as

$$\begin{cases} \psi(x) = 0. \\ \psi(u) = 1. \\ \psi(y) = r' - (s - \phi(y)), & \text{if } \phi(y) \ge 1. \\ \psi(y) = \phi(y), & \text{if } \phi(y) < 1 \text{and } y \notin N(u). \\ \psi(y) = 2, & \text{if } y \in A. \end{cases}$$

Since $r' > s$, it is clear that $\psi(w) \ne \psi(u)$ for each $w \ne u$.

Let w_1 and w_2 be two vertices in $V(G)\setminus(A \cup \{u\})$. Then, $|\psi(w_2) - \psi(w_1)| = |\phi(w_2) - \phi(w_1)|$. By the properness of ϕ, $1 \le |\psi(w_2) - \psi(w_1)| = |\phi(w_2) - \phi(w_1)| \le s - 1 < r' - 1$ for every adjacent pair $w_1, w_2 \in V(G) \setminus (A \cup \{u\})$.

Let y be a vertex in $N(u)$. Then, either $\psi(y) = 2 < r'$ whenever $y \in A$, or $r' > r' - (s - \phi(y)) = \psi(y) = \alpha + 1 + \phi(y) \ge 2$ whenever $\phi(y) \ge 1$. In either case, $1 \le \psi(y) - \psi(u) = \psi(y) - 1 < r' - 1$.

Since for any two vertices y_1, y_2 with $0 < \phi(y_1) < 1$ and $0 < \phi(y_2) < 1$, $|\phi(y_1) - \phi(y_2)| < 1$, the properness of ϕ implies that $\{y \mid \phi(y) < 1\} \supset A$ is an independent set.

Finally, let w be a vertex in A (note that $\psi(w) = 2$) and $y \ne u$ be a vertex with $\phi(y) \ge 1$. To show that ψ is a circular r'-coloring of G, it suffices to prove that $1 \le |\psi(y) - \psi(w)| \le r' - 1$ whenever $wy \in E(G)$. By the definition of α, $\phi(w) \ge 1-\alpha$. Since $r' > r'-(s-\phi(y)) = \psi(y) = \alpha+1+\phi(y) \ge 2$, $0 \le \psi(y)-\psi(w) < r'-1$. If $1 > \psi(y)-\psi(w) = \psi(y)-2 = \alpha-1+\phi(y)$, then $\phi(y) < 2-\alpha$, and hence $0 < \phi(y)-\phi(w) < 2-\alpha-(1-\alpha) = 1$. The properness of ϕ shows that ψ is a circular r'-coloring of G. That contradicts $r' = s+\alpha+1 < s+(t-1)+1 = s+t = r$ by (3), and ends the proof of our claim.

Now, we apply the claim to prove our theorem. Choose v_0 to be an arbitrary neighbor of u; beginning from v_0, there must be a sequence $v_1, v_2, \ldots, v_l$ of neighbors of u such that $0 < |[\phi(v_i), \phi(v_{i+1})]_r| \le 2 - t$, $i = 0, 1, \ldots, l \pmod l$, and $(\cup_{i=0}^{l-1}[\phi(v_i), \phi(v_{i+1})]_r) \cup [\phi(v_l), \phi(v_0)]_r = C$. Therefore, $d(u) \ge \frac{s}{2-t} = \frac{\chi_c(G\setminus u)}{\chi_c(G\setminus u)+2-\chi_c(G)}$ as required. □

Proof of Theorem 3. Since G is k-chromatic critical,

$$\frac{\chi_c(G \setminus u)}{\chi_c(G \setminus u) + 2 - \chi_c(G)} = d(u) \ge \delta(G) \ge k - 1 \ge \chi_c(G) - 1, \tag{4}$$

and

$$\begin{aligned} & \frac{\chi_c(G \setminus u)}{\chi_c(G \setminus u) + 2 - \chi_c(G)} \ge \chi_c(G) - 1 \\ \Leftrightarrow \quad & \chi_c(G \setminus u) \ge (\chi_c(G \setminus u) + 2 - \chi_c(G))(\chi_c(G) - 1) \\ \Leftrightarrow \quad & (\chi_c(G) - 2)\chi_c(G \setminus u) \le (\chi_c(G) - 2)(\chi_c(G) - 1) \\ \Leftrightarrow \quad & \chi_c(G \setminus u) \le \chi_c(G) - 1. \end{aligned} \tag{5}$$

The equalities hold in (5) iff the equalities hold in (4), i.e., $d(u) = k - 1$ and $\chi_c(G) = k$. □

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