An analogue of Dirac’s theorem on circular super-critical graphs✩

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Abstract

A graph $G$ is called circular super-critical if $\chi_c(G \setminus u) < \chi_c(G) - 1$ for every vertex $u$ of $G$. In this paper, analogous to a result of Dirac on chromatic critical graphs, a sharp lower bound on the vertex degree of circular super-critical graphs is proved. This lower bound provides a partial answer to a question of X. Zhu [The circular chromatic number of induced subgraphs, J. Combin. Theory Ser. B 92 (2004) 177–181]. Some other structural properties of circular super-critical graphs are also presented.

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1. Introduction

An $n$-coloring of $G$ is a mapping $\phi : V(G) \mapsto \{1, 2, \ldots, n\}$ such that $\phi(u) \neq \phi(v)$ for every $uv \in E(G)$. The least integer $k$ such that $G$ admits a $k$-coloring is called the chromatic number of $G$, and is denoted by $\chi(G)$. A graph is called a chromatic critical graph if $\chi(G \setminus u) < \chi(G)$ for each vertex $u$ of $G$. A $k$-chromatic critical graph is a chromatic critical graph $G$ with $\chi(G) = k$.

Let $r \geq 1$ be a real number. A circular $r$-coloring of a graph $G$ is a mapping $\psi : V(G) \mapsto [0, r)$ such that for every edge $uv$ of $G$, $1 \leq |\psi(u) - \psi(v)| \leq r - 1$ [11]. A graph is called circular $r$-colorable if it admits a circular $r$-coloring. The circular chromatic number of $G$, denoted by $\chi_c(G)$, is the least $r$ such that $G$ is circular $r$-colorable.

It was proved elsewhere [2,7] that $\chi_c(G)$ is always attained at rational number and

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G)$$

(1)

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If $\chi_c(G) = \frac{k}{d}$, then a circular $\frac{k}{d}$-coloring of $G$ is equivalent to a mapping $\psi : V(G) \rightarrow \{0, 1, 2, \ldots, k-1\}$ with $d \leq |\psi(u) - \psi(v)| \leq k - d$ for every $uv \in E(G)$.

It is well known that the chromatic number decreases by at most 1 when one vertex is removed from the graph. But this is not valid for the circular chromatic number. From (1), it is easy to check that after removing a vertex from a graph, the decrease of the circular chromatic number must be less than 2. But there are infinitely many examples showing that the decrease could be arbitrarily close to 2.

If $G$ is a chromatic critical graph, then the chromatic number decreases by exactly 1 on removing an arbitrary vertex, i.e., a $k$-chromatic critical graph $G$ is a graph with $\chi(G) = k$ and $\chi(G \setminus u) = k - 1$ for every vertex $u$ of $G$. The study of chromatic critical graphs was started by Dirac [3] who proved that for every $k$-chromatic critical graph $G$, $\delta(G) \geq k - 1$ and the subgraph induced by an arbitrary cut set of $G$ is not complete. Since $K_k$, the complete graph of order $k$, is $k$-chromatic critical, the bound $\delta(G) \geq k - 1$ is sharp.

We may define, analogous to the concept of chromatic critical graphs, a circular critical graph of a graph $G$ with $\chi_c(G \setminus u) < \chi_c(G)$ for each vertex $u$ of $G$. Here, instead of considering the circular critical graphs, we focus on a family of graphs that demands more. A graph $G$ is called circular super-critical if $\chi_c(G \setminus u) < \chi_c(G) - 1$ for each vertex $u$ of $G$. A $\frac{k}{d}$-circular super-critical graph is a circular super-critical graph with circular chromatic $\frac{k}{d}$. It is clear that there is no $r$-circular super-critical graph for rational $r \leq 3$.

Does there exist a circular super-critical graph? In [8], Zhu conjectured a negative answer to this question. In [10], Zhu disproved his conjecture by constructing an infinite family of 4-regular 4-circular super-critical graphs $G$ with $\chi_c(G - x) = \frac{8}{3}$ for every vertex $x$ of $G$, and proposed several new questions as follows.

**Question 1** ([10]). Given integer $n \geq 5$, is there a circular super-critical graph $G$ with $\chi(G) = n$?

**Question 2** ([10]). Is there a circular super-critical graph $G$ such that for every vertex $u$ of $G$, $\chi_c(G) - \chi_c(G \setminus u) \geq 2 - \varepsilon$ for some $\varepsilon < \frac{2}{3}$? Or even for any $\varepsilon > 0$?

**Question 3** ([10]). Is there a graph $G$ for which $\chi_c(G) \neq \chi(G)$ and yet there is a vertex $u$ of $G$ such that $\chi_c(G) - \chi_c(G \setminus u) > 1$?

Since $\chi_c(G \setminus u) \leq \chi_c(G) - 1$ implies $\chi(G \setminus u) = \chi(G) - 1$, every $\frac{k}{d}$-circular super-critical graph is $\lceil \frac{k}{d} \rceil$-chromatic critical. Therefore, every circular super-critical graph is 2-connected. We will show that for every circular super-critical graph $G$, $G$ is 3-connected and the complement of $G$ is 2-connected. Analogous to Dirac’s theorem that says $\delta(G) \geq k - 1$ for every $k$-chromatic critical graph $G$, a sharp lower bound on the vertex degree of circular super-critical graphs is proved.

2. Main results

**Theorem 1.** If $G$ is a circular super-critical graph, then $G$ is 3-connected and the complement of $G$ is 2-connected. Furthermore, for any cut set $S$ of $G$ of cardinality at least 3, the subgraph induced by $S$ contains at most $\frac{|S|(|S| - 1)}{2} - 2$ edges.

**Theorem 2.** If $u$ is a vertex of a graph $G$ with $\chi_c(G) - \chi_c(G \setminus u) > 1$, then $d(u) \geq \frac{\chi_c(G \setminus u)}{\chi_c(G) - \chi_c(G \setminus u) - 1}$. 

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Theorem 2 provides us with a necessary condition for a graph to be circular super-critical. If a graph $G$ is circular super-critical, then for every vertex $u$, $\chi_c(G) - \chi_c(G \setminus u) > 1$ and hence $d(u) \geq \frac{\chi_c(G \setminus u)}{\chi_c(G) - 1}$.

Since the function $\frac{x}{x - (\chi_c(G) - 1)}$ is monotone decreasing on $x \in (\chi_c(G) - 2, \chi_c(G) - 1)$, and since $\chi_c(G \setminus u) < \chi_c(G) - 1$, $\delta(G) > \left[ \frac{\chi_c(G) - 1}{\chi_c(G) - 2} \right] = \chi_c(G) - 1$ for every circular super-critical graph, where $\chi(G) - 1$ is Dirac's bound on the minimum degree of chromatic critical graphs.

The bound of Theorem 2 is best possible in the sense that for any integer $n \geq 4$, there exists a graph $G$ and a vertex $u$ of $G$ such that $\chi_c(G) = n$, $\chi_c(G \setminus u) > 1$ and $d(u) = \frac{\chi_c(G \setminus u)}{\chi_c(G) - 1}$. When $n = 4$, the graph $G$ constructed by Zhu in [10] is a 4-circular super-critical graph with $d(x) = 4 = \frac{\delta(G \setminus x)}{\chi_c(G \setminus x) - 2}$ for each vertex $x$. The other 4-chromatic example is as follows. Given integer $l \geq 2$, let $G$ be the graph obtained from the odd circuit $C_{2l+1}$ of length $2l + 1$ by adding a new vertex $u$ and joining $u$ to every vertex of $C_{2l+1}$. Then, $\chi_c(G) = 4$, $\chi_c(C_{2l+1}) = 2l + 1$, and $d(u) = 2l + 1 = \frac{2l + 1}{4} = \frac{\chi_c(G \setminus u)}{\chi_c(G) + 2 - \chi_c(G)}$.

For $n \geq 4$, let $G$ be the graph obtained from the complement of the odd circuit $C_{2n-1}$ by adding a new vertex $u$ and joining $u$ to each of the other vertices. Then, $\chi_c(G) = \chi(G) = n + 1$, $\chi_c(G \setminus u) = n - 1$ and $d(u) = 2n - 1 = \frac{2n - 1}{n + 1} = \frac{\chi_c(G \setminus u)}{\chi_c(G) + 2 - \chi_c(G)}$.

The converse of Theorem 2 may not be true. To see this, let us consider the Petersen graph $P$. It is known that $\chi_c(P) = \chi(P) = 3$ (see [9]). Since for an arbitrary vertex $x$ of $P$, $P \setminus x$ contains $C_5$ as an induced subgraph, $\chi_c(P \setminus x) \geq \frac{5}{2}$ (in fact, it is easy to check that $\chi_c(P \setminus x) = 3$). Hence $\frac{\chi_c(P \setminus x) + 2 - \chi_c(P)}{\chi_c(P \setminus x) - 1} \leq \frac{3}{1.5} \leq 2$. Therefore, $3 = d(x) \geq \frac{\chi_c(P \setminus x)}{\chi_c(P \setminus x) + 2 - \chi_c(P)}$ for every vertex $x$ of $P$. But $P$ is not circular super-critical.

Circular super-critical graphs are all chromatic critical. We may restrict our focus to chromatic critical graphs. Theorem 3 below shows that if $G$ is chromatic critical, then $d(u) = \frac{\chi_c(G \setminus u)}{\chi_c(G \setminus u) + 2 - \chi_c(G)}$ for every vertex $u$ is almost sufficient for $G$ being circular super-critical.

**Theorem 3.** Let $G$ be a $k$-chromatic critical graph ($k \geq 3$) with $d(u) = \frac{\chi_c(G \setminus u)}{\chi_c(G) - 1}$ for every vertex $u$. Then, $\chi_c(G \setminus u) \leq \chi_c(G) - 1$ and the equality holds iff $d(u) = k - 1$ and $\chi_c(G) = k$.

As a corollary of Theorem 2, we get an upper bound on the circular chromatic number of circular super-critical graphs that yields a partial answer to Question 2.

**Corollary 1.** If $G$ is a circular super-critical graph such that for some constant $\varepsilon > 0$, $\chi_c(G) - \chi_c(G \setminus u) \geq 2 - \varepsilon$ for every vertex $u$ of $G$, then $\chi_c(G) \leq \varepsilon(\delta(G) - 1) + 2$.

**Proof.** Since $G$ is circular super-critical, by Theorem 2, for every vertex $u$, $d(u) \geq \frac{\chi_c(G \setminus u)}{\chi_c(G) - 1}$, i.e.,

$$\chi_c(G \setminus u) \geq \frac{d(u)(\chi_c(G) - 2)}{d(u) - 1}$$

because $d(u) > 1$. Therefore, $\chi_c(G \setminus u_0) \geq \frac{\delta(G)(\chi_c(G) - 2)}{\delta(G) - 1}$ for some vertex $u_0$ of degree $\delta(G)$.

Since $\chi_c(G) - \chi_c(G \setminus u_0) \geq 2 - \varepsilon$, $\chi_c(G) - 2 + \varepsilon \geq \chi_c(G \setminus u_0) \geq \frac{\delta(G)(\chi_c(G) - 2)}{\delta(G) - 1}$, and hence $\chi_c(G) \leq \varepsilon(\delta(G) - 1) + 2$ as desired. □
Let $G$ be a graph of order $n$. A vertex $u$ of $G$ is called a major vertex if $d(u) = n - 1$. If $G$ contains a vertex, say $v$, of degree at least $n - 2$, then there must exist a vertex, say $w$, in $G$ such that $v$ is a major vertex in $G \setminus w$. Since a graph with a major vertex has the same circular chromatic number as chromatic number, $\chi_c(G \setminus w) = \chi(G \setminus w) \geq \chi(G) - 1 \geq \chi_c(G) - 1$.

Therefore, if $0 < \varepsilon < 1$ and $\chi_c(G) - \chi_c(G \setminus u) \geq 2 - \varepsilon$ for every vertex $u$, then $G$ is a circular super-critical graph and $\delta(G) \leq |V(G)| - 3$, and hence $\chi_c(G) \leq \varepsilon(\delta(G) - 1) + 2 \leq \varepsilon(|V(G)| - 4) + 2 \leq \varepsilon|V(G)| + 2 - 4\varepsilon < \varepsilon|V(G)| + 2$ for some $\varepsilon > 0$. In another words, Corollary 1 says that for any constant $\varepsilon > 0$, Question 2 has no solutions in graphs with $\chi(G) \geq \varepsilon|V(G)| + 3$.

3. Proofs of the theorems

The following two lemmas will be used in the proof of Theorem 1.

Lemma 1 ([5]). Let $G$ be a graph such that the complement $G^c$ of $G$ is non-Hamilton. Then $\chi_c(G^c) = \chi(G)$.

Lemma 2 ([6]). Let $G$ be a graph and $u$ and $v$ be two nonadjacent vertices of $G$ such that in any $\chi(G)$-coloring of $G$, $u$ and $v$ always receive the same color. Then, $\chi_c(G) = \chi(G)$.

To prove Theorem 1, we still need a lemma on the chromatic critical graphs. Let $S$ be a cut set of a graph $G$, $G_1$ and $G_2$ be two induced subgraphs of $G$ such that $V(G_1) \cap V(G_2) = S$ and $E(G_1) \cup E(G_2) = E(G)$. We call $G_1$ and $G_2$ a pair of $S$-components of $G$.

Lemma 3. Let $G$ be a $k$-chromatic critical graph, and $S$ be a cut set of $G$ that induces a subgraph in which all vertices are pairwise adjacent except $u$ and $v$. Then, there are two induced subgraphs $G_1$ and $G_2$ of $G$ such that

1. in any $(k - 1)$-coloring of $G_1$, $u$ and $v$ always receive the same color, and
2. in any $(k - 1)$-coloring of $G_2$, $u$ and $v$ always receive distinct colors.

The proof is easy and we omit it.

Proof of Theorem 1. Let $\chi_c(G) = r$ and $\chi(G) = \lceil r \rceil = k$. Since the circular super-criticality implies chromatic criticality, $G$ must be 2-connected. Suppose that $S$ is a cut set of $G$, and let $G_1$ and $G_2$ be a pair of $S$-components. Then, $\chi_c(G_1) < r - 1$ and $\chi_c(G_2) < r - 1$.

If $|S| = 2$, we let $S = \{u, v\}$, then $uv \notin E(G)$. By a theorem of Dirac [4] (see also Theorem 8.3 of [1]), we may suppose that in any $(k - 1)$-coloring of $G_1$, $u$ and $v$ always receive the same color. Therefore, $r - 1 > \chi_c(G_1) = \chi(G_1) = k - 1 \geq r - 1$ by Lemma 2. This contradiction shows that $G$ is $3$-connected.

If $|S| \geq 3$ and the number of edges in the subgraph induced by $S$ is no less than $\frac{|S|(|S| - 1)}{2} - 1$, we may assume that all but two of the vertices, say $u$ and $v$, of $S$ are adjacent to each other. By Lemma 3, $G$ has a $(k - 1)$-chromatic subgraph, say $G_1$, such that in any $(k - 1)$-coloring of $G_1$, $u$ and $v$ always receive the same color. A contradiction immediately follows from Lemma 2.

Let $H$ be the complement of $G$. If $H$ is separable, then $H$ is non-Hamilton and $H \setminus x$ is non-Hamilton for every cut vertex $x$ of $H$. By Lemma 1, $\chi_c(G) = \chi(G)$ and $\chi_c(G \setminus x) = \chi(G \setminus x)$ for every cut vertex $x$. This contradiction shows that $H$ is 2-connected. \[\square\]

For a real number $r > 0$ and two real numbers $p, q \in [0, r)$, we define the modulo $r$ interval from $p$ to $q$, denoted by $[p, q]_r$, to be $[p, q]$ if $p \leq q$, and to be $[p, r) \cup [0, q]$ if $p > q$. 

Proof of Theorem 2. Let $\chi_c(G) = r$, $\chi_c(G \setminus u) = s$ and let $r - s = t > 1$. Let $\phi$ be a circular $s$-coloring of $G \setminus u$.

First we claim that for each $x \in N(u)$, there exists a vertex $y \in N(u)$ such that $||\phi(x), \phi(y)||_r \leq 2 - t$. Otherwise, assume that there exists an $x \in N(u)$ such that

$$\text{for every } y \in N(u) \setminus \{x\}, ||\phi(x), \phi(y)||_r > 2 - t. \tag{2}$$

Without loss of generality, we assume that $\phi(x) = 0$. Let $A = \{y \mid y \in N(u) \text{ and } 0 < \phi(y) < 1\}$. If $A \neq \emptyset$, then let $\alpha = \max_{y \in A} \{1 - \phi(y)\}$. Otherwise, $\alpha = 0$. By (2),

$$\alpha < 1 - (2 - t) = t - 1. \tag{3}$$

Let $r' = s + \alpha + 1$, and let $\psi : V(G) \mapsto [0, r')$ be defined as

$$\begin{align*}
\psi(x) &= 0, \\
\psi(u) &= 1, \\
\psi(y) &= r' - (s - \phi(y)), \quad \text{if } \phi(y) \geq 1, \\
\psi(y) &= \phi(y), \quad \text{if } \phi(y) < 1 \text{ and } y \not\in N(u), \\
\psi(y) &= 2, \quad \text{if } y \in A.
\end{align*}$$

Since $r' > s$, it is clear that $\psi(w) \neq \psi(u)$ for each $w \neq u$.

Let $w_1$ and $w_2$ be two vertices in $V(G) \setminus (A \cup \{u\})$. Then, $|\psi(w_2) - \psi(w_1)| = |\phi(w_2) - \phi(w_1)|$.

By the properness of $\phi$, $1 \leq |\psi(w_2) - \psi(w_1)| = |\phi(w_2) - \phi(w_1)| \leq s - 1 < r' - 1$ for every adjacent pair $w_1, w_2 \in V(G) \setminus (A \cup \{u\})$.

Let $y$ be a vertex in $N(u)$. Then, either $\psi(y) = 2 < r'$ whenever $y \in A$, or $r' > r' - (s - \phi(y)) = \psi(y) = \alpha + 1 + \phi(y) \geq 2$ whenever $\phi(y) \geq 1$. In any case, $1 \leq |\psi(y) - \psi(u)| = |\psi(y) - 1 < r' - 1$.

Since for any two vertices $y_1, y_2$ with $0 < \phi(y_1) < 1$ and $0 < \phi(y_2) < 1, |\phi(y_1) - \phi(y_2)| < 1$, the properness of $\phi$ implies that $[y \mid \phi(y) < 1] \supset A$ is an independent set.

Finally, let $w$ be a vertex in $A$ (note that $\psi(w) = 2$) and $y \neq u$ be a vertex with $\phi(y) \geq 1$. To show that $\psi$ is a circular $r'$-coloring of $G$, it suffices to prove that $1 \leq |\psi(y) - \psi(w)| \leq r' - 1$ whenever $wy \in E(G)$. By the definition of $\alpha$, $\phi(w) \geq 1 - \alpha$. Since $r' > r' - (s - \phi(y)) = \psi(y) = \alpha + 1 + \phi(y) \geq 2$, $0 \leq |\psi(y) - \psi(u)| < r' - 1$. If $1 > |\psi(y) - \psi(w)| = |\psi(y) - 2 = \alpha + 1 + \phi(y)|$, then $\phi(y) < 2 - \alpha$, and hence $0 < \phi(y) - \phi(w) < 2 - \alpha - (1 - \alpha) = 1$. The properness of $\phi$ shows that $\psi$ is a circular $r'$-coloring of $G$. That contradicts $r' = s + \alpha + 1 < s + (t - 1) + 1 = s + t = r$ by (3), and ends the proof of our claim.

Now, we apply the claim to prove our theorem. Choose $v_0$ to be an arbitrary neighbor of $u$; beginning from $v_0$, there must be a sequence $v_1, v_2, \ldots, v_l$ of neighbors of $u$ such that $0 < ||\phi(v_i), \phi(v_{i+1})||_r \leq 2 - t, i = 0, 1, \ldots, l \text{ (mod } l\text{), and } (\bigcup_{i=0}^{l-1}[\phi(v_i), \phi(v_{i+1})])_r \cup \{\phi(v_l)\} = C$. Therefore, $d(u) \geq \frac{s}{2-t} = \frac{\chi_c(G \setminus u)}{\chi_c(G \setminus u) + 2 - \chi_c(G)}$ as required. \hfill \Box

Proof of Theorem 3. Since $G$ is $k$-chromatic critical,

$$\frac{\chi_c(G \setminus u)}{\chi_c(G \setminus u) + 2 - \chi_c(G)} \geq d(u) \geq \delta(G) \geq k - 1 \geq \chi_c(G) - 1, \tag{4}$$

and

$$\begin{align*}
\chi_c(G \setminus u) &\geq (\chi_c(G \setminus u) + 2 - \chi_c(G))(\chi_c(G) - 1) \\
\implies (\chi_c(G) - 2)\chi_c(G \setminus u) &\leq (\chi_c(G) - 2)(\chi_c(G) - 1) \\
\implies \chi_c(G \setminus u) &\leq \chi_c(G) - 1.
\end{align*} \tag{5}$$
The equalities hold in (5) iff the equalities hold in (4), i.e., \( d(u) = k - 1 \) and \( \chi_c(G) = k \).

□

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