1. In a recent paper S. Zucker has proved the Hodge conjecture for cubic fourfolds [4]. His proof uses the method of normal functions. Zucker's paper contains also an alternate proof, due to Clemens, based on the families of lines on a cubic threefold. In this paper we want to give still another proof, valid for any unirational fourfold.

Let $X$ be a smooth, projective variety defined over the complex numbers $C$. Consider the Hodge decomposition:

$$H^i(X, \mathbb{C}) = \sum_{p+q=i} H^{p,q}(X).$$

Let $CH^p(X)$ denote the Chow group of algebraic cycles on $X$, modulo rational equivalence, of codimension $p$. Consider the standard map:

$$\lambda^p_Q: CH^p(X) \otimes \mathbb{Q} \to H^{2p}(X, \mathbb{Q}) \cap H^{p,p}(X).$$

The Hodge $(p, p)$-conjecture states that the map $\lambda^p_Q$ is surjective, i.e. that every rational cohomology class of type $(p, p)$ comes from an algebraic cycle with rational coefficients.

**Theorem.** If $X$ is a smooth, projective unirational fourfold (i.e. a variety of dimension 4) then the Hodge $(2,2)$-conjecture is true.

**Remarks:**

1. It is well-known that the $(1,1)$ and $(3,3)$-conjecture are true for any fourfold.

2. Recall that a $n$-dimensional variety $X$ (over $C$) is called unirational if there exists a surjective rational transformation $h: P_n \to X$.

**Corollary.** The Hodge $(2,2)$-conjecture is true for a smooth cubic fourfold in projective space $P_5$.

The proof that such a cubic fourfold is unirational is the same as for a cubic threefold, see [2] App. B.

2. The proof of the theorem is based on the following two observations:
**Lemma 1** (cf [4], A2). Let $X$ and $Y$ be smooth projective varieties of the same dimensions and $f : X \to Y$ a proper surjective morphism (hence of finite degree $q$). If the Hodge $(p, p)$-conjecture is true for $X$ then it is also true for $Y$.

**Proof:** Consider $f_* f^* : H^{2p}(X, \mathbb{Q}) \to H^{2p}(Y, \mathbb{Q})$.

\[ f_* f^* = q = \text{deg}(f). \]

Also $f^*$ and $f_*$ respect the Hodge decomposition. Furthermore there are similar homomorphisms with similar properties for the Chow groups. Let $a \in H^{p,p}(Y) \cap H^{2p}(Y, \mathbb{Q})$ then by assumption $f^*(a) = \lambda_X(a)$ with $a \in CH^p(X) \otimes \mathbb{Q}$. Then

\[ a = \frac{1}{q} f_* f^*(a) = \frac{1}{q} f_* \lambda_X(a) = \frac{1}{q} \lambda_Y(f_*(a)) \text{ and } f_*(a) \in CH^p(Y) \otimes \mathbb{Q}. \]

**Lemma 2.** Let $X$ be a smooth fourfold for which the Hodge $(2,2)$-conjecture is true. Let $Y \subset X$ be a smooth surface (resp. smooth curve, resp. point). Let $X' = B_Y(X)$ be obtained by blowing up $X$ along $Y$. Then the $(2,2)$-conjecture is true for $X'$.

**Proof:** Consider the usual diagram

\[
\begin{array}{ccc}
Y' & \subset & X' \\
\downarrow g & j & \downarrow f \\
Y & \subset & X \\
\end{array}
\]

Take the case of a surface $Y$. We have

\[ H^4(X', \mathbb{Q}) = H^4(X, \mathbb{Q}) \oplus H^2(Y, \mathbb{Q}), \]

and similar relations for $H^4(X', \mathbb{C})$ and also

\[ CH^2(X') = CH^2(X) \oplus CH^1(Y). \]

Moreover the inclusion $H^2(Y, \mathbb{Q}) \subset H^4(X', \mathbb{Q})$ is given by $j_* \cdot g^*$ and we have for the Hodge decomposition

\[ H^{a,b}(Y) \to H^{a,b}(Y') \to H^{a+1,b+1}(X'). \]

The $(2,2)$-conjecture for $X'$ now follows from the $(2,2)$-fact for $X$ and the $(1,1)$-fact for $Y$.

**Proof of the Theorem:** Since $X$ is a unirational fourfold we have
a commutative diagram [3]

\[
\begin{array}{ccc}
  X' & \xrightarrow{f} & X \\
  \downarrow{g} & & \downarrow{h} \\
  P_4 & \xrightarrow{=} & X
\end{array}
\]

where \(h\) is a rational transformation, \(f\) is a proper surjective morphism of finite degree and \(g\) is obtained by successive blowing ups of smooth surfaces, curves and points. By lemma 2 the \((2,2)\)-conjecture is true for \(X'\) and hence, by lemma 1, it is true for \(X\).

3. **Remarks**

1. The same type of arguments show that on a unirational fourfold \(X\) numerical, cohomological and torsion equivalence coincide.

2. Using ideas of Bloch [1] one can show by arguments of the above type that the Hodge \((2,2)\)-conjecture holds on a smooth quartic fourfold in \(P_3\).

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**References**


